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TOPSIS with statistical distances: A new approach to MADM

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A B S T R A C T

Multiple attribute decision making (MADM) methods are very useful in choosing the best alternative among the available finite but conflicting alternatives. TOPSIS is one of the MADM methods, which is simple in its methodology and logic. In TOPSIS, Euclidean distances of each alternative from the positive and negative ideal solutions are utilized to find the best alternative. In literature, apart from Euclidean distances, the city block distances have also been tried to find the separations measures. In general, the attribute data are distributed with unequal ranges and also possess moderate to high correlations. Hence, in the present paper, use of statistical distances is proposed in place of Euclidean distances. Procedures to find the best alternatives are developed using statistical and weighted statistical distances respectively. The proposed methods are illustrated with some industrial problems taken from literature. Results show that the proposed methods can be used as new alternatives in MADM for choosing the best solutions.

1. Introduction

Multiple attribute decision making (MADM) methods involve the selection of best alternative from the available limited but conflicting alternatives. This is often achieved by finding the order of preference for all the alternatives and choosing the rank one alternative as the best for practical purposes. Hwang and Yoon (1981) cite the first MADM application by Churchman et al. (1957) which uses the simple additive weighting method to solve a decision making problem. The ever increasing interest in MADM methods can be witnessed from the oldest reviews and books like MacCrimmon (1968) and Barret (1970) to the recent research publications and books like Ullah et al. (2015), Jain and Raj (2015), Ahn (2015), Tzeng and Huang (2011) and Zardari et al.(2015).

Popular MADM methods include TOPSIS (Hwang & Yoon, 1981; Deng et al., 2000), AHP (Saaty 1980, 2000; Belton & Gear 1983; Lootsma 1999), ELECTRE (Roy 1989, 1991; Roy & Vincke 1981), VIKOR (Yu, 1973; Zeleny, 1982; Opricovic & Tzeng, 1998), PROMITHEE (Brans et al., 1984), with

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their different versions and modifications. MADM problems clubbed with vagueness in the attributes data or in the attribute weights are solved using fuzzy MADM methods (Bass & Kwakernaak, 1977; Chen & Hwang 1992; Figueira et al., 2004, etc.). Among the available MADM techniques, TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) developed by Hwang and Yoon (1981) is undoubtedly simple to understand by its straight forward approach and logic. This method relies on finding the best alternative by considering the Euclidian distances S^+ and S^- of each alternative from the positive and the negative ideal solutions, respectively. The best alternative should have minimum distance from the positive ideal solution and at the same time it must be far off from the negative ideal solution. Hence, the ratio $\frac{S^-}{S^- + S^+}$ which is a measure of relative closeness from the positive ideal solution is used as a criterion to find the best alternative.

In literature, the Euclidean as well as City block distances have been employed in calculating the separation measures (Dasarathy, 1976; Yoon, 1980). In the present work, use of statistical distances from the positive and the negative ideal solutions is proposed in TOPSIS methodology. The necessity of using statistical distance is briefed in the next section. The proposed methods employing the statistical and weighted statistical distances, named s-TOPSIS and ws-TOPSIS, respectively, are explained in Section 3. Illustration of the proposed methods with suitable examples is carried out in Section 4. Final conclusions on the proposed methodology are drawn in last Section.

2. Statistical distances

MADM methods use the data pertaining to limited alternatives which consist of multiple attributes. The data of each alternative with *n* attributes can be viewed as an n-dimensional point and the number of alternatives as different points in the n-dimensional space. In such a representation, it is well known that Euclidian distances are unsatisfactory while dealing with multiple alternatives. When the coordinates represent measurements that are subject to random fluctuations of different magnitudes, it is often desirable to weight coordinates subject to a great deal of variability less heavily than those that are not highly variable (Johnson and Wichern 2001). Also, the Euclidian distances fail to consider the correlations between the attributes when they are correlated. Both the aspects of unequal ranges in attribute data and the possible correlation between them can be simultaneously handled by statistical distances. The need to consider the statistical distances is explained by considering a few alternatives with two correlated attributes X_1 and X_2.

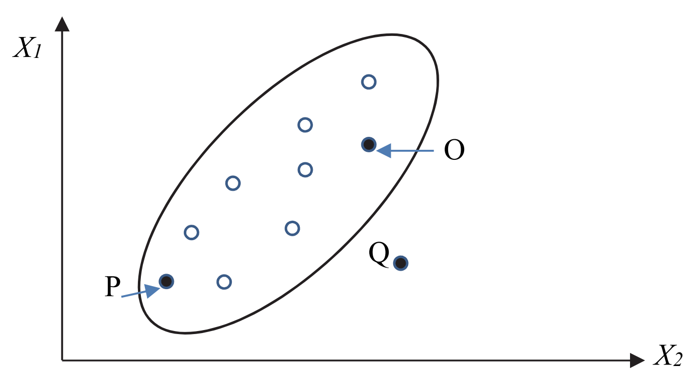

Fig. 1. Correlated attributes with unequal variablilities

Fig. 1 depicts the cluster of points in two dimensional space representing different alternatives with two attributes X_1 and X_2. The positive correlation between the two attributes and the unequal variability along the major and minor axes of the ellipse are evident from the Figure 1. In order to highlight the significance of the statistical distance, three points in the data set, namely the centre of the data set '*O*' and two points *P* and *Q*, one within the ellipse (*P*) and one outside the ellipse (*Q*) are considered. As apparent, the Euclidean distance of the point *P* from the centre '*O*' is large compared to the distance from point Q to centre O. In contrast, the statistical distance between O and P is less compared to the distance between O and Q. This is because of the fact that statistical distance considers the unequal variability of the points along the minor and major axes of the ellipse and also the correlation between the two attributes. Hence, the statistical distances are widely in used in different applications like multivariate quality control charts and discriminate analysis. The square of the statistical distance (sd) from the centre point O to any point X is given by:

$$\mathrm{sd}^2 = (X-\mu)^T S^{-1} (X-\mu), \tag{1}$$

where μ is the mean vector and S^{-1} is the inverse of the covariance matrix obtained from the attribute data of the alternatives. When used with two attributes or variables,

