ESTRATÉGIAS DE GENERALIZAÇÃO DE PADRÕES PARA MELHORAR O
PENSAMENTO ALGÉBRICO DOS ESTUDANTESSTRATEGIES OF PATTERN GENERALIZATION FOR ENHANCING STUDENTS'
ALGEBRAIC THINKING

提高学生代数思维的模式概括策略

NURWIDIYANTO^{1*}, ZHANG, Kaijun²^{1,2} Mathematics Education, School of Mathematics and Statistics, Northeast Normal University, Changchun, Jilin, P. R. China** Corresponding author*
e-mail: zil997@nenu.edu.cn

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RESUMO

A matemática é vista como uma ciência do padrão. Identificar e usar padrões é a essência do pensamento matemático para as crianças melhorarem o pensamento algébrico desde o início da fase escolar. O padrão é um arranjo de objetos com regularidades ou propriedades que podem ser generalizadas. Portanto, é importante conhecer as estratégias usadas pelos alunos na generalização de padrões e como os alunos pensam nesses processos. Este estudo é uma pesquisa descritiva com uma abordagem quantitativa-qualitativa mista, que teve como objetivo investigar o pensamento algébrico do aluno usando várias estratégias para generalizar o padrão visual. Um instrumento sobre o padrão linear de crescimento geométrico foi administrado a 75 alunos do ensino fundamental (notas 5-6) e 81 alunos do ensino médio (notas 7-8) em duas escolas particulares em Semarang, Indonésia. Os resultados mostraram que os estudantes usaram diferentes estratégias de generalização de padrões. O aluno geralmente preferia estratégia recursiva, fragmentada e funcional em cada tarefa de generalização, enquanto poucos usavam a contagem de estratégias de desenho para generalizar o padrão. O uso da estratégia recursiva diminuiu, enquanto o uso da estratégia de chunking e da estratégia funcional aumentou nos graus 5-8 para os problemas. Os resultados também mostraram que o aluno que usou a estratégia recursiva e de fragmentação preferiu alterar os padrões visuais em linhas de números. Portanto, eles adotam uma abordagem numérica, encontrando a diferença comum do padrão visual em cada etapa.

Palavras-chave: *Pensamento algébrico, Estudo transversal, Estratégia de generalização, Padrão visual.*

ABSTRACT

Mathematics is seen as a science of pattern. Identifying and using patterns is the essence of mathematical thinking for children to improve algebraic thinking from their early schooling. The pattern is an arrangement of objects that have regularities or properties that can be generalized. Therefore, it is essential to know the strategies used by students in generalizing patterns and how students think in these processes. This study is descriptive research with a mixed quantitative-qualitative approach that aimed to investigate student's algebraic thinking using various strategies to generalize the visual pattern. An instrument about the linear geometric growing pattern was administrated to 75 upper primary school students (grades 5-6) and 81 lower secondary students (grades 7-8) in two private schools in Semarang, Indonesia. The results showed that students used different pattern generalization strategies. The student generally preferred recursive, chunking, and functional approaches in each generalization task, whereas few used counting from drawing strategies to generalize patterns. The use of the recursive strategy decreased, whereas the chunking strategy and the functional strategy increased across grades 5-8 for the problems. The results also showed the student who used the recursive and chunking strategy preferred to change visual patterns into rows of numbers. Hence, they adopt a numeric approach by finding the common difference of visible pattern in each step.

Keywords: Algebraic thinking, Cross-age study, Generalization strategy, Visual pattern.

摘要

数学被看作是一门关于模式的科学。识别和使用模式是数学思维的本质，可以促进儿童从学校教育初期开始的代数思维的发展。模式是对一类具有某些规律和属性的对象的抽象。因此，了解学生概括模式的策略以及学生的思考方式是重要的。本研究采用定性与定量相结合的方法探讨学生的代数思维发展。本研究中应用线性几何增长模型为研究工具，对印度尼西亚三宝垄的两所私立学校的75名小学高年级（5-6年级）学生和81名中学生（7-8年级）进行研究，结果表明，学生使用了不同的模式概括策略，在每个概括任务中，学生一般更倾向于递归、分块和功能策略，但很少有人从绘制策略中计数来概括模式，从5-8年级，学生对递归策略的使用逐渐减少，分块策略和功能策略逐渐增加。研究结果还显示，使用递归和分块策略的学生更喜欢将数字排成行，通过发现每一步的共同差异进行模式概括。

关键词：代数思维，跨年龄研究，概括策略，视觉模式。

1. INTRODUCTION:

Mathematics is seen as a science of pattern (Resnik, 1981), and identifying and using patterns is the essence of mathematical thinking (Booker, 2009). According to Callejo and Zapatera (2017), identifying patterns is a way for children to improve algebraic thinking from early schooling. Papic *et al.* (2009) has defined patterns as an essential skill in early mathematic learning, particularly in the improvement of spatial awareness, sequencing, and ordering. Besides, Twohill (2017) argues that shape patterns may help students develop their generalizations, represent generalizations, and have syntactic guided reasoning on generalizations. The pattern plays an essential role in improving thinking, communication, association, and problem-solving (Tanişlı and Özdaş, 2009). Besides, Fox (2005) reported patterns commonly contain the mathematical ability to calculate, classify, compare, measure, estimate, and symbolize, and these processes make the mathematical knowledge and skills of the students meaningful.

