ON INVARIANT ARITHMETIC STATISTICALLY CONVERGENCE VIA WEIGHTED DENSITY

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ABSTRACT. In this paper, our concern is to introduce the concepts of invariant arithmetic convergence, invariant arithmetic statistically convergence and lacunary invariant arithmetic statistically convergence using weighted density via Orlicz function $\tilde{\phi}$. Finally, we give some relations between lacunary invariant arithmetic statistical $\tilde{\phi}$ -convergence and invariant arithmetic statistical $\tilde{\phi}$ -convergence via weighted density.

1. INTRODUCTION

The idea of statistical convergence was first introduced by Fast [12]. It became a notable topic in summability theory after the work of Fridy [14] and Šalát [34]. Statistical convergence has several applications in different fields of mathematics like number theory, trigonometric series, summability theory, statistics and probability theory, measure theory, optimization, approximation theory and rough set theory. Since then several generalizations and applications of this concept have been investigated by various authors, namely Fridy and Orhan [15], Gürdal and Huban [17], Nabiev et al. [29], Schaefer [40], and many others (see [6, 8, 10, 11, 16, 17, 18, 19, 22, 36, 38]). Also, the readers should refer to the monographs [3, 4] and [28] for the background on the sequence spaces and related topics. The idea of arithmetic convergence was introduced by Ruckle [33]. The studies on arithmetic convergence and related results can be found in [23, 24, 25, 41, 42].

In this study, the concepts of invariant arithmetic weight g -statistically $\tilde{\phi}$ -convergence, lacunary invariant arithmetic $\tilde{\phi}$ -convergence of weight g , invariant arithmetic weight g -statistically $\tilde{\phi}$ -convergence and lacunary invariant arithmetic weight g -statistically $\tilde{\phi}$ -convergence have been investigated. We also establish some relations between lacunary invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergence and invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergence.

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2. DEFINITIONS AND NOTATIONS

Statistical convergence depends on the natural density of subsets of the set $\mathbb{N}$ of positive integers. The natural density $\delta(A)$ of a subset A of $\mathbb{N}$ is defined by

$$\delta(A) = \lim_{t \rightarrow \infty} t^{-1} |\{m \leq t : m \in A\}|$$

where the vertical bars indicate the number of elements in the enclosed set. A sequence (x_m) is said to be statistically convergent to ℓ if for each $\varepsilon > 0$,

$$\lim_{t \rightarrow \infty} t^{-1} |\{m \leq t : |x_m - \ell| \geq \varepsilon\}| = 0.$$

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [31], that is, a function $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\tilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \tilde{\phi}(x) \rightarrow \infty \text{ as } x \rightarrow \infty.$$

Further, an Orlicz function $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ is said to satisfy the Δ_2 condition, if there exists an positive real number M such that $\tilde{\phi}(2x) \leq M \cdot \tilde{\phi}(x)$ for every $x \in \mathbb{R}^+$ (also see [19, 37]). In [31], Rao and Ren describes some important applications of Orlicz functions in many areas such as economics, stochastic problems etc. The reader can also refer to the paper [9] and recent monograph [4] related with various ways to generalize Orlicz sequence spaces systematically and investigate several structural properties of such spaces.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r . The space of lacunary convergent sequence N_θ was introduced by Freedman [13]. In [15] the relation between lacunary statistical convergence and statistical convergence was established among other things. And the notion of lacunary convergence has been investigated by many authors [7, 21, 26].

Several authors have studied invariant convergent sequences (see, [20, 30, 32]).

Let σ be a mapping of the positive integers into themselves. A continuous linear functional φ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ -mean if it satisfies following conditions:

- (i) $\varphi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n ,
- (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
- (iii) $\varphi(x_{\sigma(n)}) = \varphi(x_n)$ for all $x \in \ell_\infty$.

The mapping σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m , where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, φ extends the limit functional on c , the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ -mean is often called a Banach limit. The space V_σ , the set of bounded sequences whose invariant means are equal, is the set of almost convergent sequences.