$X^T = [\, x_{i1} \quad x_{i2} \,]$ represents an i th alternative

$\mu^T = [\, \mu_1 \quad \mu_2 \,]$ represents the mean values of the two attributes, and

$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$ represents the covariance matrix where

s_{11} is the variance of the first attribute, s_{22} is the variance of the second attribute and $s_{12} = s_{21}$ is the covariance between the two attributes.

3. Proposed s-TOPSIS and ws-TOPSIS methods

In the present work, TOPSIS with statistical distances (s-TOPSIS) and weighted statistical distances (ws-TOPSIS) are proposed to find the order of preference of the given alternatives. Normalization of the attribute data is an essential step in TOPSIS. In the proposed methods, the process of normalization of the data is not needed as a separate step. Despite this simplification the proposed methods may offer some difficulty in the form of multicollinearity. As evident from Eq. (1), the statistical distances are calculated using the inverse of the covariance of the decision matrix. Hence, there is a possibility that the inverse of covariance matrix may turn out to be near singular and hence an inaccurate solution. This difficulty may arise because of possible multicollinearity present in the attributes data. The multicollinearity is due to high correlations between the attributes considered in the analysis. Keat et al. (2009) suggest a simple remedy to the avoid multicollinearity. They state that a standard remedy for multicollinearity is to drop one of the variables that are closely associated with other variables. As a rule of thumb correlation coefficients (ρ) of 0.7 or more provide a basis for researchers to suspect the existence of multicollinearity.

Hence, the MADM problems based on the proposed methodology are classified into two categories.

Category 1: Problems without multicollinearity in the attribute data. In this category of problems, there is no need to drop any attribute from the analysis.

Category 2: Problems with multicollinearity in the attribute data. In this category some of the attributes have to be dropped to avoid multicollinearity.

The first category of the problems can be solved by considering all attributes of the original decision matrix. The second category of the problems needs careful elimination of some of the attributes from the analysis such that the multicollinearity is removed.

There are different approaches to detect and reduce the effect of multicollinearity. A measure of multicollinearity of the attributes data is the condition number of the correlation matrix. The condition number k is defined as

$$k = \sqrt{\frac{Maximum\ \ eigen\ value\ of\ \ the\ correlation\ matrix}{Minimum\ eigen\ value\ of\ \ the\ correlation\ matrix}} \tag{2}$$

The harmful effects of collinearity in the data become strong when the values of the condition number exceed 15, which means that the ratio of maximum and minimum eigen values exceeds 225. (Chatterjee et al., 2000).

In the present work, it is suggested to find the effect of multicollinearity by the condition number using the eigen values of the correlation matrix obtained from decision matrix. If the condition number is high, it is suggested to drop some of the attributes based on the higher correlations between them. Based on the results of applying the proposed methods to many MADM problems, it is observed that results are good if the condition number is maintained at less than 10. If the condition number is greater than 10, dropping of the attributes is suggested such that the effects of multicollinearity are reduced.

Dropping some of the variables that cause multicollinearity can be observed in literature dealing with regression problems (e.g Chang & Mastrangelo, 2011). In dealing with regression problems, the sign of the correlation coefficient do not play any role. In the case of multiple attribute decision making problems the sign of correlation coefficient plays a vital role in dropping the attributes. This is because of the fact that each attribute in MADM problems is associated with its own weight and also the attributes are either beneficial or non-beneficial. The attributes whose values are required to be maintained low are non-beneficial attributes (e.g., cost, pollution, wear etc.) and others for which higher values are required are beneficial attributes (e.g. material strength, profit, etc.). Hence, based on the magnitude of the correlations the attributes cannot be simply dropped. In the present work, the procedure to tackle the multicollinearity has been discussed initially and the proposed methods with statistical distances are discussed later.

3.1 Procedure to reduce the multicollinearity using attributes dropping

Step 1: Collect the information on all the alternatives with different attributes. If any of the attributes is qualitative, convert the qualitative values into suitable quantitative values. The attribute data is presented in the form of a matrix known as decision matrix.

Step 2: Obtain the weights to be given to each attribute. Any of the existing methods like entropy method or AHP can be used in this step.

Step 3: Find the correlation matrix of the decision matrix. Obtain the eigen values of the correlation matrix and calculate the condition number. If the condition number is less than 10, there is no need to drop any attribute from the analysis. Otherwise, proceed to Step 4.

Step 4: Find the valid and invalid correlations (for $|\rho| > 0.7$) between all the attributes. The valid correlations between different combinations of attributes are given in Table 1 below:

Table 1
Valid correlations between different combinations of attributes

Attribute combination	Valid correlation
Beneficial Vs Beneficial	Positive (+)
Non-beneficial Vs Non-beneficial	Positive (+)
Beneficial Vs Non-beneficial	Negative (-)
Non-beneficial Vs Beneficial	Negative (-)

From the Table 1, it is clear that the correlation to be valid, it has to be positive for similar attributes and it has to be negative for dissimilar attributes.

