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A Turán–Kubilius type inequality on shifted products

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Abstract. In 1934 Turán proved that if $f(n)$ is an additive arithmetic function satisfying certain conditions, then for almost all $m \leqslant n$ the value of $f(m)$ is "near" the expectation $\sum_{p\leqslant n}\frac{f(p)}{p}$. Later Kubilius sharpened this result by proving that the conditions in Turán's theorem can be relaxed, and still the same conclusion holds. In an earlier paper we studied whether this result has a sum set analogue, *i.e.*, if $f(n)$ is an additive arithmetic function and $\mathcal{A}$, $\mathcal{B}$ are "large" subsets of $\{1,2,\dots,n\}$, then for almost all $a \in \mathcal{A}$, $b \in \mathcal{B}$, the value of $f(a+b)$ is "near" the expectation? We proved such a result under an assumption which is slightly milder than Turán's condition, but is not needed in Kubilius estimate. In this paper we prove the multiplicative analogue of this theorem by proving a similar result with $ab+1$ in place of $a+b$.

1. Introduction

$\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, real numbers, resp. complex numbers. The letters p, q denote prime numbers and $\omega(n)$ denotes the number of distinct prime factors of n, while $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity.

Generalizing a theorem of Hardy and Ramanujan [5], Turán [10] proved that if $f(n)$ is a real valued additive arithmetic function with

$$f(p) = f(p^2) = \cdots = f(p^k) = \cdots \tag{1}$$

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for every prime number p and it is bounded:

$$|f(p)| = O(1), \tag{2}$$

then, writing

$$A_f(n) = \sum_{p \leqslant n} \frac{f(p)}{p}, \tag{3}$$

we have

$$\sum_{m \leqslant n} (f(m) - A_f(n))^2 = O(nA_f(n)). \tag{4}$$

In [6] KUBILIUS showed that Turán's conditions $f(n) \in \mathbb{R}$, (1) and (2) can be dropped, and still there is an inequality of type (4): if $f(n)$ is a complex valued additive arithmetic function, $A_f(n)$ is defined by (3), and we also write

$$D_f(n) = \left(\sum_{p^\alpha \leqslant n} \frac{|f(p^\alpha)|^2}{p^\alpha} \right)^{1/2}, \tag{5}$$

then we have

$$\sum_{m \leqslant n} |f(m) - A_f(n)|^2 = O(nD_f^2(n)). \tag{6}$$

This is called the Turán–Kubilius inequality.

In the last 25 years numerous papers have been written on the arithmetic properties of sum sets $\{a+b,\ a \in \mathcal{A},\ b \in \mathcal{B}\}$ (a list of these papers is presented in [8]). Typically, these results say that if $\mathcal{A}$, $\mathcal{B}$ are "large" subsets of $\{1, 2, \dots, n\}$ then a certain property of the sums simulates the behaviour of the consecutive integers $1, 2, \dots, n$. In some cases multiplicative analogues of these results also have been proved in which the sums $a+b$ are replaced by shifted products $ab+1$. In particular, ERDŐS, MAIER and SÁRKÖZY [3] showed that if $\mathcal{A}$, $\mathcal{B}$ are large subsets of $\{1, 2, \dots, n\}$, then the sums $a + b$ satisfy an Erdős–Kac type theorem (see also [1] and [9]), and later ELLIOTT and SÁRKÖZY [2] also proved the multiplicative analog of this result with shifted products $ab + 1$ in place of the sums $a + b$.

G. Halász asked the question whether the Turán–Kubilius inequality has a similar sum set analogue? (Oral communication.) In [8] we showed that, indeed, there is such an inequality which is, however, not quite as strong as (6): we proved a similar result midway between Turán's and Kubilius's inequality. In this paper our goal is to prove the multiplicative analogue of the theorem in [8], *i.e.*, we will prove a similar result with shifted products $ab + 1$ in place of the sums $a + b$. The proof will also be reminiscent of the proof in [8], however, there will be a crucial difference: while in [8] we used additive characters and the main tool was the standard form of the large sieve, here, due to the multiplicative nature of the problem we will use multiplicative characters and Gallagher's (multiplicative) character version of the large sieve [4].

2. The theorem and comments

We will prove the following theorem:

Theorem 1. *Let f be a complex valued additive arithmetic function, define*

$$K_f(m) = \max\left\{|f(p^\alpha)|\,,\ p \text{ prime},\ \alpha \in \mathbb{N},\ p^\alpha \leqslant m\right\}, \tag{7}$$

let $A_f(n)$ be defined by (3), *C a fixed positive number, $n \in \mathbb{N}$ (with $n \to +\infty$) and $\mathcal{A}$, $\mathcal{B} \subseteq \{1, 2, \ldots, n\}$ with*

$$\sqrt{|\mathcal{A}|\,|\mathcal{B}|} > n \exp\left(-C\sqrt{\log\log n}\log\log\log n\right). \tag{8}$$

Then we have

$$\frac{1}{|\mathcal{A}|\,|\mathcal{B}|}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\left|f(ab+1) - A_f(n^2+1)\right|^2 = O\left(C^2K_f^2(n^2+1)\,\log\log(n^2+1)\right). \tag{9}$$

Remarks.

(i) The special case $f(n) = \omega(n)$ and $\mathcal{A} = \mathcal{B} = \mathbb{N}$, shows that (9) is sharp.

(ii) Condition (8) is also sharp, *i.e.*, to ensure that the left hand side of (9) is $O(K_f^2(n^2+1)\log\log(n^2+1))$ one needs assumption (8). This can be shown by an example similar to the one in [8].

(iii) While Theorem 1 is sharp for $f(m) = \omega(m)$, it gives only a very weak upper bound for the left hand side of (9) if $f(m) = \Omega(m)$. The reason of this is that the prime powers p^α with small p and large α may influence the distribution of the values $\Omega(ab+1)$ (with $a \in \mathcal{A}$, $b \in \mathcal{B}$) significantly. Again this can be shown by an example similar to the one in [8].

3. Structure of the proof

Let $\mathcal{P}$ denote the set of prime powers $p^\alpha \leqslant n^2+1$, and write

$$V = \frac{n}{\sqrt{|\mathcal{A}|\,|\mathcal{B}|}}. \tag{10}$$

We split $\mathcal{P}$ into three parts:

$$\mathcal{P}_1 = \{p^\alpha : p \leqslant V,\ \alpha \geqslant 1,\ p^\alpha \leqslant n^2+1\}$$

$$\mathcal{P}_2 = \{p^\alpha : V < p,\ \alpha \geqslant 1,\ p^\alpha \leqslant (n^2+1)^{1/6}\}$$

$$\mathcal{P}_3 = \{p^\alpha : V < p,\ \alpha \geqslant 1,\ (n^2+1)^{1/6} < p^\alpha \leqslant n^2+1\},$$

so that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$ for $1 \leqslant i < j \leqslant 3$. Define the additive arithmetic functions $f_1(m)$, $f_2(m)$, $f_3(m)$ by

$$f_i(p^\alpha) = \begin{cases} f(p^\alpha) & \text{if } p^\alpha \in \mathcal{P}_i \\ 0 & \text{if } qp^\alpha \notin \mathcal{P}_i \end{cases} \qquad (\text{for } i = 1,2,3).$$

Then clearly we have $f(m) = f_1(m) + f_2(m) + f_3(m)$. Thus by using the elementary inequality $|z_1+z_2+z_3|^2 \leqslant 3(|z_1|^2+|z_2|^2+|z_3|^2)$ (where z_1, z_2, z_3 are any complex numbers) we may estimate the sum on the left hand side of (9) in the following way:

$$\frac{1}{|\mathcal{A}|\,|\mathcal{B}|}\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\bigl|f(ab+1) - A_f(n^2+1)\bigr|^2 \leqslant 3(T_1+T_2+T_3) \tag{11}$$

where

$$T_i = \frac{1}{|\mathcal{A}|\,|\mathcal{B}|}\sum_{a\in\mathcal{A}} q, \sum_{b\in\mathcal{B}}\left| f_i(ab+1) - \sum_{p\leqslant n^2+1}\frac{f_i(p)}{p}\right|^2 \quad (\text{for } i = 1,2,3). \tag{12}$$

The crucial part of the proof is the estimate of T_2 which is based on Gallagher's (multiplicative) character version of the large sieve; this estimate will be carried out in Sections 4 and 5. T_1 will be estimated in Section 6, while the (nearly trivial) estimate of T_3 and the completion of the proof of Theorem 1 will be presented in Section 7.