A set E of positive integers said to have uniform invariant density of zero if and only if the number of elements of E which lie in the set $\{\sigma(m), \sigma^2(m), \dots, \sigma^n(m)\}$ is $o(n)$ as $n \rightarrow \infty$, uniformly in m .

The idea of arithmetic convergence was firstly originated by Ruckle [33]. A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$ there is

an integer n such that for every integer m we have $|x_m - x_{\langle m, n \rangle}| < \varepsilon$, where the symbol $\langle m, n \rangle$ denotes the greatest common divisor of two integers m and n . We denote the sequence space of all arithmetic convergent sequence by AC .

The arithmetic statistically convergence and lacunary arithmetic statistically convergence was examined by Yaying and Hazarika [42].

Definition 2.1. A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences.

Thus, for $\varepsilon > 0$ and integer n

$$ASC = \left\{ x = (x_m) : \lim_{t \rightarrow \infty} \frac{1}{t} |\{m \leq t : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0 \right\}.$$

We shall write $ASC - \lim x_m = x_{\langle m, n \rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m, n \rangle}$.

Definition 2.2. A sequence $x = (x_m)$ is said to be lacunary arithmetic statistically convergent if for $\varepsilon > 0$ there is an integer n such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{m \in I_r : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0.$$

We shall write

$$ASC_\theta = \left\{ x = (x_m) : \lim_{r \rightarrow \infty} \frac{1}{h_r} |\{m \in I_r : |x_m - x_{\langle m, n \rangle}| \geq \varepsilon\}| = 0 \right\}.$$

We will use $ASC_\theta - \lim x_m = x_{\langle m, n \rangle}$ to denote the sequence (x_m) is lacunary arithmetic statistically convergent to $x_{\langle m, n \rangle}$.

In the recent times in [2], using the density of weight g where $g : \mathbb{N} \rightarrow [0, \infty)$ is a function with property that $\lim_{t \rightarrow \infty} g(t) = \infty$ and $\frac{t}{g(t)} \rightarrow 0$ as $t \rightarrow \infty$, the concept of natural density was extended as follows: The upper density of weight g was defined by

$$\bar{\delta}_g(A) = \limsup_{t \rightarrow \infty} \frac{A(1, t)}{g(t)}$$

for $A \subset \mathbb{N}$, where $A(1, t)$ denotes the number of elements in $A \cap [1, t]$. The lower density of weight g is defined in a similar manner. Then, the family

$$\mathcal{I}_g = \{A \subset \mathbb{N} : \bar{\delta}_g(A) = 0\}$$

creates an ideal. It was seen in [2] that $\mathbb{N} \in \mathcal{I}_g$ iff $\frac{t}{g(t)} \rightarrow 0$ as $t \rightarrow \infty$. Furthermore, we suppose that $t/g(t) \rightarrow 0$ as $t \rightarrow \infty$ so that $\mathbb{N} \notin \mathcal{I}_g$ and $\mathcal{I}_g$ is a proper admissible ideal of $\mathbb{N}$. We denote by G the collection of such weight functions g satisfying the above properties (see [1, 5, 27, 35, 39]).

If we take the function $g(t) = t$ we obtain the usual statistical convergence. It is clear that every convergent sequence is also weight g -statistical convergent. But the converse is not true in general.

Example 2.3. Let us define the function $g(t) = 3t$ and the sequence as

$$x_m = \begin{cases} 1, & m = t^2, t \in \mathbb{N} \\ 0, & m \neq t^2. \end{cases}$$

Then $\frac{1}{3t} |\{m \leq t : x_m \neq 0\}| \leq \frac{\sqrt{t}}{3t}$. Hence, $g - st - \lim x_m = 0$.

As a natural consequence we can introduce the following definition.

Definition 2.4. Let $x = (x_m)$ be a real valued sequence. x is arithmetic weighted g -statistical $\tilde{\phi}$ -convergent if for any given $\varepsilon > 0$ and integer n

$$\lim_{t \rightarrow \infty} \frac{\left| \left\{ m \leq t : \tilde{\phi}(x_m - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right|}{g(t)} = 0.$$

In this case we write $st \left(\tilde{\phi} \right)^g - \lim x_m = x_{\langle m, n \rangle}$.