Step 5: Choose the attribute pair with highest correlation from the correlation matrix. Find the attribute with highest number of valid correlations and drop it from the decision matrix. In case of a tie between the attributes, the attribute with lowest weight has to be dropped. When the valid number of correlations is same for any two attributes and one attribute has more invalid correlations, the attribute with less number of invalid correlations has to be dropped.

Step 6: After dropping an attribute, find the condition number with the remaining attributes. If the condition number is less than 10, stop the process. Otherwise, repeat the Step 5.

3.2 Proposed TOPSIS methodology with statistical distances (s-TOPSIS)

After obtaining the final decision matrix whose condition number is less than 10, the following procedure has to be used to find the preference order for the alternatives.

Step 1: Using the final decision matrix with *m* alternatives and *n* attributes, represent the elements of the decision matrix **A** by a_{ij} where a_{ij} denotes the value of the *j* th attribute in the *i* th alternative.

Step 2: Multiply each attribute column by the respective weight. This step gives the weighted decision matrix X of size *m*×*n*.

The element x_{ij} of the weighted decision matrix X is given by:

$$x_{ij} = a_{ij} W_j , \tag{3}$$

where W_j is the weight of the attribute j.

Step 3: The positive ideal solution (PIS) and the negative ideal solution (NIS) are to be obtained from the weighted decision matrix. PIS and NIS are two vectors with best values and worst values taken from all the attributes respectively. In PIS, the best values include the maximum values for beneficial attributes and minimum values for non-beneficial attributes. Similarly, in NIS the worst values include the minimum values for beneficial attributes and maximum values for non-beneficial attributes.

The positive ideal solution μ_b and the negative ideal solution μ_w are expressed mathematically as:

$$\mu_b = \{\max_i (x_{ij} \mid j\varepsilon J^+), \min_i (x_{ij} \mid j\varepsilon J^-)\}; \; i = 1,2, \ldots, \mathrm{m} \tag{4}$$

$$\mu_w = \{\min_i (x_{ij} \mid j\varepsilon J^+), \max_i (x_{ij} \mid j\varepsilon J^-)\}; \; i = 1,2, \ldots, \mathrm{m} \tag{5}$$

where J^+ is associated with beneficial attribute (j = 1,2,…, n)

J^- is associated with non-beneficial attribute (j = ,2,…, n)

Step 4: Calculate the statistical distances for each alternative from the PIS and NIS.

In order to calculate the statistical distances, obtain the covariance matrix (S) for the weighted decision matrix (X). Find the inverse (S^{-1}) for the covariance matrix. For any alternative ***i***, the statistical distance from the PIS is calculated as:

$$S_i^+ = [\,(X_i - \mu_b)^T \; S^{-1} \; (X_i - \mu_b)\,]^{0.5} \tag{6}$$

The statistical distance from the NIS for alternative i is calculated as:

$$S_i^- = [\,(X_i - \mu_w)^T \; S^{-1} \; (X_i - \mu_w)\,]^{0.5} \tag{7}$$

where X_i is the weighted attribute values of the i^{th} alternative.

Now, the relative closeness of an alternative *i*, to the positive ideal solution is obtained by the ratio P_i as:

$$P_i = S^- /(S^- + S^+) \tag{8}$$

3.3 TOPSIS with weighted statistical distances (ws-TOPSIS)

As suggested by Deng et al. (2000) to consider weighted Euclidean distances in modified-TOPSIS, the weighted statistical distances are introduced in the TOPSIS procedure and the method is named ws-TOPSIS. In this approach, the decision matrix A is directly used in the calculation of the covariance of decision matrix (instead of weighted decision matrix). The PIS and NIS are to be obtained from root-weighted decision matrix. The elements of root-weighted decision matrix *Y* are given by:

$$y_{ij} = a_{ij} W_j^{0.5} \tag{9}$$

The PIS and NIS represented by v_b and v_w respectively are given by:

$$v_b = \{\max_i (y_{ij} \mid j \varepsilon J^+), \min_i (y_{ij} \mid j \varepsilon J^-)\} \quad \text{i = 1,2, …, m} \tag{10}$$

$$v_w = \{\min_i (y_{ij} \mid j \varepsilon J^+), \max_i (y_{ij} \mid j \varepsilon J^-)\} \quad \text{i = 1,2, …, m} \tag{11}$$

where J^+ is associated with beneficial attribute (j = 1,2,…, n)

J^- is associated with non-beneficial attribute (j = 1,2,…, n)

Representing the covariance of the decision matrix A by V and the inverse of the covariance matrix by V^{-1}, the weighted statistical distances from PIS (D_i^+) and NIS (D_i^-) can be calculated as follows:

$$D_i^+ = [(Y_i - v_b)^T \; V^{-1} \; (Y_i - v_b)]^{0.5} \tag{12}$$

$$D_i^- = [(Y_i - v_w)^T \; V^{-1} \; (Y_i - v_w)]^{0.5} \tag{13}$$

where Y_i is the root-weighted attributes of the ith alternative.
The relative closeness of an alternative *i* to the ideal solution using weighted statistical distances is obtained as:

$$P_{wsi} = \frac{D_i^-}{D_i^- + D_i^+} \tag{14}$$

The order of preference is given by arranging the values of the relative closeness in descending order.

4. Illustrative example for Category 1

As mentioned in Section 3, there are certain decision making problems which do not need to drop any attribute from the analysis. Hence, in this section both the s-TOPSIS and ws-TOPSIS procedures are explained considering a manufacturing problem without dropping the attributes.