4. The estimate of T_2. Preliminary lemmas

We will use the following application of Gallagher's character version of the large sieve [4]:

Lemma 1. *If M, N are positive integers and $c_{M+1},\ldots,c_{M+N}$ are complex numbers then for $0<K\leqslant L$ we have*

$$\sum_{K<k\leqslant L}\frac{1}{\varphi(k)}\sideset{}{^*}\sum_{\chi \bmod k}\left|\sum_{n=M+1}^{M+N}c_n\,\chi(n)\right|^2\leqslant\left(\frac{N-1}{K}+2L-K\right)\sum_{n=M+1}^{M+N}|c_n|^2. \tag{13}$$

(the asterisk indicates a summation over the primitive Dirichlet characters χ modulo k.)

Proof. Let

$$u_k=\frac{k}{\varphi(k)}\sideset{}{^*}\sum_{\chi \bmod k}\left|\sum_{n=M+1}^{M+N}c_n\,\chi(n)\right|^2.$$

By partial summation we have

$$\sum_{K<k\leqslant L}\frac{u_k}{k}\leqslant\frac{1}{L}\sum_{K<k\leqslant L}u_k+\int_K^L\left(\sum_{K<k\leqslant t}u_k\right)\frac{dt}{t^2}.$$

Using the optimal form of the large sieve (see for example [7, Theorem 3]), Gallagher's character version of the large sieve becomes:

$$\sum_{k\leqslant t}\frac{k}{\varphi(k)}\sideset{}{^*}\sum_{\chi \bmod k}\left|\sum_{n=M+1}^{M+N}c_n\,\chi(n)\right|^2\leqslant(N-1+t^2)\sum_{n=M+1}^{M+N}|c_n|^2.$$

This leads to

$$\begin{aligned}\sum_{K<k\leqslant L}\frac{u_k}{k}&\leqslant\left(\frac{1}{L}(N-1+L^2)+\int_K^L(N-1+t^2)\frac{dt}{t^2}\right)\sum_{n=M+1}^{M+N}|c_n|^2\\&=\left(\frac{N-1}{L}+L+\frac{N-1}{K}-\frac{N-1}{L}+L-K\right)\sum_{n=M+1}^{M+N}|c_n|^2\end{aligned}$$

and (13) follows. □

For $\mathcal{A},\ \mathcal{B}\subseteq\{1,2,\ldots,n\}$ and $m\in\mathbb{N}$ we define

$$R(m)=\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ ab+1\equiv 0 \bmod m}}1-\frac{|\mathcal{A}|\,|\mathcal{B}|}{m}. \tag{14}$$

(where $\mathcal{A}\times\mathcal{B}$ denotes the set of the ordered pairs (a,b) with a belonging to $\mathcal{A}$, b belonging to $\mathcal{B}$). Using Dirichlet characters we can write

$$R(m)=\frac{1}{\varphi(m)}\sum_{\chi \bmod m}\bar{\chi}(-1)\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\chi(ab)-\frac{|\mathcal{A}|\,|\mathcal{B}|}{m}$$

Denoting by χ_0 the principal character modulo m, we write

$$R(m)=R_0(m)+R_1(m) \tag{15}$$

with

$$R_0(m)=\frac{1}{\varphi(m)}\bar{\chi}_0(-1)\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\chi_0(ab)-\frac{|\mathcal{A}|\,|\mathcal{B}|}{m} \tag{16}$$

and

$$R_1(m)=\frac{1}{\varphi(m)}\sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}}\bar{\chi}(-1)\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\chi(ab). \tag{17}$$

Notation 1. In the sequel p and q will always denote prime numbers.

Lemma 2. *We have*

$$\sum_{p^\alpha\in\mathcal{P}_2}|R_0(p^\alpha)|\ll|\mathcal{A}|\,|\mathcal{B}| \tag{18}$$

and

$$\sum_{p^\alpha\in\mathcal{P}_2}\sum_{\substack{q^\beta\in\mathcal{P}_2\\ q\neq p}}\left|R_0(p^\alpha q^\beta)\right|\ll|\mathcal{A}|\,|\mathcal{B}|\log\log(n^2+1).+ \tag{19}$$

Proof. For $m\in\mathbb{N}$ we deduce from (16) that

$$\begin{aligned}|R_0(m)|&\leqslant\frac{1}{\varphi(m)}\left|\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ (ab,m)=1}}1-|\mathcal{A}|\,|\mathcal{B}|\right|+\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)|\mathcal{A}|\,|\mathcal{B}|\,,\\ &=\frac{1}{\varphi(m)}\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ (ab,m)>1}}1+\left(\frac{1}{\varphi(m)}-\frac{1}{m}\right)|\mathcal{A}|\,|\mathcal{B}|\,.\end{aligned}$$

Taking $m=p^\alpha$ we first observe that

$$\sum_{p^\alpha\in\mathcal{P}_2}\left(\frac{1}{\varphi(p^\alpha)}-\frac{1}{p^\alpha}\right)\leqslant\sum_p\left(\frac{1}{p-1}-\frac{1}{p}\right)\sum_{\alpha\geqslant1}\frac{1}{p^{\alpha-1}}\leqslant 2.$$

Hence

$$\sum_{p^\alpha \in \mathcal{P}_2} |R_0(p^\alpha)| \ll \sum_{p>V} \sum_{\alpha \geqslant 1} \frac{1}{p^{\alpha-1}(p-1)} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 + |\mathcal{A}|\,|\mathcal{B}|$$

$$\ll \sum_{p>V} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 + |\mathcal{A}|\,|\mathcal{B}|\,. \tag{20}$$

Observing that

$$\sum_{V<p\leqslant V^2} \frac{1}{p} = \log\log(V^2) - \log\log V + O(1) = O(1)$$

we get

$$\sum_{p>V} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 \ll \sum_{p>V^2} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 + |\mathcal{A}|\,|\mathcal{B}|\,.$$

But

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 \leqslant \sum_{\substack{a\in\mathcal{A} \\ p \mid a}} |\mathcal{B}| + \sum_{\substack{b\in\mathcal{B} \\ p \mid b}} |\mathcal{A}| \leqslant \frac{2n^2}{p}$$

It follows that

$$\sum_{p>V} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 \ll \sum_{p>V^2} \frac{n^2}{p^2} + |\mathcal{A}|\,|\mathcal{B}| \ll \frac{n^2}{V^2} + |\mathcal{A}|\,|\mathcal{B}|\,,$$

and by (10) we obtain

$$\sum_{p>V} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 \ll |\mathcal{A}|\,|\mathcal{B}|\,. \tag{21}$$

Inserting this estimate in (20) we get (18).

Similarly taking $m = p^\alpha q^\beta$ with $p \neq q$ we have

$$\sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \left(\frac{1}{\varphi(p^\alpha q^\beta)} - \frac{1}{p^\alpha q^\beta} \right)$$

$$\leqslant \sum_{V<p\leqslant n^2+1} q, \sum_{V<q\leqslant n^2+1} \left(\frac{1}{(p-1)(q-1)} - \frac{1}{pq} \right) \sum_{\alpha\geqslant 1} q, \sum_{\beta\geqslant 1} \frac{1}{p^{\alpha-1}q^{\beta-1}}$$

$$\leqslant 4 \sum_{V<p\leqslant n^2+1} \sum_{V<q\leqslant n^2+1} \left(\frac{1}{p-1} \left(\frac{1}{q-1} - \frac{1}{q} \right) + \frac{1}{q} \left(\frac{1}{p-1} - \frac{1}{p} \right) \right)$$
$$\ll \log\log(n^2+1).$$