Remark. If we take the functions $\tilde{\phi}(x) = |x|$ and $g(t) = t$, then convergent concepts in above definition coincide with arithmetic statistically convergent in [42].

3. MAIN RESULTS

Following the above definitions and results, we aim in this section to introduce some new notions of invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergent. In addition to these definition, natural inclusion theorems shall also be presented.

Definition 3.1. A sequence $x = (x_m)$ is said to be invariant arithmetic weight g -convergent if for an integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \sum_{m=1}^t x_{\sigma^m(u)} = x_{\langle m, n \rangle},$$

uniformly in $u = 1, 2, \dots$

In this case we write $x_m \rightarrow x_{\langle m, n \rangle} (AV_\sigma)^g$ and the set of all invariant arithmetic weight g -convergent sequences will be demonstrated by $(AV_\sigma)^g$.

Definition 3.2. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be strongly invariant arithmetic $\tilde{\phi}$ -convergent of weight g if for an integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \sum_{m=1}^t \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) = 0,$$

uniformly in $u = 1, 2, \dots$

In this case we write $x_m \rightarrow x_{\langle m, n \rangle} [AV_\sigma] \left(\tilde{\phi} \right)^g$ to denote the sequence (x_m) is strongly invariant arithmetic $\tilde{\phi}$ -convergent of weight g to $x_{\langle m, n \rangle}$ and the set of all invariant arithmetic $\tilde{\phi}$ -convergent of weight g sequences will be demonstrated by $[AV_\sigma] \left(\tilde{\phi} \right)^g$.

Definition 3.3. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be invariant arithmetic weight g -statistically $\tilde{\phi}$ -convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| = 0,$$

uniformly in $u = 1, 2, \dots$

We shall use $AS_\sigma C(\tilde{\phi})^g$ to denote the set of all invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergent sequences. Thus, we define

$$AS_\sigma C(\tilde{\phi})^g = \left\{ x = (x_m) : \text{for some } x_{\langle m, n \rangle}, AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle} \right\}.$$

In this case we write $AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle} (AS_\sigma C)$.

For the next result we assume the convex Orlicz function which satisfies Δ_2 -condition.

Theorem 3.4. *Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function and $x = (x_m)$ and $y = (y_m)$ be two sequences.*

(i) *If $AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ and α is any real constant, then $AS_\sigma C(\tilde{\phi})^g - \lim (\alpha x_m) = \alpha x_{\langle m, n \rangle}$.*

(ii) *If $AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ and $AS_\sigma C(\tilde{\phi})^g - \lim y_m = y_{\langle m, n \rangle}$, then $AS_\sigma C(\tilde{\phi})^g - \lim (x_m + y_m) = (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})$.*

Proof. Since $\tilde{\phi}$ satisfies the Δ_2 -condition, then there exists $M > 0$ such that $\tilde{\phi}(2x) \leq M\tilde{\phi}(x)$, for every $x \in \mathbb{R}$.

(i) The result is obvious when $\alpha = 0$. Let $q \in \mathbb{N}$ such that $|\alpha| \leq 2^q$ and $AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$, then for integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) < \varepsilon \right\} \right| = 1.$$

Let us consider such m for which $\tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) < \frac{\varepsilon}{2^q}$, then

$$\begin{aligned} \tilde{\phi}(\alpha x_{\sigma^m(u)} - \alpha x_{\langle m, n \rangle}) &= \tilde{\phi}(|\alpha| (x_{\sigma^m(u)} - x_{\langle m, n \rangle})) \leq \tilde{\phi}(2^q (x_{\sigma^m(u)} - x_{\langle m, n \rangle})) \\ &\leq 2^q \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \leq \varepsilon. \end{aligned}$$

Thus for integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(\alpha x_{\sigma^m(u)} - \alpha x_{\langle m, n \rangle}) < \varepsilon \right\} \right| = 1,$$

i.e.

