SOME CHARACTERIZATIONS OF DICHOTOMY FOR IMPULSIVE DYNAMIC SYSTEMS

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ABSTRACT. We study the problem of dichotomy and boundedness for impulsive dynamic equations on arbitrary closed subset of real numbers. The spectral decomposition theorem gives all our main results. The obtained results are fundamentally new, even for the classical case.

1. INTRODUCTION

Dynamical systems has many applications see, for example ([15, 16, 21]) and references therein. In 1960, Millman et al. [11], began to study impulsive differential systems (IDS). To design the impulsive control protocol, the theory of IDS is a standard mathematical method. Research in IDS has gained momentum ([17, 18, 19]). Such a system provides not only realistic models but also presents many interesting mathematical problems. Wang et al. [20], reviewed and addressed some recent developments of impulsive control theory, fundamental theory on impulsive control systems.

An IDS is described by three components:

- (1) A differential system, which governs the state of the system between impulses.
- (2) An impulse condition, which models an impulsive jump defined by a jump function at the instant an impulse occurs.
- (3) A jump rule, which defines a set of jump events in which the impulse equation is active.

Exponential type dichotomy, which traces back to [12] and generalizes the concept of hyperbolicity from autonomous to nonautonomous linear systems, has been playing an ever more critical role in the study of nonautonomous dynamical systems such as dynamic equations on time scales [13], ordinary differential equations [4, 5, 6], and difference equations [1].

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The study of dichotomy for dynamic equations on time scales is quite recent [22]. There are several approaches in the literature to study dichotomy, one of which is the spectral decomposition approach [22].

The presence of impulses in the dynamic equations on time scales leads to complications with the concept of the solution. One major difficulty is the spectral decomposition theorem for impulsive dynamic systems on time scales. To overcome this difficulty, we generalized the spectral decomposition approach to the impulsive case.

In this paper, we investigate the problem of exponential dichotomy and boundedness of solutions of the following Cauchy problems:

$$\begin{cases} a^\Delta(x) = Aa(x) + f(x)Qb, & x \neq x_k, \\ \Delta a(x_k^+) = I_k(a(x)), & x = x_k \\ a(x_0) = a_0 \end{cases}$$

and

$$\begin{cases} a^\Delta(x) = -Aa^\sigma(x) + f(x)(I - Qb), & x \neq x_k \\ \Delta a(x_k^+) = I_k(a(x)), & x = x_k \\ a(x_0) = a_0, \end{cases}$$

where $\mathbb{T}$ is a time scale, A is a regressive matrix b is non-zero vector in $\mathbb{R}^m$, $f(s)$ is a bounded and right-dense continuous function on $\mathbb{T}$, Q is a projection on $\mathbb{C}^m$ and I_k is vector-valued impulsive function defined on $\mathbb{R}^n$.

Impulsive dynamic system is an important area of study. There are many qualitative results obtained for above systems [7, 8, 9].

The organization of the paper is as follows: Section 2 presents the basic knowledge of time scale calculus, impulsive linear dynamic systems, including the general definition of impulsive transition matrix and its fundamental properties. Section 3 introduces the concept of dichotomy analysis and criteria for impulsive systems with different impulsive conditions. Finally, concluding remarks and propositions for future works are given.

2. PRELIMINARIES

To keep this article self-contained, this section contains some preliminary definitions and the associated notation [2, 3]. For a nonempty closed subset of real numbers (time scale) $\mathbb{T}$, let us consider the following operators and step size function:

$$\begin{aligned} x^\sigma &= \sigma(x) := \inf\{s \in \mathbb{T} : s > x\}; \\ x^\rho &= \rho(x) := \sup\{s \in \mathbb{T} : s < x\}, \end{aligned}$$

and

$$\mu : \mathbb{T} \rightarrow \mathbb{R}^+ : \mu(x) := x^\sigma - x.$$

It is easy to see that for $x \in \mathbb{T}$, $x^\sigma \geq x$ and $x^\rho \leq x$.

Throughout this paper, we always assume that $\sup \mathbb{T} = \infty$ and for any $x \in \mathbb{T}$, let $\mathbb{T}_{(x)} := [x, \infty) \cap \mathbb{T}$ and $\mathbb{T}_+ := \mathbb{T}_{(0)}$.