Hence

$$\sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q\neq p}} \left| R_0(p^\alpha q^\beta) \right| \ll \sum_{V<p\leqslant n^2+1} \sum_{\substack{V<q\leqslant n^2+1 \\ q\neq p}} \sum_{\alpha\geqslant 1} \sum_{\beta\geqslant 1} \frac{1}{\varphi\left(p^\alpha q^\beta\right)} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ (ab,pq)>1}} 1$$
$$+ |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1),$$

and observing that for $p \neq q$,

$$\sum_{\alpha\geqslant 1} \sum_{\beta\geqslant 1} \frac{1}{\varphi\left(p^\alpha q^\beta\right)} = \frac{1}{(p-1)(q-1)} \sum_{\alpha\geqslant 1} \sum_{\beta\geqslant 1} \frac{1}{p^{\alpha-1} q^{\beta-1}} \ll \frac{1}{pq}$$

we get

$$\sum_{p^\alpha \in \mathcal{P}_2} q, \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q\neq p}} \left| R_0(p^\alpha q^\beta) \right| \ll \sum_{V<p} \frac{1}{p} \left(\sum_{\substack{q\leqslant n^2+1 \\ q\neq p}} \frac{1}{q} \right) \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1$$
$$+ \sum_{V<q} \frac{1}{q} \left(\sum_{\substack{p\leqslant n^2+1 \\ p\neq q}} \frac{1}{p} \right) \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ q \mid ab}} 1 + |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1),$$

thus by symmetry of the roles of p and q

$$\sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q\neq p}} \left| R_0(p^\alpha q^\beta) \right| \ll \left(\sum_{p>V} \frac{1}{p} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p \mid ab}} 1 + |\mathcal{A}|\,|\mathcal{B}| \right) \log\log(n^2+1).$$

Using (21) we get (19). □

Lemma 3. *We have*

$$\sum_{p^\alpha \in \mathcal{P}_2} |R(p^\alpha)| \ll |\mathcal{A}|\,|\mathcal{B}| \tag{22}$$

and

$$\sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q\neq p}} \left| R(p^\alpha q^\beta) \right| \ll |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1). \tag{23}$$

PROOF. By (15) and Lemma 2 it suffice to show the estimates (22) and (23) with R_1 in place of R. We first observe that we may assume

$$V < (n^2+1)^{1/6} \tag{24}$$

for otherwise $\mathcal{P}_2 = \emptyset$ thus (22) and (23) are trivially true.

Let $\mathcal{M}_1 = \mathcal{P}_2$ and $\mathcal{M}_2 = \{p^\alpha q^\beta,\ p^\alpha \in \mathcal{P}_2,\ q^\beta \in \mathcal{P}_2,\ p \neq q\}$. For $i \in \{1,2\}$ we can write

$$\begin{aligned}\sum_{m\in\mathcal{M}_i} |R_1(m)| &= \sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \Bigg| \sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}} \bar{\chi}(-1) \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \chi(ab)\Bigg| \\ &\leqslant \sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}} \Bigg|\sum_{a\in\mathcal{A}} \chi(a)\Bigg| \Bigg|\sum_{b\in\mathcal{B}} \chi(b)\Bigg|.\end{aligned}$$

By Cauchy's inequality this is bounded above by

$$\Bigg(\sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}} \Bigg|\sum_{a\in\mathcal{A}} \chi(a)\Bigg|^2\Bigg)^{1/2} \Bigg(\sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}} \Bigg|\sum_{b\in\mathcal{B}} \chi(b)\Bigg|^2\Bigg)^{1/2}.$$

By symmetry it is sufficient to consider the first parenthesis. If the Dirichlet character χ modulo m is induced by the primitive character χ_1 modulo k (with $k \mid m$) then $\chi(n) = \chi_1(n)$ for all n such that $(n,m) = 1$. Therefore

$$\sum_{a\in\mathcal{A}} \chi(a) = \sum_{\substack{a\in\mathcal{A}\\ (a,m)=1}} \chi(a) = \sum_{\substack{a\in\mathcal{A}\\ (a,m)=1}} \chi_1(a)$$

and

$$\sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}} \Bigg|\sum_{a\in\mathcal{A}} \chi(a)\Bigg|^2 = \sum_{m\in\mathcal{M}_i} \frac{1}{\varphi(m)} \sum_{\substack{k\mid m\\ k>1}} \sideset{}{^*}\sum_{\chi_1 \bmod k} \Bigg|\sum_{\substack{a\in\mathcal{A}\\ (a,m)=1}} \chi_1(a)\Bigg|^2$$

where the asterisk indicates summation over the primitive characters χ_1 modulo k.

For $i = 1$ we have $k = p^\alpha$ with $\alpha \geqslant 1$ and $m = p^{\alpha+\beta}$ with $\beta \geqslant 0$. We need to estimate

$$\sum_{p^\alpha\in\mathcal{P}_2} \sum_{\substack{\beta\geqslant 0\\ p^{\alpha+\beta}\in\mathcal{P}_2}} \frac{1}{\varphi(p^{\alpha+\beta})} \sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha} \Bigg|\sum_{\substack{a\in\mathcal{A}\\ (a,p)=1}} \chi_1(a)\Bigg|^2$$

For $(a,p)>1$ we have $\chi_1(a)=0$ so we can drop the condition $(a,p)=1$. Moreover

$$\sum_{\substack{\beta\geqslant 0\\ p^{\alpha+\beta}\in\mathcal{P}_2}}\frac{1}{\varphi(p^{\alpha+\beta})}\leqslant\frac{1}{p^{\alpha-1}(p-1)}\sum_{\beta\geqslant 0}\frac{1}{p^\beta}\leqslant\frac{2}{\varphi(p^\alpha)},$$

hence

$$\sum_{m\in\mathcal{M}_1}\frac{1}{\varphi(m)}\sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}}\left|\sum_{a\in\mathcal{A}}\chi(a)\right|^2\ll\sum_{p^\alpha\in\mathcal{P}_2}\frac{1}{\varphi(p^\alpha)}\sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha}\left|\sum_{a\in\mathcal{A}}\chi_1(a)\right|^2.$$

Using (13) with $K=V$, $L=(n^2+1)^{1/6}$ we obtain

$$\sum_{m\in\mathcal{M}_1}\frac{1}{\varphi(m)}\sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}}\left|\sum_{a\in\mathcal{A}}\chi(a)\right|^2\ll\Big(\frac{n}{V}+n^2+1)^{1/6}\Big)\,|\mathcal{A}|\,.$$

Using (24) we have $(n^2+1)^{1/6}\leqslant 2n/(n^2+1)^{1/6}<2n/V$, hence

$$\sum_{m\in\mathcal{M}_1}\frac{1}{\varphi(m)}\sum_{\substack{\chi \bmod m\\ \chi\neq\chi_0}}\left|\sum_{a\in\mathcal{A}}\chi(a)\right|^2\ll\frac{n}{V}\,|\mathcal{A}|\,.$$

A similar estimate is valid when $\mathcal{A}$ is replaced by $\mathcal{B}$. Combining these two estimates we get

$$\sum_{m\in\mathcal{M}_1}|R_1(m)|\ll\Big(\frac{n}{V}\,|\mathcal{A}|\Big)^{1/2}\Big(\frac{n}{V}\,|\mathcal{B}|\Big)^{1/2}=|\mathcal{A}|\,|\mathcal{B}|\,,$$

hence we get (22).