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(\alpha x_{\sigma^m(u)} - \alpha x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| = 0.$$

Hence $AS_\sigma C(\tilde{\phi})^g - \lim (\alpha x_m) = \alpha x_{\langle m, n \rangle}$.

(ii) Let $AS_\sigma C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ and $AS_\sigma C(\tilde{\phi})^g - \lim y_m = y_{\langle m, n \rangle}$, i.e.

$$\begin{aligned} &\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ &= 0 = \lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(y_{\sigma^m(u)} - y_{\langle m, n \rangle}) \geq \varepsilon \right\} \right|, \end{aligned}$$

or

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) < \varepsilon \right\} \right| \\ &= 1 = \lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(y_{\sigma^m(u)} - y_{\langle m, n \rangle}) < \varepsilon \right\} \right|, \end{aligned}$$

for integer n . Let us consider such m for which both of $\tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) < \frac{\varepsilon}{2M}$ and $\tilde{\phi}(y_{\sigma^m(u)} - y_{\langle m, n \rangle}) < \frac{\varepsilon}{2M}$ are true. Hence we have

$$\begin{aligned} & \tilde{\phi}((x_{\sigma^m(u)} + y_{\sigma^m(u)}) - (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})) \\ &= \tilde{\phi}((x_{\sigma^m(u)} - x_{\langle m, n \rangle}) + (y_{\sigma^m(u)} - y_{\langle m, n \rangle})) \\ &\leq \tilde{\phi}(2(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) + 2(y_{\sigma^m(u)} - y_{\langle m, n \rangle})) \\ &\leq M \cdot (\tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) + \tilde{\phi}(y_{\sigma^m(u)} - y_{\langle m, n \rangle})) \leq \varepsilon. \end{aligned}$$

Thus for integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}((x_{\sigma^m(u)} + y_{\sigma^m(u)}) - (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})) < \varepsilon \right\} \right| = 1,$$

i.e.

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}((x_{\sigma^m(u)} + y_{\sigma^m(u)}) - (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})) \geq \varepsilon \right\} \right| = 0.$$

Hence $AS_{\sigma}C(\tilde{\phi})^g - \lim(x_m + y_m) = (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})$. $\square$

Definition 3.5. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is called to be lacunary invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{r \rightarrow \infty} \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| = 0,$$

uniformly in $u = 1, 2, \dots$

We shall use $AS_{\sigma\theta}C(\tilde{\phi})^g$ to indicate the set of all lacunary invariant arithmetic weight g -statistical $\tilde{\phi}$ -convergent sequences. Thus, we define

$$AS_{\sigma\theta}C(\tilde{\phi})^g = \left\{ x = (x_m) : \text{for some } x_{\langle m, n \rangle}, AS_{\sigma\theta}C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle} \right\}.$$

In this case we write $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ or $x_m \rightarrow x_{\langle m, n \rangle} (AS_{\sigma\theta}C(\tilde{\phi})^g)$.

By using the same techniques as in Theorem 3.4, we can prove the following theorem.

Theorem 3.6. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function and $x = (x_m)$ and $y = (y_m)$ be two sequences.

(i) If $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ and α is any real constant, then $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim(\alpha x_m) = \alpha x_{\langle m, n \rangle}$.

(ii) If $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim x_m = x_{\langle m, n \rangle}$ and $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim y_m = y_{\langle m, n \rangle}$, then $AS_{\sigma\theta}C(\tilde{\phi})^g - \lim(x_m + y_m) = (x_{\langle m, n \rangle} + y_{\langle m, n \rangle})$.