4.1 Example problem using s-TOPSIS

The step by step procedure of the proposed method is explained by considering the machinability evaluation of six different metal alloys problem from Konig and Erinski (1983) which is solved by Rao (2007) using Graph Theory and Matrix Approach (GTMA) and TOPSIS. Each alloy is evaluated for its machinability with three attributes, namely, the one-hour cutting speed VC, the specific cutting force CF and the cutting power input CI. Table 2 shows the objective data of the problem.

Table 2
Objective data of the alternative alloys

Work material	VC(m/min)	CF (N/m^2)	PI (kW)
W1	710	400	28
W2	900	415	38
W3	1630	440	59
W4	1720	235	43
W5	120	1150	8
W6	160	1750	19

W1: GK-AlSi10Mg (aluminum-silicon die-cast alloy) W2: GK-AlSi6Cu4 (aluminum-silicon die-cast alloy)
W3: GK-AlMg5 (aluminum-magnesium die-cast alloy) W4: GK-MgAl9Zn (magnesium-aluminum die-cast alloy)
W5: GG26 (gray cast iron with lamellar graphite) W6: C35 (low carbon steel)

The solution to the above MADM problem using s-TOPSIS is obtained as under:

Step 1: Obtain the decision matrix of size 6×3 from the objective data of Table 2. Find the eigen values of the correlation matrix and obtain the condition number.

The correlations between attributes of the present problem are given below:

$$\begin{array}{cc} & \begin{array}{ccc} VC & CF & PI \end{array} \\ \begin{array}{c} VC \\ CF \\ PI \end{array} & \begin{bmatrix} 1 & -0.79902 & 0.92283 \\ -0.79902 & 1 & -0.69224 \\ 0.92283 & -0.69224 & 1 \end{bmatrix} \end{array}$$

The eigenvalues of the above correlation matrix are : $\begin{bmatrix} 2.6131 \\ 0.060524 \\ 0.32638 \end{bmatrix}$

The condition number from the above eigenvalues is 6.57. As the condition number is less than 10, the original decision matrix becomes the final decision matrix without dropping any attribute.

Step 2: The decision matrix is to be multiplied by the respective weights for each attribute. The weight for each attribute is taken as W_{VC} = 0.7142, W_{CF} = 0.1429 and W_{PI} = 0.1429 from Rao (2007) for comparison purpose. After multiplying each attribute by the respective weight, the following weighted decision matrix given in Table 3 is obtained.

Table 3
Weighted decision matrix of the alternatives

Work material	VC(m/min)	CF (N/m^2)	PI (kW)
W1	507.08	57.16	4.0012
W2	642.78	59.304	5.4302
W3	1164.1	62.876	8.4311
W4	1228.4	33.581	6.1447
W5	85.704	164.34	1.1432
W6	114.27	250.07	2.7151

Step 3: The positive ideal solution is obtained from the attributes data by choosing the largest values

from the beneficial attributes and smallest values from the non-beneficial attributes. Out of the three attributes, CF and PI are non-beneficial and VC is the only beneficial variable.

Hence, for the present problem, the positive ideal solution is:

$\mu_b = [1228.4 \quad 3.581 \quad 1.432]^T$

The negative ideal solution is given as:

$\mu_w = [85.704 \quad 250.7 \quad 8.4311]^T$

Step 4:

In order to calculate the statistical distances from the PIS and NIS, the covariance matrix and inverse of the covariance matrix are to be calculated. The covariance of the weighted decision matrix, S for the attribute data of the Table 3 is given by:

$$S = \begin{bmatrix} 2.4414e+005 & -33415 & 1182 \\ -33415 & 7163.6 & -151.89 \\ 1182 & -151.89 & 6.7204 \end{bmatrix}$$

The inverse of the covariance of the weighted decision matrix, S^{-1} is given by:

$$S^{-1} = \begin{bmatrix} 4.1327e-005 & 7.421e-005 & -0.0055918 \\ 7.421e-005 & 0.00040129 & -0.0039832 \\ -0.0055918 & -0.0039832 & 1.0423 \end{bmatrix}$$

The square of the statistical distance for the first alternative from the positive ideal solution is calculated using Eq. (6) as given below:

$$(S_1^+)^2 = [-724.34 \quad 23.578 \quad 2.858] \times \begin{bmatrix} 4.1327e-005 & 7.421e-005 & -0.0055918 \\ 7.421e-005 & 0.00040129 & -0.0039832 \\ -0.0055918 & -0.0039832 & 1.0423 \end{bmatrix}$$

$$\times [-724.34 \quad 23.578 \quad 2.858]^T = 50.236$$

The statistical distance of the first alternative W1 from the PIS is given by $S_1^+ = 7.0877$

Similarly, the square of the statistical distance for the first alternative W1 from the negative ideal solution is given by:

$$(S_1^-)^2 = [421.38 \quad -192.91 \quad -4.4299] \times \begin{bmatrix} 4.1327e-005 & 7.421e-005 & -0.0055918 \\ 7.421e-005 & 0.00040129 & -0.0039832 \\ -0.0055918 & -0.0039832 & 1.0423 \end{bmatrix}$$

$$\times [421.38 \quad -192.91 \quad -4.4299]^T = 44.73$$

The statistical distance of the first alternative W1 from the NIS is given by $S_1^- = 6.6881$

The relative closeness of the alternative W1 from PIS is given by:

$$P_i = \frac{S^-}{S^- + S^+} = \frac{6.6881}{6.6881 + 7.0877} = 0.48549$$

Similarly, the relative closeness values for all the alternatives from the PIS are calculated and summarized in Table 4.