For $i=2$ we may have $k=p^\alpha$ $(\alpha\geqslant 1)$, in which case $m=p^{\alpha+\alpha'}q^\beta$ $(\alpha'\geqslant 0$, $\beta\geqslant 1$, $q\neq p)$ or $k=p^\alpha q^\beta$ $(\alpha\geqslant 1$, $\beta\geqslant 1$, $p\neq q)$, in which case $m=p^{\alpha+\alpha'}q^{\beta+\beta'}$ $(\alpha'\geqslant 0$, $\beta'\geqslant 0)$ thus

$$\sum_{m\in\mathcal{M}_2}\frac{1}{\varphi(m)}\sum_{\substack{k\,|\,m\\ k>1}}\sideset{}{^*}\sum_{\chi_1 \bmod k}\left|\sum_{\substack{a\in\mathcal{A}\\ (a,m)=1}}\chi_1(a)\right|^2\leqslant U_1+U_2 \tag{25}$$

with

$$U_1=\sum_{p^\alpha\in\mathcal{P}_2}\sum_{\substack{q^\beta\in\mathcal{P}_2\\ q\neq p}}\sum_{\alpha'\geqslant 0}\frac{1}{\varphi(p^{\alpha+\alpha'}q^\beta)}\sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha}\left|\sum_{\substack{a\in\mathcal{A}\\ (a,pq)=1}}\chi_1(a)\right|^2$$

and

$$U_2 = \sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \sum_{\alpha' \geqslant 0} \sum_{\beta' \geqslant 0} \frac{1}{\varphi(p^{\alpha+\alpha'} q^{\beta+\beta'})} \sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha q^\beta} \Bigg| \sum_{\substack{a \in \mathcal{A} \\ (a,pq)=1}} \chi_1(a) \Bigg|^2$$

In U_1 we have $\chi_1(a) = 0$ whenever $(a,p) > 1$, hence we may replace the condition $(a,pq) = 1$ by $(a,q) = 1$. The sum over α' contributes by a constant factor. This leads to

$$U_1 \ll \sum_{q^\beta \in \mathcal{P}_2} \frac{1}{\varphi(q^\beta)} \sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ p \neq q}} \frac{1}{\varphi(p^\alpha)} \sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha} \Bigg| \sum_{\substack{a \in \mathcal{A} \\ (a,q)=1}} \chi_1(a) \Bigg|^2,$$

hence extending the summation from p^α to every k with $V < k \leqslant (n^2+1)^{1/6}$ we get

$$U_1 \ll \sum_{q^\beta \in \mathcal{P}_2} \frac{1}{\varphi(q^\beta)} \sum_{V < k \leqslant (n^2+1)^{1/6}} \frac{1}{\varphi(k)} \sideset{}{^*}\sum_{\chi_1 \bmod k} \Bigg| \sum_{\substack{a \in \mathcal{A} \\ (a,q)=1}} \chi_1(a) \Bigg|^2,$$

and applying (13) on the summation over k with $K = V$ and $L = (n^2+1)^{1/6}$ we get

$$U_1 \ll \sum_{q^\beta \in \mathcal{P}_2} \frac{1}{\varphi(q^\beta)} \left(\frac{n}{V} + (n^2+1)^{1/6} \right) \sum_{\substack{a \in \mathcal{A} \\ (a,q)=1}} 1.$$

We drop the condition $(a,q) = 1$ and by (24) we have $(n^2+1)^{1/6} \leqslant 2n/(n^2+1)^{1/6} < 2n/V$, and

$$\sum_{q^\beta \in \mathcal{P}_2} \frac{1}{\varphi(q^\beta)} \ll \sum_{q \leqslant (n^2+1)^{1/6}} \frac{1}{q} \ll \log\log(n^2+1)$$

we obtain

$$U_1 \ll \frac{n}{V} \, |\mathcal{A}| \log\log(n^2+1). \tag{26}$$

In U_2 we have $\chi_1(a) = 0$ whenever $(a,pq) > 1$, hence we may remove the condition $(a,pq) = 1$. The sums over α' and β' contribute by a constant factor. This leads to

$$U_2 \ll \sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \frac{1}{\varphi(p^\alpha q^\beta)} \sideset{}{^*}\sum_{\chi_1 \bmod p^\alpha q^\beta} \Bigg| \sum_{a \in \mathcal{A}} \chi_1(a) \Bigg|^2$$

hence

$$U_2 \ll \sum_{V^2 < k \leqslant (n^2+1)^{1/3}} \frac{1}{\varphi(k)} \sideset{}{^*}\sum_{\chi_1 \bmod k} \left| \sum_{a\in\mathcal{A}} \chi_1(a) \right|^2$$

and applying (13) with $K = V^2$ and $L = (n^2+1)^{1/3}$ we get

$$U_2 \ll \left(\frac{n}{V^2} + (n^2+1)^{1/3} \right) |\mathcal{A}| .$$

By (24) we have $(n^2+1)^{1/3} \leqslant 2n/(n^2+1)^{1/6} < 2n/V$, thus

$$U_2 \ll \frac{n}{V} |\mathcal{A}| . \tag{27}$$

By (26) and (27) from (25) we obtain

$$\sum_{m\in\mathcal{M}_2} \frac{1}{\varphi(m)} \sum_{\substack{k \mid m \\ k>1}} \sideset{}{^*}\sum_{\chi_1 \bmod k} \left| \sum_{\substack{a\in\mathcal{A} \\ (a,m)=1}} \chi_1(a) \right|^2 \ll \frac{n}{V} |\mathcal{A}| \log\log(n^2+1).$$

The same estimate can be obtained with $\mathcal{B}$ in place of $\mathcal{A}$. Combining these two estimates we get

$$\begin{aligned} \sum_{m\in\mathcal{M}_2} |R_1(m)| &\ll \left(\frac{n}{V} |\mathcal{A}| \log\log(n^2+1) \right)^{1/2} \left(\frac{n}{V} |\mathcal{B}| \log\log(n^2+1) \right)^{1/2} \\ &= |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1), \end{aligned}$$

hence we get (23). □

Lemma 4. *For any complex valued additive arithmetic function f_2 such that $f_2(p^\alpha) = 0$ whenever $p^\alpha \notin \mathcal{P}_2$ and $n \geqslant 8$ we have*

$$\left| \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} f_2(ab+1) - |\mathcal{A}|\,|\mathcal{B}| \sum_{p^\alpha \leqslant n^2+1} \frac{f_2(p^\alpha)}{p^\alpha} \left(1 - \frac{1}{p} \right) \right| \ll K_{f_2}(n^2+1)\ |\mathcal{A}|\,|\mathcal{B}| \tag{28}$$

where K_{f_2} is defined by (7).

Proof. Let

$$S_1 = \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} f_2(ab+1). \tag{29}$$

Since f_2 is an additive arithmetic function we have

$$S_1 = \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \sum_{p^\alpha \| ab+1} f_2(p^\alpha) = \sum_{p^\alpha \leqslant n^2+1} f_2(p^\alpha) \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1}} 1,$$

and

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1}} 1 = \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \mid ab+1}} 1 - \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^{\alpha+1} \mid ab+1}} 1.$$

Using (14) we have

$$\begin{aligned}\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1}} 1 &= \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha} - \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^{\alpha+1}} + R(p^\alpha) - R(p^{\alpha+1}) \\ &= \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha}\left(1 - \frac{1}{p}\right) + R(p^\alpha) - R(p^{\alpha+1}) \qquad (30)\end{aligned}$$

and

$$\begin{aligned}S_1 = |\mathcal{A}|\,|\mathcal{B}| &\sum_{p^\alpha \leqslant n^2+1} \frac{f_2(p^\alpha)}{p^\alpha}\left(1 - \frac{1}{p}\right) \\ &+ \sum_{p^\alpha \leqslant n^2+1} f_2(p^\alpha) R(p^\alpha) - \sum_{p^\alpha \leqslant n^2+1} f_2(p^\alpha) R(p^{\alpha+1})\end{aligned}$$

so that using (7) and $f_2(p^\alpha) = 0$ whenever $p^\alpha \notin \mathcal{P}_2$ we obtain

$$\begin{aligned}&\left| S_1 - |\mathcal{A}|\,|\mathcal{B}| \sum_{p^\alpha \leqslant n^2+1} \frac{f_2(p^\alpha)}{p^\alpha}\left(1 - \frac{1}{p}\right) \right| \\ &\qquad \leqslant K_{f_2}(n^2+1)\left(\sum_{p^\alpha\in\mathcal{P}_2} |R(p^\alpha)| + \sum_{p^\alpha\in\mathcal{P}_2} \left|R(p^{\alpha+1})\right| \right). \qquad (31)\end{aligned}$$

By (22) we have

$$\sum_{p^\alpha\in\mathcal{P}_2} |R(p^\alpha)| \ll |\mathcal{A}|\,|\mathcal{B}|$$

and

$$\begin{aligned}\sum_{p^\alpha\in\mathcal{P}_2} \left|R(p^{\alpha+1})\right| &= \sum_{\substack{p^\alpha\in\mathcal{P}_2 \\ p^{\alpha+1}\notin\mathcal{P}_2}} \left|R(p^{\alpha+1})\right| + \sum_{p^{\alpha+1}\in\mathcal{P}_2} \left|R(p^{\alpha+1})\right| \\ &\ll \sum_{\substack{p^\alpha\in\mathcal{P}_2 \\ p^{\alpha+1}\notin\mathcal{P}_2}} \left|R(p^{\alpha+1})\right| + |\mathcal{A}|\,|\mathcal{B}|\,.\end{aligned}$$