A function $f : \mathbb{T}_{(x)} \rightarrow \mathbb{R}$ is called right-dense continuous (rd-continuous), if $\lim_{x \rightarrow x_0^-} f(x)$ exists at left-dense points and $\lim_{x \rightarrow x_0^+} f(x) = f(x_0)$ at right-dense

points. The delta derivative of f is defined by

$$f^\Delta(x) = \lim_{\substack{s \rightarrow x \\ s \neq x^\sigma}} \frac{f(x^\sigma) - f(s)}{x^\sigma - s}.$$

For a finite dimensional Banach space $\mathbb{X}$ the following spaces appear mostly:

$$\begin{aligned} C_{rd}(\mathbb{T}(\tau), \mathbb{X}) &: = \{f : \mathbb{T}(\tau) \rightarrow \mathbb{X} \mid f \text{ is } rd\text{-continuous}\}, \\ C_{rd}^1(\mathbb{T}(\tau), \mathbb{X}) &: = \{f : \mathbb{T}(\tau) \rightarrow \mathbb{X} \mid f \Delta\text{-derivative is in } C_{rd}(\mathbb{T}(\tau), \mathbb{X})\}, \\ \mathcal{R}(\mathbb{T}(\tau), \mathbb{X}) &: = \left\{f \in C_{rd}(\mathbb{T}(\tau), \mathbb{X}) \mid (I_{\mathbb{X}} + \mu(x)f(x))^{-1} \text{ exists for all } x \in \mathbb{T}(\tau)\right\} \\ \mathcal{R}_U(\mathbb{T}(\tau), \mathbb{C}) &: = \{f \in \mathcal{R}(\mathbb{T}(\tau), \mathbb{C}) \mid \text{If } \exists \alpha \text{ with } \alpha^{-1} \leq |1 + (\mu f)(x)| \forall x \in \mathbb{T}(\tau)\} \\ \mathcal{R}^+(\mathbb{T}(\tau), \mathbb{R}) &: = \{f \in \mathcal{R}(\mathbb{T}(\tau), \mathbb{R}) \mid I_{\mathbb{X}} + \mu(x)f(x) > 0 \text{ for all } x \in \mathbb{T}(\tau)\}, \end{aligned}$$

where $\mathbb{X} = \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ or $\mathbb{M}_n(\mathbb{R})$ and $I_{\mathbb{X}}$ is corresponding multiplicative identity of $\mathbb{X}$.

It is well known that for $A \in \mathcal{R}(\mathbb{T}_+, \mathbb{M}_n(\mathbb{R}))$, the exponential matrix $e_A(x, s)$ at initial time $s \in \mathbb{T}_+$ is the unique solution of the following matrix initial value problem (IVP)

$$X^\Delta = AX, \quad X(x) = I_{\mathbb{M}_n(\mathbb{R})}$$

and $a(x) = e_A(x, s)a_0$, $x \geq s$, is the unique solution of IVP

$$a^\Delta = Aa, \quad a(x) = a_0$$

and $a(x) = e_{\ominus A^T}(x, s)a_0$, $x \geq s$ is the unique solution of the following adjoint IVP

$$a^\Delta = -A^T a^\sigma, \quad a(s) = a_0.$$

Recently, Zada et. al. [22], proved the following spectral decomposition theorem on time scales:

For $A \in \mathcal{R}(\mathbb{T}_+, \mathbb{M}_n(\mathbb{R}))$ and for each $z \in \mathbb{C}^n \exists w_i \in \mathcal{L}_i := \ker(A - \lambda_i I)^{n_i}$ ($i = 1, 2, \dots, k$)

$$e_A(x, 0)z = e_A(x, 0)w_1 + e_A(x, 0)w_2 + \dots + e_A(x, 0)w_k, \quad x \in \mathbb{T}.$$

Furthermore, if $w_i(x) := e_A(x, 0)w_i$ then $w_i(s) \in \ker(A - \lambda_i)^{n_i} \forall x \in \mathbb{T}$ and $\exists \mathbb{C}^n$ -valued polynomials $y_i(x)$ with $\deg(y_i) \leq n_i - 1$:

$$w_i(x) = e_{\lambda_i}(x, 0)y_i(x), \quad x \in \mathbb{T} \text{ and } (i = 1, 2, \dots, k).$$

It follows that

$$e_A(x, 0)z = e_{\lambda_1}(x, 0)y_1(x) + e_{\lambda_2}(x, 0)y_2(x) + \dots + e_{\lambda_k}(x, 0)y_k(x), \quad x \in \mathbb{T}. \quad (2.1)$$

For $I_k(a(x_k)) = I + B_k$, consider the following homogenous linear impulsive dynamic systems

$$\begin{cases} a^\Delta = Aa, & x \in \mathbb{T}_+, \quad x \neq x_k \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \dots \end{cases} \quad (2.2)$$

where $B_k \in M_n(\mathbb{R})$, for all k , $A \in \mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$, $0 = x_0 < x_1 < x_2 < \dots < x_k < \dots$, with $\lim_{k \rightarrow \infty} x_k = \infty$, $a(x_k^+)$ represents the right limit of $a(x)$ at $x = x_k$ (with $a(x_k^+) = a(x)$ if $x_k^\sigma > x_k$ for all k).

Throughout we assumed that, $x_k^\sigma = x_k$ for all k .