Table 4
Statistical distances and relative closeness for all alternatives

Materials	W1	W2	W3	W4	W5	W6
S_i^+	7.0877	7.6525	7.69	5.1063	6.2169	7.3321
S_i^-	6.6881	5.9312	5.6714	8.172	7.303	5.993
P_i	0.48549	0.43664	0.42446	0.61544	0.54017	0.44975

From the relative closeness values given in Table 4, it can be observed that the order of preference obtained by the proposed s-TOPSIS method is:

W4 > W5 > W1 > W6 > W2 >W3

4.2 Example problem using ws-TOPSIS

In this section, the same machinability evaluation problem solved in Section 4.1 is solved using weighted statistical distances.

The covariance V of the decision matrix of Table 2 is given by:

$$V = \begin{bmatrix} 4.7863e+005 & -3.2741e+005 & 11582 \\ -3.2741e+005 & 3.5081e+005 & -7438 \\ 11582 & -7438 & 329.1 \end{bmatrix}$$

The inverse of the covariance of the decision matrix, V^{-1} is given by:

$$V^{-1} = \begin{bmatrix} 2.1084e-005 & 7.5738e-006 & -0.0005707 \\ 7.5738e-006 & 8.1946e-006 & -8.1338e-005 \\ -0.0005707 & -8.1338e-005 & 0.021285 \end{bmatrix}$$

The root weighted decision matrix which can be obtained by multiplying each attribute column by the square root of the respective weight as given in Table 5.

Table 5
Root-weighted decision matrix for all alternatives

Work material	VC(m/min)	CF (N/m^2)	PI (kW)
W1	600.02	151.21	10.585
W2	760.59	156.88	14.365
W3	1377.5	166.33	22.303
W4	1453.6	88.835	16.255
W5	101.41	434.72	3.0242
W6	135.22	661.54	7.1824

The PIS and NIS obtained from the above Table 5 are given below:

$$v_b = [1453.6 \quad 88.835 \quad 3.0242]^T \qquad v_w = [101.41 \quad 661.54 \quad 22.303]^T$$

The weighted statistical distance for the first alternative W1 from the PIS is calculated as under:

$$(D_1^+)^2 = [-853.55 \quad 62.373 \quad 7.5604] \times \begin{bmatrix} 2.1084e-005 & 7.5738e-006 & -0.0005707 \\ 7.5738e-006 & 8.1946e-006 & -8.1338e-005 \\ -0.0005707 & -8.1338e-005 & 0.021285 \end{bmatrix} \times [-853.55 \quad 62.373 \quad 7.5604]^T = 23.089$$

$$D_1^+ = 4.8051$$

Similarly, the weighted statistical distance for the first alternative W1 from the PIS is calculated as under:

$$(D_1^-)^2 = [498.61 \quad -510.33 \quad -11.719] \times \begin{bmatrix} 2.1084e-005 & 7.5738e-006 & -0.0005707 \\ 7.5738e-006 & 8.1946e-006 & -8.1338e-005 \\ -0.0005707 & -8.1338e-005 & 0.021285 \end{bmatrix}$$

$$\times [498.61 \quad -510.33 \quad -11.719]^T = 12.14$$

$$D_1^- = 3.4843$$

The relative closeness of the alternative W1 from PIS is given by:

$$P_{ws1} = \frac{D_1^-}{D_1^- + D_1^+} = \frac{3.4843}{3.4843 + 4.8051} = 0.42033$$

Similarly, the relative closeness values for all the alternatives from the PIS are calculated and summarized in Table 6.

Table 6
Weighted statistical distances and relative closeness for all alternatives

Materials	W1	W2	W3	W4	W5	W6
D_i^+	4.0851	4.5857	3.0698	1.9303	5.6954	5.8419
D_i^-	3.4842	3.5875	5.1735	6.249	2.7607	2.3397
P_{wsi}	0.42033	0.43893	0.6276	0.764	0.32647	0.28597

From the relative closeness values given in Table 6, it can be observed that the order of preference obtained by the proposed ws-TOPSIS using weighted statistical distances is:

W4 > W3 > W2 > W1 > W5 > W6

For comparison purposes, the solutions obtained by Rao (2007) using different MADM methods along with the solutions obtained using proposed methods for the same problem have been presented in Table 7.

Table 7
A comparison of the solutions by some of the MADM methods

S.No.	Method	Solution
1.	GTMA	***W4> W3 > W2> W1 > W5> W6***
2.	AHP and its Versions	***W4> W3 > W2> W1> W5> W6***
3	TOPSIS and modified TOPSIS	***W4> W3> W2> W1> W5> W6***
4.	s-TOPSIS	***W4> W5> W1> W6> W2>W3***
5.	ws-TOPSIS	***W4> W3> W2> W1> W5> W6***

From the above results, it can be observed that all the methods proposed the W4 as the best alternative in the above problem. However, when compared with other methods s-TOPSIS provided some deviations in the solution. It is worth noticing that for the present problem ws-TOPSIS provided identical solution as any of the other methods under consideration.

5. Illustrative examples for Category 2

In this section, some decision making problems involving the dropping of some of the attributes from the analysis are considered for illustration.