Using (14) we have

$$\left|R(p^{\alpha+1})\right| \leqslant \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ ab+1\equiv 0 \bmod p^{\alpha+1}}} 1 + \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^{\alpha+1}}.$$

If $ab+1\equiv 0 \bmod p^{\alpha+1}$ then $(a,p^{\alpha+1})=1$ and a admits an inverse $i(a)$ modulo $p^{\alpha+1}$. Therefore

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ ab+1\equiv 0 \bmod p^{\alpha+1}}} 1 = \sum_{\substack{a\in\mathcal{A}\\ (a,p^{\alpha+1})=1}} \sum_{\substack{b\in\mathcal{B}\\ b\equiv -i(a) \bmod p^{\alpha+1}}} 1 \leqslant \sum_{\substack{a\in\mathcal{A}\\ (a,p^{\alpha+1})=1}} \frac{n}{p^{\alpha+1}} \leqslant \frac{n\,|\mathcal{A}|}{p^{\alpha+1}},$$

and exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ we obtain

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ ab+1\equiv 0 \bmod p^{\alpha+1}}} 1 \leqslant \min\left(\frac{n\,|\mathcal{A}|}{p^{\alpha+1}}, \frac{n\,|\mathcal{B}|}{p^{\alpha+1}}\right) \leqslant \frac{n}{p^{\alpha+1}}\sqrt{|\mathcal{A}|\,|\mathcal{B}|},$$

so that

$$\sum_{\substack{p^{\alpha}\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \left|R(p^{\alpha+1})\right| \leqslant \left(n\sqrt{|\mathcal{A}|\,|\mathcal{B}|} + |\mathcal{A}|\,|\mathcal{B}|\right) \sum_{\substack{p^{\alpha}\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \frac{1}{p^{\alpha+1}}.$$

If $p^{\alpha}\in\mathcal{P}_2$ and $p^{\alpha+1}\notin\mathcal{P}_2$ then by the definition of $\mathcal{P}_2$ we have $V<p$ and $p^{\alpha}\leqslant (n^2+1)^{1/6} < p^{\alpha+1}$ so that α is uniquely defined. Hence

$$\sum_{\substack{p^{\alpha}\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \frac{1}{p^{\alpha+1}} \leqslant \sum_{p>V} \frac{1}{p^2} \ll \frac{1}{V}. \tag{32}$$

By these estimates and the definition of V given by (10) we get

$$\sum_{\substack{p^{\alpha}\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \left|R(p^{\alpha+1})\right| \ll \frac{n\sqrt{|\mathcal{A}|\,|\mathcal{B}|} + |\mathcal{A}|\,|\mathcal{B}|}{V} \ll |\mathcal{A}|\,|\mathcal{B}|\,.$$

We deduce that

$$\sum_{p^{\alpha}\in\mathcal{P}_2} |R(p^{\alpha})| + \sum_{p^{\alpha}\in\mathcal{P}_2} \left|R(p^{\alpha+1})\right| \ll |\mathcal{A}|\,|\mathcal{B}|\,. \tag{33}$$

(28) follows from (31) and (33). $\square$

Lemma 5. *For any complex valued additive arithmetic function f_2 such that $f_2(p^\alpha)=0$ whenever $p^\alpha \notin \mathcal{P}_2$ and $n \geqslant 8$ we have*

$$\left|\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}|f_2(ab+1)|^2 - |\mathcal{A}|\,|\mathcal{B}|\left|\sum_{p^\alpha\leqslant n^2+1}\frac{f_2(p^\alpha)}{p^\alpha}\left(1-\frac{1}{p}\right)\right|^2\right| \ll K_{f_2}^2(n^2+1)\;|\mathcal{A}|\,|\mathcal{B}|\;\log\log(n^2+1) \quad (34)$$

where K_{f_2} is defined by (7).

Proof. Let

$$S_2=\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}|f_2(ab+1)|^2. \quad (35)$$

Since f_2 is an additive arithmetic function we can write

$$S_2=\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}}\left|\sum_{p^\alpha\|ab+1}f_2(p^\alpha)\right|^2$$

and expanding the square we get

$$S_2=\sum_{p^\alpha\leqslant n^2+1}|f_2(p^\alpha)|^2\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ p^\alpha\|ab+1}}1+\sum_{\substack{p^\alpha q^\beta\leqslant n^2+1\\ p\neq q}}f_2(p^\alpha)\overline{f_2(q^\beta)}\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ p^\alpha\|ab+1\\ q^\beta\|ab+1}}1. \quad (36)$$

First we will give an upper bound for the first term. Using (30) we can write

$$\begin{aligned}\sum_{p^\alpha\leqslant n^2+1}|f_2(p^\alpha)|^2&\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ p^\alpha\|ab+1}}1\\ &=\sum_{p^\alpha\leqslant n^2+1}|f_2(p^\alpha)|^2\left(\frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha}\left(1-\frac{1}{p}\right)+R(p^\alpha)-R(p^{\alpha+1})\right)\\ &\leqslant K_{f_2}^2(n^2+1)\sum_{p^\alpha\in\mathcal{P}_2}\left(\frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha}\left(1-\frac{1}{p}\right)+|R(p^\alpha)|+\left|R(p^{\alpha+1})\right|\right)\end{aligned}$$

Now

$$\begin{aligned}\sum_{p^\alpha\in\mathcal{P}_2}\frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha}\left(1-\frac{1}{p}\right)&\leqslant|\mathcal{A}|\,|\mathcal{B}|\sum_{p\leqslant n^2+1}\sum_{\alpha\geqslant1}\frac{1}{p^\alpha}\left(1-\frac{1}{p}\right)\\ &=|\mathcal{A}|\,|\mathcal{B}|\sum_{p\leqslant n^2+1}\frac{1}{p}\ll|\mathcal{A}|\,|\mathcal{B}|\;\log\log(n^2+1)\end{aligned}$$

while by (33) we have

$$\sum_{p^\alpha \in \mathcal{P}_2} |R(p^\alpha)| + \sum_{p^\alpha \in \mathcal{P}_2} \left|R(p^{\alpha+1})\right| \ll |\mathcal{A}|\,|\mathcal{B}|\,.$$

Using these two estimates we obtain from (36)

$$\begin{aligned} S_2 = &\sum_{\substack{p^\alpha q^\beta \leqslant n^2+1 \\ p\neq q}} f_2(p^\alpha)\overline{f_2(q^\beta)} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1 \\ q^\beta \| ab+1}} 1 \\ &+ O(K_{f_2}^2(n^2+1)\ |\mathcal{A}|\,|\mathcal{B}|\ \log\log(n^2+1)). \end{aligned} \tag{37}$$

For $q \neq p$,

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1 \\ q^\beta \| ab+1}} 1 = \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha q^\beta \mid ab+1}} 1 - \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^{\alpha+1} q^\beta \mid ab+1}} 1 - \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha q^{\beta+1} \mid ab+1}} 1 + \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^{\alpha+1} q^{\beta+1} \mid ab+1}} 1,$$

thus using (14) we get

$$\begin{aligned} \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ p^\alpha \| ab+1 \\ q^\beta \| ab+1}} 1 = &\frac{|\mathcal{A}|\,|\mathcal{B}|}{p^\alpha q^\beta}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) \\ &+ R(p^\alpha q^\beta) - R(p^{\alpha+1} q^\beta) - R(p^\alpha q^{\beta+1}) + R(p^{\alpha+1} q^{\beta+1}). \end{aligned} \tag{38}$$