Corresponding to (2.2) consider the following IVP

$$\begin{cases} a^\Delta = Aa, & x \in \mathbb{T}_+, \quad x \neq x_k \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \dots, \\ a(\tau^+) = \eta, & \tau \geq 0. \end{cases} \quad (2.3)$$

Let us consider the transition matrix $G_A(x, s)$, $0 \leq s \leq x$, of (2.2) associated with impulsive sequence $\{B_k, x_k\}_{k=1}^\infty$ is defined as

$$G_A(x, s) = \begin{cases} e_A(x, s) & \text{if } x_{k-1} \leq s \leq x \leq x_k; \\ e_A(x, x_k^+)(I + B_k)e_A(x_k, s) & \text{if } x_{k-1} \leq s < x_k < x \leq x_{k+1}; \\ e_A(x, x_k^+) \left[\prod_{s < x_j \leq x} (I + B_j) e_A(x_j, x_{j-1}^+) \right] & \\ (I + B_i) e_A(x_i, s), & \text{if } x_{i-1} \leq s < x_i < \dots < x_k < x \leq x_{k+1}, \end{cases}$$

Example 2.1. Let us consider the following IVP

$$\begin{cases} a^\Delta = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} a, \quad x \in \mathbb{T}_+, \quad x \neq x_{2k} \\ a(x_{2k}^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(x_{2k}), \quad k = 1, 2, \dots, \\ a(\tau^+) = \eta, \quad \tau \geq 0. \end{cases} \quad (2.4)$$

where $A = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$, $(I + B_k) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix}$, $\lambda_2 > \lambda_1 > 0$ and $x_k := k$.

Then the impulsive transition matrix $G_A(x, s)$, $0 \leq s \leq x$, of (2.4) associated with impulsive sequence $\{B_k, x_k\}_{k=1}^\infty$ is

$$G_A(x, s) = \begin{cases} \begin{pmatrix} e_{-\lambda_1}(x, s) & 0 \\ 0 & e_{-\lambda_2}(x, s) \end{pmatrix} & \text{if } x_{k-1} \leq s \leq x \leq x_k; \\ \begin{pmatrix} e_{-\lambda_1}(x, x_k^+) e_{-\lambda_1}(x_k, s) a_{2k} & 0 \\ 0 & e_{-\lambda_2}(x, x_k^+) e_{-\lambda_2}(x_k, s) b_{2k} \end{pmatrix} & \text{if } x_{k-1} \leq s < x_k < x \leq x_{k+1}; \\ \begin{pmatrix} e_{-\lambda_1}(x, x_k^+) a_{2i} e_{-\lambda_1}(x_i, s) & 0 \\ \prod_{s < x_j \leq x} (e_{-\lambda_1}(x_j, x_{j-1}^+) a_{2j}) & 0 \\ 0 & e_{-\lambda_2}(x, x_k^+) a_{2i} e_{-\lambda_2}(x_i, s) \\ & \prod_{s < x_j \leq x} (e_{-\lambda_2}(x_j, x_{j-1}^+) a_{2j}) \end{pmatrix} & \text{if } x_{i-1} \leq s < x_i < \dots < x_k < x \leq x_{k+1}. \end{cases}$$

Remark. It is easy to see that spectral decomposition theorem may not holds for $G_A(x, s)$. For example, if B_k is arbitrary non-diagonal matrix and $AB_k \neq B_k A$, then $\mathcal{L}_j$ is not $G_A(x, s)$ -invariant subspace of $\mathbb{C}^n$.

In this circumstance, we need to modify spectral decomposition theorem for impulsive transition matrix $G_A(x, s)$, $0 \leq s \leq x$, associated with $\{B_k, x_k\}_{k=1}^\infty$.

(A₁): Assume that $A \in \mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, $k = 1, 2, \dots$.

(A₂): Let (A₁) holds and there exist a positive constant θ such that $x_{k+1} - x_k < \theta$, $k = 1, 2, \dots$.

Lemma 2.2. Let (A₁) holds such that $AB_k = B_k A$ and $\mathcal{L}_j$ is $(I + B_k)$ -invariant subspace of $\mathbb{C}^n$ for each $k = 1, 2, 3, \dots$. For each $z \in \mathbb{C}^n \exists w_i \in \mathcal{L}_i := \ker(A - \lambda_i I)^{n_i}$ ($i = \overline{1, k}$):

$$G_A(x, 0)z = G_A(x, 0)w_1 + G_A(x, 0)w_2 + \dots + G_A(x, 0)w_k, \quad x \in \mathbb{T}.$$

Furthermore, if $w_i(x) := G_A(x, 0) w_j$ then $w_j(x) \in \ker(A - \lambda_i)^{n_i}$ for all $x \in \mathbb{T}$ and $\exists \mathbb{C}^n$ -valued polynomials $y_i(x)$ with $\deg(y_i) \leq n_i - 1$:

$$w_j(x) = e_{\lambda_i}(x, 0) y_i(x), \quad x \in \mathbb{T} \text{ and } (i = \overline{1, k}).$$