Definition 3.7. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be strongly lacunary invariant arithmetic $\tilde{\phi}$ -convergent of weight g if for an integer n

$$\lim_{r \rightarrow \infty} \frac{1}{g(h_r)} \sum_{m \in I_r} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) = 0,$$

uniformly in $u = 1, 2, \dots$

In this case we write $x_m \rightarrow x_{\langle m, n \rangle} (AL_{\sigma\theta}) (\tilde{\phi})^g$ to denote the sequence (x_m) is strongly lacunary invariant arithmetic $\tilde{\phi}$ -convergent of weight g to $x_{\langle m, n \rangle}$ and the set of all strongly lacunary invariant arithmetic $\tilde{\phi}$ -convergent of weight g sequences will be denoted $(AL_{\sigma\theta}) (\tilde{\phi})^g$.

Definition 3.8. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $\{x_m\}$ is said to be $\tilde{\phi}$ -bounded if there exists $M > 0$ such that $\tilde{\phi}(x_m) \leq M$ for $m \in \mathbb{N}$. We denote the space of all bounded sequences by $\ell_\infty^{\tilde{\phi}}$.

Now, we give some inclusion relations between $AL_{\sigma\theta} (\tilde{\phi})^g$ -convergence and $AS_{\sigma\theta}C (\tilde{\phi})^g$ -convergence and demonstrate that these are equivalent for bounded sequences. We also study relation between $AS_{\sigma\theta}C (\tilde{\phi})^g$ -convergence and $AS_\sigma C (\tilde{\phi})^g$ -convergence.

Theorem 3.9. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function and $\theta = \{k_r\}$ be a lacunary sequence. Then, the following statements hold:

- (i) $x_m \rightarrow x_{\langle m, n \rangle} (AL_{\sigma\theta}) (\tilde{\phi})^g$ implies $x_m \rightarrow x_{\langle m, n \rangle} (AS_{\sigma\theta}C) (\tilde{\phi})^g$,
- (ii) $x_m \in \ell_\infty^{\tilde{\phi}}$ and $x_m \rightarrow x_{\langle m, n \rangle} (AS_{\sigma\theta}C) (\tilde{\phi})^g$ imply $x_m \rightarrow x_{\langle m, n \rangle} (AL_{\sigma\theta}) (\tilde{\phi})^g$,
- (iii) $(AS_{\sigma\theta}C) (\tilde{\phi})^g \cap \ell_\infty = (AL_{\sigma\theta}) (\tilde{\phi})^g$.

Proof. (i) Let $\varepsilon > 0$ and $x_m \rightarrow x_{\langle m, n \rangle} (AL_{\sigma\theta}) (\tilde{\phi})^g$. Then, we can write

$$\begin{aligned} \sum_{m \in I_r} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) &\geq \sum_{\substack{m \in I_r \\ \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon}} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \\ &\geq \varepsilon \cdot \left| \left\{ m \in I_r : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \end{aligned}$$

and so

$$\frac{1}{\varepsilon g(h_r)} \sum_{t \in I_r} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \frac{1}{g(h_r)} \left| \left\{ t \in I_r : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right|.$$

which gives the result.

(ii) Suppose that $x_m \rightarrow x_{\langle m, n \rangle} (AS_{\sigma\theta}C) (\tilde{\phi})^g$ and $x_m \in \ell_\infty^{\tilde{\phi}}$. If $x_m \in \ell_\infty^{\tilde{\phi}}$, then, there exists a positive integer M such that

$$\tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \leq M$$

for all m and u . Given $\varepsilon > 0$, we get

$$\begin{aligned} \frac{1}{g(h_r)} \sum_{m \in I_r} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) &= \frac{1}{g(h_r)} \sum_{\substack{m \in I_r \\ \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \frac{\varepsilon}{2}}} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \\ &\quad + \frac{1}{g(h_r)} \sum_{\substack{t \in I_r \\ \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) < \frac{\varepsilon}{2}}} \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \\ &\leq \frac{M}{g(h_r)} \left| \left\{ m \in I_r : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \frac{\varepsilon}{2} \right\} \right| + \frac{\varepsilon}{2}. \end{aligned}$$

from which the result follows.