Writing

$$S_2' = \sum_{\substack{p^\alpha q^\beta \leqslant n^2+1 \\ p\neq q}} \frac{f_2(p^\alpha)\overline{f_2(q^\beta)}}{p^\alpha q^\beta}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)$$

and

$$R_2 = \sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ q^\beta \in \mathcal{P}_2 \\ p\neq q}} \left(\left|R(p^\alpha q^\beta)\right| + \left|R(p^{\alpha+1} q^\beta)\right| + \left|R(p^\alpha q^{\beta+1})\right| + \left|R(p^{\alpha+1} q^{\beta+1})\right|\right),$$

we obtain from (37) and (38) that

$$|S_2 - |\mathcal{A}|\,|\mathcal{B}|\,S_2'| \ll K_{f_2}^2(n^2+1)\ (|\mathcal{A}|\,|\mathcal{B}|\ \log\log(n^2+1) + R_2). \tag{39}$$

By (23) we have

$$\sum_{p^\alpha \in \mathcal{P}_2} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ p \neq q}} \left| R(p^\alpha q^\beta) \right| \ll |\mathcal{A}|\,|\mathcal{B}| \,\log\log(n^2+1).$$

Using (14) we have

$$\left| R(p^{\alpha+1} q^\beta) \right| \leqslant \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ ab+1\equiv 0 \bmod p^{\alpha+1}q^\beta}} 1 + \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^{\alpha+1}q^\beta}$$

and counting trivially (using the fact that only a and b which are invertible modulo $p^{\alpha+1}q^\beta$ are counted) we can write

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ ab+1\equiv 0 \bmod p^{\alpha+1}q^\beta}} 1 \leqslant \min\left(\frac{n\,|\mathcal{A}|}{p^{\alpha+1}q^\beta}, \frac{n\,|\mathcal{B}|}{p^{\alpha+1}q^\beta} \right) \leqslant \frac{n}{p^{\alpha+1}q^\beta} \sqrt{|\mathcal{A}|\,|\mathcal{B}|},$$

thus using (32)

$$\sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ p^{\alpha+1}\notin\mathcal{P}_2}} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \left| R(p^{\alpha+1} q^\beta) \right|$$
$$\ll n\sqrt{|\mathcal{A}|\,|\mathcal{B}|} \sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ p^{\alpha+1}\notin\mathcal{P}_2}} \frac{1}{p^{\alpha+1}} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \frac{1}{q^\beta} \ll \frac{n\sqrt{|\mathcal{A}|\,|\mathcal{B}|}}{V} \log\log(n^2+1),$$

and by the definition of V given by (10) we get

$$\sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ p^{\alpha+1}\notin\mathcal{P}_2}} \sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q \neq p}} \left| R(p^{\alpha+1} q^\beta) \right| \ll |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1).$$

Similarly

$$\sum_{\substack{q^\beta \in \mathcal{P}_2 \\ q^{\beta+1}\notin\mathcal{P}_2}} \sum_{\substack{p^\alpha \in \mathcal{P}_2 \\ q \neq p}} \left| R(p^{\alpha} q^{\beta+1}) \right| \ll |\mathcal{A}|\,|\mathcal{B}| \log\log(n^2+1).$$

Using (14) we have

$$\left| R(p^{\alpha+1} q^{\beta+1}) \right| \leqslant \sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B} \\ ab+1\equiv 0 \bmod p^{\alpha+1}q^{\beta+1}}} 1 + \frac{|\mathcal{A}|\,|\mathcal{B}|}{p^{\alpha+1}q^{\beta+1}}$$

and counting trivially (using the fact that only a and b which are invertible modulo $p^{\alpha+1}q^{\beta+1}$ are counted) we can write

$$\sum_{\substack{(a,b)\in\mathcal{A}\times\mathcal{B}\\ ab+1\equiv 0 \bmod p^{\alpha+1}q^{\beta+1}}} 1 \leqslant \min\left(\frac{n\,|\mathcal{A}|}{p^{\alpha+1}q^{\beta+1}}, \frac{n\,|\mathcal{B}|}{p^{\alpha+1}q^{\beta+1}}\right) \leqslant \frac{n}{p^{\alpha+1}q^{\beta+1}}\sqrt{|\mathcal{A}|\,|\mathcal{B}|},$$

we obtain

$$\sum_{\substack{p^\alpha\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \sum_{\substack{q^\beta\in\mathcal{P}_2\\ q^{\beta+1}\notin\mathcal{P}_2}} absR(p^{\alpha+1}q^{\beta+1}) \ll \sum_{\substack{p^\alpha\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \sum_{\substack{q^\beta\in\mathcal{P}_2\\ q^{\beta+1}\notin\mathcal{P}_2}} \frac{n}{p^{\alpha+1}q^{\beta+1}}\sqrt{|\mathcal{A}|\,|\mathcal{B}|},$$

thus using (32) we get

$$\sum_{\substack{p^\alpha\in\mathcal{P}_2\\ p^{\alpha+1}\notin\mathcal{P}_2}} \sum_{\substack{q^\beta\in\mathcal{P}_2\\ q^{\beta+1}\notin\mathcal{P}_2}} \left|R(p^{\alpha+1}q^{\beta+1})\right| \ll \frac{n\sqrt{|\mathcal{A}|\,|\mathcal{B}|}}{V^2} = \frac{|\mathcal{A}|\,|\mathcal{B}|}{V} \ll |\mathcal{A}|\,|\mathcal{B}|\,.$$

Thus it follows from (39) that

$$|S_2 - |\mathcal{A}|\,|\mathcal{B}|\,S_2'| \ll K_{f_2}^2(n^2+1)\,|\mathcal{A}|\,|\mathcal{B}|\,\log\log(n^2+1).$$

In order to prove (34) it is sufficient to show that

$$\left|\sum_{p^\alpha\leqslant n^2+1} \frac{f_2(p^\alpha)}{p^\alpha}\left(1-\frac{1}{p}\right)\right|^2 = S_2' + O(K_{f_2}^2(n^2+1)).$$

We write

$$\begin{aligned}\left|\sum_{p^\alpha\leqslant n^2+1} \frac{f_2(p^\alpha)}{p^\alpha}\left(1-\frac{1}{p}\right)\right|^2 &= \sum_{\substack{p^\alpha\leqslant n^2+1\\ q^\beta\leqslant n^2+1}} \frac{f_2(p^\alpha)\overline{f_2(q^\beta)}}{p^\alpha q^\beta}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\\ &= \sum_{\substack{p^\alpha\leqslant n^2+1\\ q^\beta\leqslant n^2+1\\ p=q}} \cdots + \sum_{\substack{p^\alpha q^\beta\leqslant n^2+1\\ p\neq q}} \cdots + \sum_{\substack{p^\alpha\leqslant n^2+1\\ q^\beta\leqslant n^2+1\\ p^\alpha q^\beta> n^2+1\\ p\neq q}} \cdots\,.\end{aligned}$$

By the definition of f_2 we have $f_2(p^\alpha)\overline{f_2(q^\beta)} \neq 0$ only if $p^\alpha \leqslant (n^2+1)^{1/6}$ and $q^\beta \leqslant (n^2+1)^{1/6}$. This implies that the third sum above is empty ($p^\alpha q^\beta > n^2+1$

is not possible). S_2' is the second sum above. The first sum can be majorized easily:

$$\left| \sum_{\substack{p^\alpha \leqslant n^2+1 \\ q^\beta \leqslant n^2+1 \\ p=q}} \frac{f_2(p^\alpha)\overline{f_2(q^\beta)}}{p^\alpha q^\beta} \left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right) \right|$$

$$\leqslant K_{f_2}^2(n^2+1) \sum_p \left(1-\frac{1}{p}\right)^2 \sum_{\alpha \geqslant 1} \frac{1}{p^\alpha} \sum_{\beta \geqslant 1} \frac{1}{p^\beta}$$

$$= K_{f_2}^2(n^2+1) \sum_p \frac{1}{p^2} = O(K_{f_2}^2(n^2+1)).$$

This completes the proof of (34). □

5. Completion of the estimate of T_2

Our first step is to replace the function $A_{f_2}(n) = \sum_{p \leqslant n} \frac{f_2(p)}{p}$ in the definition of T_2 by $E_{f_2}(n)$ defined by

$$E_{f_2}(n) = \sum_{p^\alpha \leqslant n} \frac{f_2(p^\alpha)}{p^\alpha} \left(1-\frac{1}{p}\right). \tag{40}$$