It follows that

$$G_A(x, 0)z = e_{\lambda_1}(x, 0) y_1(x) + e_{\lambda_2}(x, 0) y_2(x) + \cdots + e_{\lambda_k}(x, 0) y_k(x), \quad x \in \mathbb{T}.$$

Proof. Similar to [22, Theorem 4]. $\square$

Lemma 2.3. [10] *Let (A_1) holds, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$ the IVP (2.3) has a unique solution:*

$$a(x) = G_A(x, \tau)\eta, \quad x \geq \tau.$$

Corollary 2.4. [10] *Let (A_1) holds, then*

$$\|G_A(x, \tau)\| \leq \prod_{\tau < x_k \leq x} (1 + \|B_k\|) e^{\|A\|(x-\tau)}$$

for $\tau, x \in \mathbb{T}_+$ with $x \geq \tau$. Moreover, for any $\tau, x \in \mathbb{T}_+$, there exists $M > 0$ such that we have $\|G_A(r, s)\| \leq M$ on the set

$$\{(r, s) \in \mathbb{T}_+ \times \mathbb{T}_+; \tau \leq s \leq r \leq x\}.$$

Lemma 2.5. [10] *Let (A_1) holds, then*

- (i) $G_A(x, x) = I, \quad x \geq 0;$
- (ii) $G_A(x, s) = G_A^{-1}(s, x), \quad 0 \leq s \leq x;$
- (iii) $G_A(\sigma(x), s) = [I + \mu(x)A] G_A(x, s), \quad 0 \leq s \leq x;$
- (iv) $G_A(x, s)G_A(s, r) = G_A(x, r), \quad 0 \leq r \leq s \leq x;$
- (v) $\frac{\partial}{\partial s} G_A(x, s) = -G_A(x, \sigma(s))A, \quad s \neq x_k;$
- (vi) $G_A^{-1}(x, s) = G_{\ominus A^T}^T(x, s), \quad 0 \leq s \leq x.$

Consider the following nonhomogenous IVP

$$\begin{cases} a^\Delta = Aa + h(x), & x \in \mathbb{T}_+, \quad x \neq x_k, \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \dots \\ a(\tau^+) = \eta, & \tau \geq 0, \end{cases} \quad (2.5)$$

where $h : \mathbb{T}_+ \rightarrow \mathbb{R}^n$ is a given function.

Lemma 2.6. [10] *Let (A_1) holds with $h \in \mathcal{R}(\mathbb{T}_+, \mathbb{R}^n)$, then for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$, the IVP (2.5) has a unique solution:*

$$a(x) = G_A(x, \tau)\eta + \int_{\tau}^x G_A(x, \sigma(s))h(s)\Delta s, \quad x \geq \tau.$$

Moreover for each $(\tau, \eta) \in \mathbb{T}_+ \times \mathbb{R}^n$, the IDS

$$\begin{cases} a^\Delta = -A^T a^\sigma + h(x), & x \geq \tau, \quad x \neq x_k \\ a(x_k^+) = (I + C_k)a(x_k), & k = 1, 2, \dots \\ a(\tau^+) = \eta, & \tau \geq 0. \end{cases} \quad (2.6)$$

has at most one solution, and

$$a(x) = G_{\ominus A^T}(x, \tau)\eta + \int_{\tau}^x G_{\ominus A^T}(x, s)h(s)\Delta s, \quad x \geq \tau \quad (2.7)$$

where $C_k := -B^T(I + B_k^T)^{-1}$, $k = 1, 2, \dots$.

3. DICHOTOMY

The goal of the section is to study the dichotomy of the solution of linear impulsive dynamic systems.

Definition 3.1. [10, Definition 5.1] *The IDS (2.3) is said to be exponentially stable if $(\exists) \lambda > 0$ with $-\lambda \in \mathcal{R}(\mathbb{T}_+, \mathbb{R})$: for every $\tau \in \mathbb{T}_+$, $(\exists) N = N(\tau) \geq 1$: the solution of (2.3) through $(\tau, a(\tau))$ satisfies*

$$\|a(x)\| \leq N \|a(\tau)\| e_{-\lambda}(x, \tau) \text{ for all } x \in \mathbb{T}_+.$$

Using the idea of [14], here first we decompose $\mathbb{C}$ into three parts and then with the help of this decomposition we decompose $\mathbb{C}^m$ into three spectral subspaces which will help us to use the idea of dichotomy.