5.1 Product design selection using s-TOPSIS and ws-TOPSIS

As a first example, a product design selection problem from Besharati et al. (2006) has been considered. The problem is to select a power electronic device based on three attributes, namely, junction temperature (JT), thermal cycles to failure (CF) and manufacturing cost (MC). Out of the three attributes CF is the only beneficial attribute. Rao (2007) solved the same problem using GTMA, AHP, TOPSIS and modified TOPSIS methods. In order to have comparison of the results with the proposed s-TOPSIS and ws-TOPSIS, the attribute weights are chosen as in Rao (2007). The weights are W_{JI} = 0.1047, W_{CF} = 0.2582 and W_{MC} = 0.6371. The design alternatives for the problem are given in Table 8.

Table 8
Description of the design alternatives (Besharati 2006)

Design Number	Junction Temperature (JT)	Cycles to Failure (CF)	Manufacturing Cost (MC)
1	126	22000	85
2	105	38000	99
3	138	14000	65
4	140	13000	60
5	147	10600	52
6	116	27000	88
7	112	32000	92
8	132	17000	75
9	122	23500	85
10	135	15000	62

The correlation matrix of the decision matrix of the Table 8 is given below:

$$\begin{array}{c} \\ JT \\ CF \\ MC \end{array} \begin{array}{c} \begin{array}{ccc} JT & CF & MC \end{array} \\ \begin{bmatrix} 1 & -0.9880 & -0.9716 \\ -0.9880 & 1 & 0.9508 \\ -0.9716 & 0.9508 & 1 \end{bmatrix} \end{array}$$

The eigen values of the above correlation matrix are given by: $\begin{bmatrix} 2.9403 \\ 0.0083 \\ 0.0514 \end{bmatrix}$

The condition number calculated from the above eigen values is 18.8 which is larger than 10.

Since the condition number is high in the present problem, the final decision matrix has to be obtained by following the procedure of Section 3.1. The following steps are followed to avoid multicollinearity.

Step 1: The initial decision matrix is a 3×10 matrix with three attributes namely, JT, CF and MC.

Step 2: The weights of the above attributes are taken as: W_{JI} = 0.1047, W_{CF} = 0.2582 and W_{MC} = 0.6371.

Step 3: The correlation matrix and the eigen values of the correlation matrix are obtained and the initial condition number is 18.8 which is high compared to 10.

Step 4: From the above correlation matrix, the number of valid and invalid correlations are obtained as given in Table 9. The beneficial and non-beneficial attributes are indicated with positive and negative symbols for ease of recognition.

Table 9
The number of valid and invalid correlations between the attributes

Attributes	Valid correlations	Invalid correlations	Number of valid correlations
JT (-)	CF (-0.9880)	MC(-0.9712)	1
CF(+)	JT(-0.9880)	MC(0.9508)	1
MC(-)		JT(-0.9712), CF(0.9508)	0

From Table 9, it can be observed that both the attributes JT and CF are having valid and invalid correlations equally.

Step 5: Choosing the pair of attributes with highest correlation, the attributes CF and JT are obtained with highest correlation of -0.9880. Hence, the number of valid correlations is compared between these two attributes. As there is a tie between the valid and invalid correlations, the attribute with lowest weight has to be dropped. Since the attribute JT has lowest weight in the present problem it is dropped from the analysis.

Step 6: With CF and MC attributes, the condition number is calculated. As the eigen values of the correlation matrix with these attributes are 1.9508 and 0.049224, the condition number obtained is 6.2953 which is less than 10. Hence, the final decision matrix is having only two attributes, namely CF and MC.

With two attributes CF and MC, the s-TOPSIS and ws-TOPSIS methods are applied. The weighted and root weighted decision matrices are obtained as in Tables 10 and 11.

Table 10
The weighted decision matrix

Design Number	Cycles to Failure (CF)	Manufacturing Cost (MC)
1	5680.4	54.154
2	9811.6	63.073
3	3614.8	41.411
4	3356.6	38.226
5	2736.9	33.129
6	6971.4	56.065
7	8262.4	58.613
8	4389.4	47.782
9	6067.7	54.154
10	3873.0	39.500

Table 11
The root-weighted decision matrix

Design Number	Cycles to Failure (CF)	Manufacturing Cost (MC)
1	11179	67.846
2	19309	79.020
3	7113.9	51.882
4	6605.7	47.891
5	5386.2	41.506
6	13720	70.240
7	16260	73.433
8	8638.3	59.864
9	11941	67.846
10	7622	49.487

Following the usual notation, the following PIS and NIS are obtained from Tables 9 and 10 respectively in order to apply s-TOPSIS and ws-TOPSIS methods.

$$\mu_b = [9811.6, 33.129]^T \qquad \mu_w = [2736.9, 63.073]^T \qquad \nu_b = [19309, 41.506]^T \qquad \nu_w = [5386.2, 79.02]^T$$

The relative closeness values calculated for both s-TOPSIS and ws-TOPSIS methods are given in Tables 12 and 13.

Table 12
Relative closeness values of each alternative with s-TOPSIS

Design Number	1	2	3	4	5	6	7	8	9	10
S_i^+	12.35	9.61	11.18	10.55	9.85	11.19	10.25	12.11	11.81	10.23
S_i^-	6.88	9.85	8.13	8.80	9.61	8.06	9.06	7.13	7.41	9.08
P_i	0.358	0.506	0.421	0.455	0.494	0.419	0.469	0.371	0.386	0.470

Table 13
Relative closeness values of each alternative with ws-TOPSIS

Design Number	1	2	3	4	5	6	7	8	9	10
D_i^+	19.53	12.05	20.17	19.64	19.38	16.79	14.35	20.54	18.48	18.72
D_i^-	11.53	19.38	11.02	11.62	12.05	14.30	16.85	10.55	12.58	12.45
P_{wsi}	0.371	0.617	0.355	0.372	0.383	0.459	0.540	0.339	0.405	0.399

The order of preferences obtained from both s-TOPSIS and ws-TOPSIS are summarized along with the solutions from Rao (2007) in Table 14.