We have

$$E_{f_2}(n) - A_{f_2}(n) = \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \frac{f_2(p^\alpha)}{p^\alpha} - \sum_{p^\alpha \leqslant n} \frac{f_2(p^\alpha)}{p^{\alpha+1}},$$

so that

$$|E_{f_2}(n) - A_{f_2}(n)| \leqslant 2K_{f_2}(n) \sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \frac{1}{p^\alpha}.$$

Observing that

$$\sum_{\substack{p^\alpha \leqslant n \\ \alpha \geqslant 2}} \frac{1}{p^\alpha} \leqslant \sum_p \sum_{\alpha \geqslant 2} \frac{1}{p^\alpha} = \sum_p \frac{1}{p(p-1)} \leqslant \sum_{n \geqslant 2} \frac{1}{n(n-1)} = 1,$$

we obtain

$$|E_{f_2}(n) - A_{f_2}(n)| \leqslant 2K_{f_2}(n).$$

Using the inequality $|u+v|^2 \leqslant 2\,|u|^2 + 2\,|v|^2$ with $u = f_2(ab+1) - E_{f_2}(n^2+1)$ and $v = E_{f_2}(n^2+1) - A_{f_2}(n^2+1)$ (so that $|v| \leqslant 2K_{f_2}(n^2+1)$), we get

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \left|f_2(ab+1) - A_{f_2}(n^2+1)\right|^2 \\ \leqslant 2sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \left|f_2(ab+1) - E_{f_2}(n^2+1)\right|^2 + 8\,|\mathcal{A}|\,|\mathcal{B}|\,K_{f_2}^2(n^2+1). \quad (41)$$

Now we will prove

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \left|f_2(ab+1) - E_{f_2}(n^2+1)\right|^2 \ll K_{f_2}^2(n^2+1)\ |\mathcal{A}|\,|\mathcal{B}|\ \log\log(n^2+1). \quad (42)$$

We have

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \left|f_2(ab+1) - E_{f_2}(n^2+1)\right|^2 \\ = S_2 - S_1\,\overline{E_{f_2}(n^2+1)} - \overline{S_1}\,E_{f_2}(n^2+1) + |\mathcal{A}|\,|\mathcal{B}|\left|E_{f_2}(n^2+1)\right|^2$$

where S_1 and S_2 are defined by (29) and (35) respectively. We can rewrite this as

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \left|f_2(ab+1) - E_{f_2}(n^2+1)\right|^2 \\ = \left(S_2 - |\mathcal{A}|\,|\mathcal{B}|\left|E_{f_2}(n^2+1)\right|^2\right) - \left(S_1 - |\mathcal{A}|\,|\mathcal{B}|\,E_{f_2}(n^2+1)\right)\,\overline{E_{f_2}(n^2+1)} \\ - \left(\overline{S_1} - |\mathcal{A}|\,|\mathcal{B}|\,\overline{E_{f_2}(n^2+1)}\right)\,E_{f_2}(n^2+1).$$

By Lemma 4 we have

$$\left|S_1 - |\mathcal{A}|\,|\mathcal{B}|\,E_{f_2}(n^2+1)\right| \ll K_{f_2}(n^2+1)\ |\mathcal{A}|\,|\mathcal{B}|$$

and by Lemma 5 we have

$$\left|S_2 - |\mathcal{A}|\,|\mathcal{B}|\left|E_{f_2}(n^2+1)\right|^2\right| \ll K_{f_2}^2(n^2+1)\ |\mathcal{A}|\,|\mathcal{B}|\ \log\log(n^2+1),$$

thus we obtain

$$\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \left|f_2(ab+1) - E_{f_2}(n^2+1)\right|^2 \\ \ll K_{f_2}(n^2+1)(K_{f_2}(n^2+1)\log\log(n^2+1) + \left|E_{f_2}(n^2+1)\right|)\ |\mathcal{A}|\,|\mathcal{B}|\,.$$

Now observing that

$$\left|E_{f_2}(n^2+1)\right| \leqslant \left|A_{f_2}(n^2+1)\right| + 2K_{f_2}(n^2+1)$$
$$\leqslant K_{f_2}(n^2+1) \sum_{p\leqslant n^2+1} \frac{1}{p} + 2K_{f_2}(n^2+1) \ll K_{f_2}(n^2+1)\log\log(n^2+1),$$

we get (42). It follows from (41) and (42) that

$$T_2 = O(K_{f_2}^2(n^2+1)\log\log(n^2+1)). \tag{43}$$

6. The estimate of T_1

Let $\omega_V(m)$ be the number of distinct prime factors of m not exceeding V:

$$\omega_V(m) = \sum_{\substack{p\leqslant V\\ p\,|\,m}} 1.$$

Then for all $m \leqslant n^2+1$ we have

$$|f_1(m)| = \left|\sum_{\substack{p^\alpha\|m\\ p^\alpha\in\mathcal{P}_1}} f_1(p^\alpha)\right| \leqslant \sum_{\substack{p^\alpha\|m\\ p^\alpha\in\mathcal{P}_1}} |f_1(p^\alpha)|$$

so that

$$|f_1(m)| \leqslant K_{f_1}(n^2+1) \sum_{\substack{p^\alpha\|m\\ p^\alpha\in\mathcal{P}_1}} 1 = K_{f_1}(n^2+1) \sum_{\substack{p\,|\,m\\ p\leqslant V}} 1 = K_{f_1}(n^2+1)\,\omega_V(m).$$

Moreover by (7), (10) and (8) we have

$$\left|\sum_{p\leqslant n^2+1} \frac{f_1(p)}{p}\right| \leqslant K_{f_1}(n^2+1) \sum_{p\leqslant V} \frac{1}{p}$$
$$\ll K_{f_1}(n^2+1)\log\log V \ll K_{f_1}(n^2+1)\log\log\log n.$$

Using the inequality $|z_1+z_2|^2 \leqslant 2(|z_1|^2+|z_2|^2)$ it follows that for $a\in\mathcal{A}$, $b\in\mathcal{B}$ we have

$$\left| f_1(ab+1) - \sum_{p \leqslant n^2+1} \frac{f_1(p)}{p} \right|^2 \leqslant 2\,|f_1(ab+1)|^2 + 2 \left| \sum_{p \leqslant n^2+1} \frac{f_1(p)}{p} \right|^2$$
$$\leqslant 2K_{f_1}^2(n^2+1)\omega_V^2(ab+1) + O(K_{f_1}^2(n^2+1)(\log\log\log n)^2),$$

so that

$$\begin{aligned} T_1 &= \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \left| f_1(ab+1) - \sum_{p\leqslant n^2+1} \frac{f_1(p)}{p} \right|^2 \\ &\leqslant \frac{2K_{f_1}^2(n^2+1)}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \omega_V^2(ab+1) + O(K_{f_1}^2(n^2+1)(\log\log\log n)^2). \end{aligned}$$

We split this double sum in two parts:

$$T_1 \leqslant 2K_{f_1}^2(n^2+1)(X_1+X_2) \tag{44}$$

where

$$X_1 = \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{\substack{a\in\mathcal{A} \quad b\in\mathcal{B} \\ \omega_V(ab+1)\leqslant 5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(ab+1)$$

and

$$X_2 = \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{\substack{a\in\mathcal{A} \quad b\in\mathcal{B} \\ \omega_V(ab+1)> 5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(ab+1).$$

Then clearly we have

$$X_1 \leqslant \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} (5C(\log\log(n^2+1))^{1/2})^2 = 25C^2 \log\log(n^2+1). \tag{45}$$

In order to estimate X_2, we may assume that

$$|\mathcal{A}| \leqslant |\mathcal{B}|\,. \tag{46}$$

Then we have

$$\begin{aligned} X_2 &= \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{\substack{b\in\mathcal{B} \\ \omega_V(ab+1)>5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(ab+1) \\ &\leqslant \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{\substack{m\leqslant n \\ \omega_V(am+1)>5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(am+1) \\ &\leqslant \frac{1}{|\mathcal{B}|} \max_{a\leqslant n} \sum_{\substack{m\leqslant n \\ \omega_V(am+1)>5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(am+1). \end{aligned} \tag{47}$$