For further discussion, we have to divide $\mathbb{C}$ into three sets as follows.

$$\begin{aligned} E_{\mathbb{C}}(\mathbb{T}) &= \left\{ \lambda \in \mathbb{C} : \limsup_{S \rightarrow \infty} \frac{1}{S - s_0} \int_{s_0}^S \lim_{u \rightarrow \mu(s)} \frac{\log |1 + u\lambda|}{s} \Delta s < 0 \right\}, \\ E_{\mathbb{C}}^+(\mathbb{T}) &= \left\{ \lambda \in \mathbb{C} : \limsup_{S \rightarrow \infty} \frac{1}{S - s_0} \int_{s_0}^S \lim_{u \rightarrow \mu(s)} \frac{\log |1 + u\lambda|}{s} \Delta s > 0 \right\}, \\ E_{\mathbb{C}}^0(\mathbb{T}) &= \left\{ \lambda \in \mathbb{C} : \limsup_{S \rightarrow \infty} \frac{1}{S - s_0} \int_{s_0}^S \lim_{u \rightarrow \mu(s)} \frac{\log |1 + u\lambda|}{s} \Delta s = 0 \right\}. \end{aligned}$$

Clearly, $\mathbb{C} = E_{\mathbb{C}}(\mathbb{T}) \cup E_{\mathbb{C}}^+(\mathbb{T}) \cup E_{\mathbb{C}}^0(\mathbb{T})$. By using this decomposition of $\mathbb{C}$, we can state the following definition.

Definition 3.2. *Let $\sigma(A)$ denote the spectrum of the regressive matrix A . The system $z^\Delta(s) = Az(s); z(s_0) = z_0$ is said to be exponentially stable if all eigenvalues of matrix A are uniformly regressive and $\sigma(A) \subset E_{\mathbb{C}}(\mathbb{T})$ and it is said to be expansive if $\sigma(A) \subset E_{\mathbb{C}}^+(\mathbb{T})$. If $\sigma(A) \cap E_{\mathbb{C}}^0(\mathbb{T}) = \emptyset$, then the system is said to be dichotomic.*

More explicitly the decomposition of $\mathbb{C}^m = \bigoplus_{j=1}^k \mathcal{L}_j$ can be written as

$$\mathbb{C}^m = W_s(A) \oplus W_0(A) \oplus W_u(A),$$

where

$$\begin{aligned} W_s(A) &= \bigoplus_{j=1, \lambda_j \in E_{\mathbb{C}}(\mathbb{T})}^k \ker(A - \lambda_j I)^{n_j} \\ W_0(A) &= \bigoplus_{j=1, \lambda_j \in E_{\mathbb{C}}^0(\mathbb{T})}^k \ker(A - \lambda_j I)^{n_j} \end{aligned}$$

and

$$W_u(A) = \bigoplus_{j=1, \lambda_j \in E_{\mathbb{C}}^+(\mathbb{T})}^k \ker(A - \lambda_j I)^{n_j}$$

Now if A is a dichotomic matrix, then $W_0(A) = \{0\}$ and so $\mathbb{C}^m = W_s(A) \oplus W_u(A)$.

Lemma 3.3. [10] *Let (A_2) holds. If the solution of IVP*

$$\begin{cases} a^\Delta = Aa, & x \in \mathbb{T}_\tau, \quad x \neq x_k \\ a(x_k^+) = (I + B_k)a(x_k), & k = 1, 2, \dots \\ a(\tau^+) = \eta, & \tau \geq 0 \end{cases} \quad (3.1)$$

is bounded, then $(\exists) N = N(\tau) \geq 1$, λ with $-\lambda \in \mathcal{R}^+$:

$$\|G_A(x, \tau)\| \leq Ne_{-\lambda}(x, \tau)$$

for all $x \in \mathbb{T}_{(\tau)}$.

Corollary 3.4. [10] *Let (A_2) hold, then bounded solution of (3.1) implies that the solution of (2.3) is exponential stable.*

Corollary 3.5. [10] *Let (A_2) holds, and there exists b , such that*

$$\sup_{k \geq 1} \|B_k\| \leq b, \quad k = 1, 2, \dots \quad (3.2)$$

If the solution of IVP (3.1) is bounded, then $\exists$ constants N , λ with $-\lambda \in \mathcal{R}^+$:

$$\|G_A(x, \tau)\| \leq Ne_{-\lambda}(x, \tau) \text{ for all } x \in \mathbb{T}_{(\tau)}.$$

Lemma 3.6. [3] *For any constant λ with $-\lambda \in \mathcal{R}^+$, we have that*

$$e_{-\lambda}(x, \tau) \leq e^{-\lambda(x-\tau)}, \text{ for all } \tau, x \in \mathbb{T} \text{ with } x \in \mathbb{T}_{(\tau)}. \quad (3.3)$$

Theorem 3.7. *Let (A_2) holds with $AB_k = B_kA$ and $\mathcal{L}_j$ is $(I + B_k)$ -invariant subspace of $\mathbb{C}^m$ for each $k = 1, 2, 3, \dots$. Then the system*

$$\begin{cases} a^\Delta = Aa, \\ a(x_k^+) = (I + B_k)a(x_k), \\ a(0) = 0 \end{cases} \quad (3.4)$$

is exponentially stable if and only if, $b \in \mathbb{C}^m$ and each bounded function f the unique solution of the Cauchy problem

$$\begin{cases} a^\Delta = Aa + f(s)b, \\ a(x_k^+) = (I + B_k)a(x_k), \\ a(0) = 0, \end{cases} \quad (3.5)$$

is bounded.