Let $\theta = \{k_r\}$ be given and define x_m to be $1, 2, \dots, [\sqrt{h_r}]$ for $m = \sigma^t(u)$, $t = k_{r-1} + 1, k_{r-1} + 2, \dots, k_{r-1} + [\sqrt{h_r}]$; $u \geq 1$, and $x_m = 0$ otherwise (where $[\cdot]$ denotes the greatest integer function). Note that x is not bounded. Further, for $0 < \varepsilon < 1$ we get

$$\frac{1}{g(h_r)} \left| \left\{ m \in I_r : \tilde{\phi}(x_{\sigma^m(u)} - 0) \geq \varepsilon \right\} \right| = \frac{[\sqrt{h_r}]}{g(h_r)} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

i.e. $x_m \rightarrow 0 (AS_{\sigma\theta}C) \left(\tilde{\phi} \right)^g$. But

$$\frac{1}{g(h_r)} \sum_{m \in I_r} \tilde{\phi}(x_{\sigma^m(u)} - 0) = \frac{1}{g(h_r)} \left(\frac{[\sqrt{h_r}] ([\sqrt{h_r}] + 1)}{2} \right) \rightarrow \frac{1}{2} \neq 0 \text{ as } r \rightarrow \infty,$$

hence, $x_m \not\rightarrow 0 (AL_{\sigma\theta}) \left(\tilde{\phi} \right)^g$. Thus, inclusion (i) is proper and this example denotes that the boundedness condition can not be omitted from the hypothesis (ii).

(iii) This is an immediate consequence of (i) and (ii). $\square$

We now give a lemma which will be used in the proof of Theorem 3.11.

Lemma 3.10. *Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. Assume for given $\varepsilon_1 > 0$ and every $\varepsilon > 0$, there exists t_0 and u_0 and there is an integer n such that*

$$\frac{1}{g(t)} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| < \varepsilon_1$$

for all $t \geq t_0$ and $u \geq u_0$, then, $x = (x_m) \in AS_{\sigma}C \left(\tilde{\phi} \right)^g$.

Proof. Let $\varepsilon_1 > 0$ be given. For each $\varepsilon > 0$, select t'_0, u_0 such that

$$\frac{1}{g(t)} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| < \frac{\varepsilon_1}{2} \quad (1)$$

for all $t \geq t'_0$ and $u \geq u_0$. It is enough to prove that there exists t''_0 such that $t \geq t''_0$ and $0 \leq u \leq u_0$,

$$\frac{1}{t} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| < \varepsilon_1. \quad (2)$$

Since taking $t_0 = \max \{t'_0, t''_0\}$, (2) will hold for $t \geq t_0$ and for all u , which gives the result.

Once u_0 has been choosen, $0 \leq u \leq u_0$, u_0 is fixed. So put

$$S = \left| \left\{ 0 \leq m \leq u_0 - 1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right|.$$

Now taking $0 \leq u \leq u_0$ and $t \geq u_0$, (1) we have

$$\begin{aligned} & \frac{1}{g(t)} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{g(t)} \left| \left\{ 0 \leq m \leq u_0-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & + \frac{1}{g(t)} \left| \left\{ u_0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{S}{g(t)} + \frac{1}{g(t)} \left| \left\{ u_0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{S}{g(t)} + \frac{\varepsilon_1}{2}, \end{aligned}$$

and taking t , sufficiently large, we can write $\frac{S}{g(t)} + \frac{\varepsilon_1}{2} < \varepsilon_1$, which gives (2), and therefore, the result follows. $\square$

Theorem 3.11. *Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. $AS_{\sigma\theta}C(\tilde{\phi})^g = AS_{\sigma}C(\tilde{\phi})^g$ for every lacunary sequence θ .*

Proof. Let $x \in AS_{\sigma\theta}C(\tilde{\phi})^g$. Then, from Definition 3.5, given $\varepsilon_1 > 0$, there exist r_0 and $x_{\langle m, n \rangle}$ such that

$$\frac{1}{g(h_r)} \left| \left\{ 0 \leq m \leq h_r-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| < \varepsilon_1$$

for $r \geq r_0$ and $u = k_{r-1} + 1 + u$, $u \geq 0$.