Table 14
A comparison of the results obtained from different methods

S.No.	Method	Solution
1.	GTMA	***2>7>6>9>1>10>8>4>3>5***
2.	AHP	***5>4>10>2>3>7>6>8>9>1***
3	TOPSIS	***5>10>4>3>2>7>8>6>9>1***
4	Modified TOPSIS	***2>7>6>5>10>4>9>3>1>8***
5	s-TOPSIS	***2>5>10>7>4>3>6>9>8>1***
6	ws-TOPSIS	***2>7>6>9>10>5>4>1>3>8***
7	Besharati et al (2006)	***5>10>4>3>7>6>2>8>9>1***

From the results of the Table 14, it can be observed that both s-TOPSIS and ws-TOPSIS methods provided the design **2** as the best alternative. This result is in agreement with the results of GTMA, modified TOPSIS methods. Also, the first three alternatives suggested by GTMA, modified TOPSIS and ws-TOPSIS are same. But, the last choice given by GTMA is somewhat alarming since some of the popular methods (AHP and TOPSIS) suggested this alternative as the first preference. Modified TOPSIS and ws-TOPSIS suggest the alternative 8 as the last choice. Similarly, TOPSIS and s-TOPSIS suggest the alternative 1 as the last choice. It is evident from the results that the last choice cannot be alternative 5, as the MC value is minimum for this alternative and the weight given to MC is very high, and also some popular methods suggested this as the first alternative. The proposed methods s-TOPSIS and ws-TOPSIS have given the design alternative **5** the due position, and alternatives 1 and 8 are given the last preference. AHP, TOPSIS and modified TOPSIS also gave alternative 1 or 8 as the last preference.

5.2 Machine group selection in FMC using s-TOPSIS and ws-TOPSIS

As a second example, the machine group selection problem from Wang et al.(2000) is considered in this section. The problem is solved using both statistical and weighted statistical distances. Table 14 consists of the data of the four attributes for ten machine groups. The attributes purchase cost (TC), total floor space (FS), total number of machines (MN) are considered as non-beneficial attributes and the productivity (P) is the only beneficial attribute.

Table 15
Objective data of the machine groups

Alternative	Total purchasing cost (TC) ($)	Total floor space (FS) (m^2)	MN	Productivity (P) (mm/min)
1	581818	54.49	3	5500
2	595454	49.73	3	4500
3	586060	51.24	3	5000
4	522727	45.71	3	5800
5	561818	52.66	3	5200
6	543030	74.46	4	5600
7	522727	75.42	4	5800
8	486970	62.62	4	5600
9	509394	65.87	4	6400
10	513333	70.67	4	6000

The correlation matrix of the decision matrix of the Table 15 is given below:

$$\begin{array}{c} \\ TC \\ FS \\ MN \\ P \end{array} \begin{array}{c} \begin{array}{cccc} TC & FS & MN & P \end{array} \\ \begin{bmatrix} 1 & -0.53392 & -0.7726 & -0.79338 \\ -0.53392 & 1 & 0.91963 & 0.55868 \\ -0.7726 & 0.91963 & 1 & 0.66886 \\ -0.79338 & 0.55868 & 0.66886 & 1 \end{bmatrix} \end{array}$$

The eigenvalues of the above correlation matrix are: $\begin{bmatrix} 3.1302 \\ 0.6177 \\ 0.23068 \\ 0.021428 \end{bmatrix}$

The condition number calculated using the above eigen values is 11.354 which is greater than 10. Hence, the procedure of Section 3.1 is followed to drop some of the variables.

Step 1: The initial decision matrix is a 4×10 matrix with four attributes namely, TC, FS, MN and P.

Step 2: The weights of the above attributes are taken as: W_{TC} = 0.467, W_{FS} = 0.16, W_{MN} = 0.095 and W_{FS} = 0.278 for comparison purpose from Rao (2007).

Step 3: The correlation matrix and the eigenvalues of the correlation matrix are obtained and the condition number is 11.354 which is high compared to 10.

Step 4: From the correlation matrix of the initial decision matrix, the valid and invalid correlations are obtained for all attributes, and are given in the Table 16 below:

Table 16
The number of valid and invalid correlations between the attributes

Attributes	Valid correlations	Invalid correlations	Number of valid correlations
TC (-)	P (-0.79338)	MN (-0.7726)	1
FS (-)	MN (0.91963)		1
MN (-)	FS (0.91963)	TC (-0.7726)	1
P (+)	TC (-0.79338)		1

Step 5: The attributes with highest correlation are FS and MN with ρ = 0.91963. From Table 15, it can be observed that MN and FS are having equal number of valid correlations. Since the attribute MN has an additional invalid correlation, the attribute FS is dropped from the decision matrix.

Step 6: With the remaining attributes TC, MN and P in the decision matrix, the eigenvalues of the correlation matrix are calculated. The eigenvalues and the condition number are $[\,2.4912 \quad 0.17681 \quad 0.33201]^T$ and 3.7576 respectively. As the condition number is less than 10, the problem can be solved with three attributes in the decision matrix.

In order to apply s-TOPSIS and ws-TOPSIS, the weighted and root-weighted decision matrices are obtained and are given in Tables 17 and 18.