Proof. Let $A \in \mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$. Consider the system

$$\begin{cases} a^\Delta(x) = Aa(x), \\ a(x_k^+) = (I + B_k)a(x_k), \quad a(0) = 0, \end{cases}$$

is exponentially stable. Then by Definition 3.1, we have

$$\|a(x)\| \leq \alpha \|a_0\| e_{-\gamma}(x, x_0),$$

for all $x, x_0 \in \mathbb{T}$. We need to prove that for each $b \in \mathbb{C}^m$, (3.5) is bounded. Then from Theorem 2.6, the system (3.5) has atmost one solution with

$$a(x) = \int_0^x G_A(x, \sigma(s)) f(s)b \Delta s. \quad (3.6)$$

Applying norm on equation (3.6), we have

$$\begin{aligned}
\|a(x)\| &= \sup_{x \in \mathbb{T}} \left| \int_0^x G_A(x, \sigma(s)) f(s) b \Delta s \right| \leq \sup_{x \in \mathbb{T}} \left| \int_0^x G_A(x, \sigma(s)) \Delta s \right| \|f(s) b\| \\
&= \sup_{x \in \mathbb{T}} \left| A^{-1} \int_0^x A G_A(x, \sigma(s)) \Delta s \right| C = C A^{-1} \sup_{x \in \mathbb{T}} |G_A(x, 0) - I| \\
&\leq C A^{-1} \|G_A(x, 0) - I\| \leq C A^{-1} \|G_A(x, 0)\| + C A^{-1} \\
&\leq C A^{-1} \alpha \|a_0\| e_{-\gamma}(x, 0) + C A^{-1} \\
&\leq C A^{-1} \alpha \|a_0\| e^{-\gamma x} + C A^{-1}.
\end{aligned} \tag{3.7}$$

It follows $a(\cdot)$ is bounded.

Sufficiency: Suppose to the contrary that system (3.4) is not exponentially stable. Then for $\nu \in \{1, 2, \dots, k\}$ there exist an eigenvalue λ_ν of A such that $\lambda_\nu \notin \mathbb{E}_{\mathbb{C}}(\mathbb{T})$. From [22, Remark 1], Thus $\operatorname{Re}(\lambda_\nu) \geq 0$ which implies that $|1 + u\lambda_\nu| \geq 1$ for all non-negative $u \in \mathbb{R}$. Letting $f(x) = C$ and $b = 0 + 0 + \dots + 0 + b_\nu + 0 + \dots + 0$, then by spectral decomposition theorem implies that

$$G_A(s, \sigma(x))b = e_{\lambda_\nu}(x, \sigma(s))p_\nu(s),$$

where $p_\nu(s)$ is a polynomial of degree less than or equal to $m_\nu - 1$. In this solution of (3.5) can be written as

$$\begin{aligned}
a(x) &= \int_0^x G_A(x, \sigma(s)) f(s) b \Delta s \\
&= \int_0^x e_{\lambda_\nu}(x, \sigma(s)) p_\nu(s) C \Delta s \\
&= C \int_0^x \frac{e_{\lambda_\nu}(x, s)}{1 + \mu \lambda_\nu} p_\nu(s) \Delta s
\end{aligned}$$

Since $\operatorname{Re}(\lambda_\nu) \geq 0$, so $a(x) \rightarrow \infty$ as $x \rightarrow \infty$ that is unique solution of (3.5) is unbounded, which is a contradiction. Hence the system (3.4) is exponentially stable. $\square$

Example 3.8. Let us consider the IVP given in Example 2.1 with $A = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}$,

$(I + B_k) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix}$, $\lambda_2 > \lambda_1 > 0$ and $x_k := k$. Moreover, we choose $(a_{2k})_{k \geq 1}$ and $(b_{2k})_{k \geq 1}$ such that $a_{ij} := a_{2(i+1)} a_{2(i+2)} \dots a_{2(i+j)} \leq a$ and $b_{ij} := b_{2(i+1)} b_{2(i+2)} \dots b_{2(i+j)} \leq b$ for each fixed $i \geq 0$ and for $j = 1, 2, \dots$.