The last sum can be rewritten as

$$\sum_{\substack{m\leqslant n\\ \omega_V(am+1)>5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(am+1) = \sum_{t>5C(\log\log(n^2+1))^{1/2}} \sum_{\substack{m\leqslant n\\ \omega_V(am+1)=t}} t^2$$
$$= \sum_{t>5C(\log\log(n^2+1))^{1/2}} t^2 \sum_{\substack{m\leqslant n\\ \omega_V(am+1)=t}} 1. \tag{48}$$

Note that if the inner sum is non-empty, *i.e.* there is an $m \leqslant n$ with $\omega_V(am+1) = t$, then t must satisfy

$$t = \omega_V(am+1) \leqslant \max_{k\leqslant n^2+1} \omega_V(k) \leqslant \pi(V) < V,$$

thus in the last double sum we may restrict ourselves to t values with

$$t < V. \tag{49}$$

If an integer m is counted in the last inner sum in (48), then there are primes $q_1 < q_2 < \cdots < q_t \leqslant V$ such that $q_1 q_2 \cdots q_t \mid am+1$. Thus this sum is

$$\sum_{\substack{m\leqslant n\\ \omega_V(am+1)=t}} 1 \leqslant \sum_{q_1<q_2<\cdots<q_t\leqslant V} \sum_{\substack{m\leqslant n\\ q_1q_2\cdots q_t\,|\,am+1}} 1 \leqslant \sum_{q_1<q_2<\cdots<q_t\leqslant V} \left(\frac{n}{q_1q_2\cdots q_t}+1\right)$$
$$\leqslant n\left(\sum_{q\leqslant V}\frac{1}{q}\right)^t \cdot \frac{1}{t!} + \binom{\pi(V)}{t} \leqslant n\frac{((1+o(1))\log\log V)^t}{t!} + V^t.$$

Inserting this estimate in (48), by (49) we get

$$\sum_{\substack{m\leqslant n\\ \omega_V(am+1)>5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(am+1)$$
$$\ll n \sum_{t>5C(\log\log(n^2+1))^{1/2}} \frac{((1+o(1))\log\log V)^t}{(t-2)!} + \sum_{t<V} t^2 V^t. \tag{50}$$

By definition (10) of V, (8) and Stirling's formula, for large n the first term is

$$n \sum_{t>5C(\log\log(n^2+1))^{1/2}} \frac{((1+o(1))\log\log V)^t}{(t-2)!}$$
$$\ll n \sum_{t>5C(\log\log(n^2+1))^{1/2}} ((\frac{1}{2}+o(1))\log\log\log n)^t \left(\frac{3}{t}\right)^{t-2}$$

$$\ll n \sum_{t > 5C(\log\log(n^2+1))^{1/2}} \left(\frac{2\log\log\log n}{t}\right)^t t^2.$$

Denote the general term of the last sum by A_t, and write

$$t_0 = \left\lfloor 5C(\log\log(n^2+1))^{1/2} \right\rfloor + 1.$$

Then for $t \geqslant t_0$ we have

$$\begin{aligned} \frac{A_{t+1}}{A_t} &\ll (\log\log\log n)\left(\frac{t}{t+1}\right)^{t+1}\frac{1}{t} = (\log\log\log n)\left(1 - \frac{1}{t+1}\right)^{t+1}\frac{1}{t} \\ &\ll \frac{\log\log\log n}{t} \ll \frac{\log\log\log n}{t_0} = o(1). \end{aligned}$$

It follows that the first term in (50) can be estimated as

$$\begin{aligned} n &\sum_{t > 5C(\log\log(n^2+1))^{1/2}} \frac{((1+o(1))\log\log V)^t}{(t-2)!} \\ &\ll n\sum_{t \geqslant t_0} A_t \ll nA_{t_0} = n\left(\frac{2\log\log\log n}{t_0}\right)^{t_0} t_0^2 \\ &= n\exp\left(t_0(\log(2\log\log\log n) - \log t_0) + 2\log t_0\right) \\ &= n\exp\left(-(1+o(1))\ t_0\log t_0\right) \\ &\ll n\exp\left(-(1+o(1))\frac{5}{2}C(\log\log n)^{1/2}\log\log\log n\right) \end{aligned} \tag{51}$$

and again by(10) and (8), the second term is

$$\begin{aligned} &\sum_{t<V} t^2V^t < \sum_{t<V} t^2V^{V+2} < V^{V+3} \\ &< \exp\left(C\sqrt{\log\log n}\log\log\log n\left(\exp\left(C\sqrt{\log\log n}\log\log\log n\right)+3\right)\right) \\ &= n^{o(1)}. \end{aligned} \tag{52}$$

By (50), (51) and (52)

$$\begin{aligned} &\sum_{\substack{m \leqslant n \\ \omega_V(am+1) > 5C(\log\log(n^2+1))^{1/2}}} \omega_V^2(am+1) \\ &\ll n\exp\left(-(1+o(1))\,\frac{5}{2}\,C(\log\log n)^{1/2}\log\log\log n\right) \quad \text{for all } a \leqslant n. \end{aligned} \tag{53}$$

It follows from (8), (46), (47) and (53) that

$$X_2 \ll \frac{n}{\sqrt{|\mathcal{A}|\,|\mathcal{B}|}} \exp\Big(-2C(\log\log n)^{1/2}\log\log\log n\Big)$$
$$\ll \exp\Big(-C(\log\log n)^{1/2}\log\log\log n\Big) = o(1). \tag{54}$$

Combining (44), (45) and (54) we obtain

$$T_1 = O(C^2 K_{f_1}^2(n^2+1)\log\log(n^2+1)). \tag{55}$$

7. The estimate of T_3 and the completion of the proof of Theorem 1

If $m \leqslant n^2+1$ then

$$|f_3(m)| = \Big|\sum_{p^\alpha \| m} f_3(p^\alpha)\Big| \leqslant \sum_{p^\alpha \| m} |f_3(p^\alpha)|$$

so that, since $f_3(p^\alpha) = 0$ whenever $p^\alpha \notin \mathcal{P}_3$ and using (7),

$$|f_3(m)| \leqslant \sum_{\substack{p^\alpha \in \mathcal{P}_3 \\ p^\alpha \| m}} K_{f_3}(n^2+1) = K_{f_3}(n^2+1) \sum_{\substack{p^\alpha \in \mathcal{P}_3 \\ p^\alpha \| m}} 1. \tag{56}$$

Here the last sum is $\leqslant 5$ since otherwise we had

$$m \geqslant \prod_{\substack{p^\alpha \in \mathcal{P}_3 \\ p^\alpha \| m}} p^\alpha > \prod_{\substack{p^\alpha \in \mathcal{P}_3 \\ p^\alpha \| m}} (n^2+1)^{1/6} \geqslant \Big((n^2+1)^{1/6}\Big)^6 = n^2+1$$

which contradicts our assumption $m \leqslant n^2+1$. Thus it follows from (56) that

$$|f_3(ab+1)| \leqslant 5\ K_{f_3}(n^2+1) \quad \text{for all } a \in \mathcal{A},\ b \in \mathcal{B}. \tag{57}$$

Moreover we have

$$\Big|\sum_{p \leqslant n^2+1} \frac{f_3(p)}{p}\Big| \leqslant \sum_{p \leqslant n^2+1} \frac{|f_3(p)|}{p} \leqslant K_{f_3}(n^2+1) \sum_{p \in \mathcal{P}_3} \frac{1}{p}$$

so that

$$\Big|\sum_{p \leqslant n^2+1} \frac{f_3(p)}{p}\Big| \leqslant K_{f_3}(n^2+1) \sum_{(n^2+1)^{1/6} < p \leqslant n^2+1} \frac{1}{p} = O(K_{f_3}(n^2+1)). \tag{58}$$

It follows from (57) and (58) that

$$T = \frac{1}{|\mathcal{A}|\,|\mathcal{B}|} \sum_{a\in\mathcal{A}} \sum_{b\in\mathcal{B}} \left| O(K_{f_3}(n^2+1)) + O(K_{f_3}(n^2+1)) \right|^2 = O(K_{f_3}^2(n^2+1)). \quad (59)$$

(9) follows from (11) (43), (55) and (59), observing that

$$K_f(n^2+1) = \max(K_{f_1}(n^2+1), K_{f_2}(n^2+1), K_{f_3}(n^2+1)),$$

and this completes the proof of Theorem 1.

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