Table 17
The weighted decision matrix

Alternative	TC	MN	P
1	271710	0.285	1529
2	278080	0.285	1251
3	273690	0.285	1390
4	244110	0.285	1612.4
5	262370	0.285	1445.6
6	253600	0.380	1556.8
7	244110	0.380	1612.4
8	227410	0.380	1556.8
9	237890	0.380	1779.2
10	239730	0.380	1668

Table 18
The root-weighted decision matrix

Alternative	TC	MN	P
1	397600	0.92466	2899.9
2	406920	0.92466	2372.7
3	400500	0.92466	2636.3
4	357220	0.92466	3058.1
5	383930	0.92466	2741.7
6	371090	1.23290	2952.6
7	357220	1.23290	3058.1
8	332780	1.23290	2952.6
9	348110	1.23290	3374.4
10	350800	1.23290	3163.5

From the Tables 17 and 18, the following PIS and NIS values are obtained to apply the s-TOPSIS and ws-TOPSIS methods.

$$\mu_b = [227410 \quad 0.285 \quad 1779.2]^T \qquad \mu_w = [278080 \quad 0.38 \quad 1251]^T$$
$$\nu_b = [332780 \quad 0.92466 \quad 3374.4]^T \qquad \nu_w = [406920 \quad 1.2329 \quad 2372.7]^T$$

The relative closeness values calculated for both s-TOPSIS and ws-TOPSIS methods are given in Table 19 and Table 20.

Table 19
Relative closeness values of each alternative with s-TOPSIS

Design Number	1	2	3	4	5	6	7	8	9	10
S_i^+	16.17	27.69	19.32	2.88	12.09	26.98	19.66	17.43	14.29	16.09
S_i^-	21.65	9.12	13.81	37.66	20.02	8.18	12.73	21.14	23.86	16.65
P_i	0.573	0.248	0.417	0.929	0.624	0.233	0.393	0.548	0.626	0.508

Table 20
Relative closeness values of each alternative with ws-TOPSIS

Design Number	1	2	3	4	5	6	7	8	9	10
D_i^+	7.81	10.56	8.27	1.13	4.85	5.87	12.73	21.14	23.86	16.65
D_i^-	3.77	0.87	1.88	8.82	3.51	2.79	4.79	10.09	7.91	6.19
P_{wsi}	0.325	0.076	0.185	0.887	0.419	0.323	0.576	0.775	0.774	0.706

The order of preferences obtained from both s-TOPSIS and ws-TOPSIS are summarized along with the solutions from Rao (2007) in Table 21.

Table 21
A comparison of the results obtained from different methods

S.No.	Method	Solution
1.	GTMA	***4>5>1>3>2>9>8>10>7>6***
2.	AHP	***4>9>8>10>5>1>7>3>6>2***
3	TOPSIS	***4>9>8>10>7>5>1>6>3>2***
4	Modified TOPSIS	***4>9>5>1>8>3>10>2>7>6***
5	s-TOPSIS	***4>9>5>1>8>10>3>7>2>6***
6	ws-TOPSIS	***4>8>9>10>7>5>1>6>3>2***

From the results of the Table 21, it is evident that all methods proposed the alternative 4 as the first choice. The results of the modified TOPSIS and s-TOPSIS methods are very much similar with the first five preferences by both the methods being the same. TOPSIS with weighted statistical distances (ws-TOPSIS) gives alternative 9 the third position in preference, the second preferred being the alternative 8. No other method except ws-TOPSIS gives second preference to alternative 8. From the data of the Table 15, it can be observed that alternative 8 cannot be an inferior solution, since the attribute TC with highest weight 0.467 is having the least value among all alternatives. From Table 20, it can be observed that the relative closeness index is almost the same for alternatives 8 and 9, which shows that ws-TOPSIS has high preference for alternative 9 also. The fourth preference by AHP, TOPSIS and ws-TOPSIS methods are also matching with alternative 10. All the methods proposed either alternative 2 or 6 as the last preference.

6. Conclusions

In the present work, a new approach to multiple attribute decision making has been suggested by introducing the statistical distance in place of Euclidian distance for TOPSIS. Use of statistical distance is suggested to take care of the unequal variability in the attributes as well as the possible correlations between the attributes. In this approach, unlike other MADM methods, normalization of the attribute data is not required since it is inherent in statistical distance.

The new approach with statistical and weighted statistical is applied on different manufacturing/industrial problems and the solutions obtained are reported. The use of statistical distances entails the use of inverse of the covariance matrix obtained from the attribute data of the alternatives. When the attributes are highly correlated, it leads to multicollinearity by which the inverse of a matrix becomes near singular and the possibility of finding better solution becomes difficult. To overcome this problem, a methodology is suggested to eliminate the multicollinearity. With the results obtained for different MADM problems, it has been concluded to maintain the condition number less than 10 for better results. The solutions obtained would be good depending on how well the multicollinearity is eliminated with minimum loss of data in use. Any possible improvement in the suggested methodology to eliminate the effects of multicollinearity would surely improve the solutions. The main limitation with the proposed methodology is that better solutions can be obtained when the number of attributes is less than or equal to the number of alternatives. When the number of alternatives is very low compared to the number of attributes, the proposed method loses more data and hence cannot be relied on. In other cases, the proposed methods can be considered as better alternatives to solve MADM problems. The novel point in the proposed methodology is the consideration of correlations among the attributes. This provides a basis to extend the existing methods of deciding the weights of the attributes by considering the correlations among them. Also, it helps in reducing the complexity of the problems by eliminating highly correlated attributes.

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