It is easy to see that $AB_k = B_k A$ and $\mathcal{L}_j$ is $(I + B_k)$ -invariant subspace of $\mathbb{C}^m$ for each $k = 1, 2, 3, \dots$, and for $0 < \theta < 1$, $x_{k+1} - x_k < \theta$, $k = 1, 2, \dots$. It implies that for $b \in \mathbb{C}^m$ and each bounded function f the unique solution of the Cauchy problem

$$\begin{cases} a^\Delta = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} a + f(s)b, & x \in \mathbb{T}_+, \ x \neq x_{2k} \\ a(x_{2k}^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(x_{2k}), & k = 1, 2, \dots, \\ a(0) = 0, \ \tau \geq 0. \end{cases}$$

$a(x) = \int_0^x G_A(x, \sigma(s)) f(s) b \Delta s$ is bounded, where

$$G_A(x, s) = \begin{cases} \begin{pmatrix} e_{-\lambda_1}(x, s) & 0 \\ 0 & e_{-\lambda_2}(x, s) \end{pmatrix} \\ \text{if } x_{k-1} \leq s \leq x \leq x_k; \\ \begin{pmatrix} e_{-\lambda_1}(x, x_k^+) & e_{-\lambda_1}(x_k, s) a_{2k} & 0 \\ 0 & e_{-\lambda_2}(x, x_k^+) & e_{-\lambda_2}(x_k, s) b_{2k} \end{pmatrix} \\ \text{if } x_{k-1} \leq s < x_k < x \leq x_{k+1}; \\ \begin{pmatrix} e_{-\lambda_1}(x, x_k^+) a_{2i} e_{-\lambda_1}(x_i, s) & 0 \\ \prod_{s < x_j \leq x} (e_{-\lambda_1}(x_j, x_{j-1}^+) a_{2j}) & 0 \\ 0 & e_{-\lambda_2}(x, x_k^+) a_{2i} e_{-\lambda_2}(x_i, s) \\ & \prod_{s < x_j \leq x} (e_{-\lambda_2}(x_j, x_{j-1}^+) a_{2j}) \end{pmatrix} \\ \text{if } x_{i-1} \leq s < x_i < \dots < x_k < x \leq x_{k+1}. \end{cases}$$

From Theorem 3.7, it follows that the system

$$\begin{cases} a^\Delta = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} a, \quad x \neq x_{2k} \\ a(x_{2k}^+) = \begin{pmatrix} a_{2k} & 0 \\ 0 & b_{2k} \end{pmatrix} a(x_{2k}), \quad k = 1, 2, \dots, \\ a(0) = 0 \end{cases}$$

is exponentially stable

Theorem 3.9. Suppose that $B_k \in M_n(\mathbb{R})$, $k = 1, 2, \dots$, $A \in \mathcal{R}(\mathbb{T}_+, M_n(\mathbb{R}))$ $AB_k = B_k A$ and $\mathcal{L}_j$ is $(I + B_k)$ -invariant subspace of $\mathbb{C}^m$ for each $k = 1, 2, 3, \dots$, and there exist a positive constant θ such that $x_{k+1} - x_k < \theta$, $k = 1, 2, \dots$. Then the system (3.4) is dichotomic if and only if there exists a projection Q having the property

$$G_A(x, 0)Q = QG_A(x, 0),$$

for all $x \geq 0$ such that for each non-zero vector $b \in \mathbb{C}^m$ and each bounded function f such that $G_A(s, 0)f(s) = f(s)G_A(s, 0)$, the solutions of the following Cauchy problems

$$\begin{cases} a^\Delta = Aa + f(s)Qb \\ a(x_k^+) = a(x_k^-) + B_k a(x_k) \\ a(0) = 0. \end{cases} \quad (3.8)$$

and

$$\begin{cases} a^\Delta = -Aa^\sigma + f(s)(I - Q)b, \\ a(x_k^+) = a(x_k^-) + B_k a(x_k) \\ a(0) = 0. \end{cases} \quad (3.9)$$

are bounded.

Proof. Necessity: Suppose that A is dichotomic. By [22, Remark 2] $\mathbb{C}^m = W_s(A) \oplus W_\mu(A)$. Let us define $Q : \mathbb{C}^m \rightarrow \mathbb{C}^m$ by $Q(w) := w_s$, where $w = w_s + w_\mu$, $w_s \in W_s$ and $w_\mu \in W_\mu$. Clearly Q is a projection. Moreover for all $w \in \mathbb{C}^m$ and all $s \geq 0$,

we have

$$\begin{aligned}
QG_A(s, 0)w &= QG_A(s, 0)(w_s + w_\mu) \\
&= Q(G_A(s, 0)w_s + G_A(s, 0)w_\mu) \\
&= G_A(s, 0)w_s \\
&= G_A(s, 0)Qw,
\end{aligned}$$

where the fact that $W_s(A)$ is an $G_A(s, 0)$ -invariant subspace, thus

$$QG_A(s, 0)w = G_A(s, 0)Qw.$$

Now the solution of (3.8) is given by

$$a(x) = \int_0^x G_A(x, \sigma(s))f(s)Qb\Delta s. \quad (3.10)$$

From equation (3.10), we consider $G_A(x, \sigma(s))f(s)Qb$, we can write

$$\begin{aligned}
G_A(x, \sigma(s))f(s)Qb &= f(s)G_A(x, \sigma(s))Qb \\
&= f(s)Q(e_{\lambda_1}(x, \sigma(s))p_1(s) + e_{\lambda_2}(x, \sigma(s))p_2(s) \\
&\quad + \cdots + e_{\lambda_\nu}(x, \sigma(s))p_\nu(s) + \cdots + e_{\lambda_k}(x, \sigma(s))p_k(s) \\
&= f(s)Q(e_{\lambda_1}(x, \sigma(s))p_1(s) + e_{\lambda_2}(x, \sigma(s))p_2(s) \\
&\quad + \cdots + e_{\lambda_\nu}(x, \sigma(s))p_\nu(s),
\end{aligned}$$