Let $g(t) \geq g(h_r)$, write $g(t) = jg(h_r) + q$, where $0 \leq q < g(h_r)$, j is an integer. Since $g(t) \geq g(h_r)$, $j \geq 1$. Hence we have

$$\begin{aligned} & \frac{1}{g(t)} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{g(t)} \left| \left\{ 0 \leq m \leq g(t)-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{g(t)} \left| \left\{ 0 \leq m \leq (j+1)g(h_r)-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{g(t)} \sum_{k=0}^j \left| \left\{ kg(h_r) \leq m \leq (k+1)g(h_r)-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \\ & \leq \frac{g(h_r)}{g(t)} (j+1) \varepsilon_1 \leq 2 \frac{g(h_r)}{g(t)} j \varepsilon_1, \quad (j \geq 1) \end{aligned}$$

and since $\frac{jg(h_r)}{g(t)} \leq 1$,

$$\frac{1}{g(t)} \left| \left\{ 0 \leq m \leq t-1 : \tilde{\phi}(x_{\sigma^m(u)} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| \leq 2\varepsilon_1.$$

Then, by the Lemma 1, $AS_{\sigma\theta}C(\tilde{\phi})^g \subset AS_{\sigma}C(\tilde{\phi})^g$. It is easy to see that $AS_{\sigma}C(\tilde{\phi})^g \subset AS_{\sigma\theta}C(\tilde{\phi})^g$. $\square$

By using the same techniques as in Theorem 3.11, we can prove the following theorem.

Theorem 3.12. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. $AL_{\sigma\theta}(\tilde{\phi})^g \Leftrightarrow [AV_\sigma](\tilde{\phi})^g$ for every lacunary sequence θ .

If we take $\sigma(u) = u + 1$, the above Definitions 3.1 – 3.7 reduce the following definitions :

Definition 3.13. A sequence $x = (x_m)$ is said to be almost arithmetic S -convergent of weight g if for an integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \sum_{m=1}^t x_{m+u} = x_{\langle m, n \rangle},$$

uniformly in $u = 1, 2, \dots$

Definition 3.14. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be strongly almost arithmetic $S(\tilde{\phi})$ -convergent of weight g if for an integer n

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \sum_{m=1}^t \tilde{\phi}(x_{m+u} - x_{\langle m, n \rangle}) = 0,$$

uniformly in $u = 1, 2, \dots$

Definition 3.15. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be almost arithmetic weight g -statistically $S(\tilde{\phi})$ -convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{t \rightarrow \infty} \frac{1}{g(t)} \left| \left\{ m \leq t : \tilde{\phi}(x_{m+u} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| = 0,$$

uniformly in $u = 1, 2, \dots$

Definition 3.16. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is called to be lacunary almost arithmetic weight g -statistical $S(\tilde{\phi})$ -convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{r \rightarrow \infty} \frac{1}{g(h_r)} \left| \left\{ m \in I_r : \tilde{\phi}(x_{m+u} - x_{\langle m, n \rangle}) \geq \varepsilon \right\} \right| = 0,$$

uniformly in $u = 1, 2, \dots$

Definition 3.17. Let $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ be an Orlicz function. A sequence $x = (x_m)$ is said to be strongly lacunary almost arithmetic convergent $S(\tilde{\phi})$ -convergent of weight g if for an integer n

$$\lim_{r \rightarrow \infty} \frac{1}{g(h_r)} \sum_{m \in I_r} \tilde{\phi}(x_{m+u} - x_{\langle m, n \rangle}) = 0,$$

uniformly in $u = 1, 2, \dots$

In case $\sigma(u) = u + 1$, we have the following statement.

Remark. Similar inclusions to Theorems 3.9, 3.11 and 3.12 hold between lacunary almost arithmetic $\tilde{\phi}$ -convergent of weight g sequences, strongly lacunary almost arithmetic weight g -statistical $\tilde{\phi}$ -convergent sequences, almost arithmetic weight g -statistically $\tilde{\phi}$ -convergent sequences and strongly almost arithmetic $\tilde{\phi}$ -convergent of weight g sequences.

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