where $\lambda_i \in W_s(A)$ for $i \in \{1, 2, \dots, \nu\}$ and $\lambda_j \in W_\mu(A)$ for $j \in \{1, 2, \dots, k\}$. It is clear that the solution of (3.8) is bounded.

Sufficiency: Suppose to the contrary that A is not dichotomic. Then $W_0(A) \neq 0$ and so there exists an eigenvalue λ_l of A such that $\lambda_l \in \mathbb{E}_\mathbb{C}^0(\mathbb{T})$. Since $b \in \mathbb{C}^m$, so letting $b = 0 + 0 + 0 + \cdots + b_l + 0 + 0$, then either $b_l \in W_s(A)$ or $b_l \in W_\mu(A)$. If $l \leq \nu$, then $b_l \in W_s(A)$ and if $l > \nu$, then $b_l \in W_\mu(A)$.

Case 1. Assume $b_l \in W_s(A)$. Then by Lemma 2.2, we have

$$\begin{aligned}
G_A(x, \sigma(s))f(s)Qb &= G_A(x, \sigma(s))f(s)Qb_l \\
&= f(s)e_{\lambda_l}(x, \sigma(s))p_l(s),
\end{aligned}$$

where $p_l(\cdot)$ is a polynomial of degree less than or equal to $m_l - 1$. So, the solution of (3.8) becomes

$$a(x) = \int_0^x G_A(x, \sigma(s))f(s)Qb\Delta s, \quad (3.11)$$

where $p_l(\cdot)$ is at most $m_l - 1$ degree polynomial. Let $f(s) = C$, then equation (3.11) becomes

$$a(x) = \int_0^x C e_{\lambda_l}(x, \sigma(s))p_l(s)\Delta s$$

or

$$a(x) = C \int_0^x e_{\lambda_l}(x, \sigma(s))p_l(s)\Delta s.$$

Since $Re(\lambda_\nu) \geq 0$, so $a(x) \rightarrow \infty$ as $x \rightarrow \infty$ that is unique solution of (3.8) is unbounded, which is contradiction. Hence the system A is dichotomic.

Case2: Suppose $b_l \in W_\mu(A)$ So the solution of (3.9) is given by

$$w(x) = \int_0^x G_{\ominus A}(x, s)f(s)(I - Q)b\Delta s.$$

By Lemma 2.2, we have

$$\begin{aligned} & G_{\ominus A}(x, \sigma(s))f(s)(I - Q)b \\ &= G_{\ominus A}(x, \sigma(s))f(s)(I - Q)b_l \\ &= f(s)e_{\ominus A}(x, \sigma(s))(I - Q)b_l \\ &= f(s)e_{\ominus \lambda_l}(x, \sigma(s))p_l(s), \end{aligned}$$

where $p_l(s)$ is a polynomial of degree less than or equal to $m_l - 1$. So, the solution of (3.9) becomes

$$w(x) = \int_0^x f(s)e_{\ominus \lambda_l}(x, \sigma(s))p_l(s)\Delta s, \quad (3.12)$$

Let $f(s) = C$, then equation (3.12) becomes

$$w(x) = \int_0^x Ce_{\ominus \lambda_l}(x, \sigma(s))p_l(s)\Delta s$$

or

$$w(x) = C \int_0^x e_{\ominus \lambda_l}(x, \sigma(s))p_l(s)\Delta s.$$

Clearly $e_{\ominus \lambda_l}(x, \sigma(s)) \rightarrow 1$ as $x \rightarrow \infty$ so, $w(x) \rightarrow \infty$ as $x \rightarrow \infty$ that is unique solution of (3.9) is unbounded, which is a contradiction. Hence in this case we also arrived at a contradiction the system (3.4) is dichotomic. The proof is complete. $\square$

4. CONCLUSION

We have presented some new criteria for admitting the exponential dichotomy of impulsive dynamic systems. If the assumption of $(I + B_k)$ -invariant subspace of $\mathbb{C}^m$ is dropped this can be an open problem to formulate Spectral Decomposition Theorem for impulsive transition matrix $G_A(x, s)$, $0 \leq s \leq x$, associated with $\{B_k, x_k\}_{k=1}^\infty$.

This topic warrants further investigation. In particular, it should be established connections between the Hyers-Ulam stability of the linear impulsive dynamic systems and its dichotomy. Also, it would be interesting to know the conditions for Hyers-Ulam stability.

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