Demonstrating mathematics proficiency is a vital prerequisite for academic improvement and career readiness, which becomes the essential predictor of future academic progress (Gamoran and Hannigan, 2000).

The construction of the general term for the pattern is a vital aspect of algebraic thinking (Twohill, 2017; Walkowiak, 2014). The application of algebraic thinking is also inflexible, which leads to a lack of confidence and initiative of students in algebra learning and even the assumption that algebra is complicated. In the early algebra learning of elementary school, students' thinking is only based on "arithmetic thinking" (Radford,

2018), where more attention is in the calculation (Carpenter *et al.*, 2005). Therefore, when more symbols are involved, students will be trapped by inherent arithmetic thinking and feel that the problem cannot be solved. For instance, Demonty *et al.*, (2018) found that early secondary school students are still experiencing difficulties and continuing their mistakes in understanding algebraic thinking, especially in constructing pattern generalizations.

The students' generalization strategies using various activities such as numerical, geometric, and repeating patterns help students use symbols to express generalizations and develop meaning for symbolic generalizations (Alajmi, 2016; Lannin *et al.*, 2006). Understanding mathematical symbols and concepts are an essential component of mathematical knowledge (Skemp, 1987). Expanded vocabularies open learners to understand differences and commonalities between items by describing their attributes. For this reason, to build elementary and secondary student's skills to generalize, the researcher was encouraged to explore the strategies that students use through the examination of a linear geometric growing pattern. The approach referred to in this study focuses on what policy students use to generalize patterns and how they generalize patterns using these strategies?

Therefore, this study aimed to investigate algebraic thinking of the students using various strategies to generalize the visual pattern.

2. BACKGROUND:

2.1. Describing Algebra and Algebraic Thinking

Many people considered algebra as a

simple “symbolic game” or a short chapter in the history of mathematics. The role of algebraic thinking is not just to solve algebraic or mathematical problems. Still, the successes of developments in education, science, economics, business, and trade are also inseparable from the role of algebraic thinking (Blanton *et al.*, 2018; Booker, 2009; Moses and Cobb, 2001). Therefore, algebra is made as one of the five content principles and standards in the NCTM (National Council of Teacher of Mathematics) that students must learn from kindergarten to secondary level (NCTM, 2000) to integrate the experience of algebraic thinking from early grades on. Thus, algebra can be applied in mathematics learning in elementary and secondary schools in all countries (Wilkie and Clarke, 2016).

According to Watson (2010), Algebra is a way to express generalizations about numbers, quantities, relationships, and functions. Similarly, Taylor-Cox (2003) defined that algebra is a generalization of arithmetic ideas where unknown values and variables can be found to solve problems. Problems that can be solved by algebra are not only theoretical problems but also problems of daily life in various contexts. Algebra itself is a set of facts and techniques and a way of thinking (Lew, 2004). Therefore, algebra can be applied to develop the students' mathematical thinking abilities, called algebraic thinking. Algebraic thinking is a fundamental element of mathematical thinking and reasoning (Windsor, 2010). Algebraic thinking involves a variety of cognitive strategies to help understand more complex mathematical concepts. Algebraic thinking develops when one discovers and states structure to solve problems related to numbers or models of various situations.

2.2. Pattern Generalization

Algebra and all branches of mathematics are about pattern generalization (Lee, 1996). Several types of mathematics patterns are number patterns, geometric or pictorial growing patterns, patterns in computational procedures, linear and quadratic patterns, and repeating patterns (Zazkis and Liljedahl, 2002). This study discusses visual patterns, namely images of lines that change from one image to the next, whose changes can be predicted (Walkowiak, 2014). The teacher commonly uses a growing visual pattern in helping students predict the following pattern and generalize. Understanding of patterns, relations, and functions are a strand of algebraic reasoning that continues to exist in the principles and standards of algebra learning in schools at all

levels (NCTM, 2000).

Generalization is one of the fundamental activities in algebraic thinking (Walkowiak, 2014). Mathematical developments depend on the application of stereotypes. According to Tall (2011), generalization in mathematics is “looking for a bigger picture,” paying attention to small groups to look for larger groups, expanding concepts in larger areas. In his book, Tall (2002) also mentioned that generalization strategies are used in mathematics to show processes in a broader context and help someone know the results of problem-solving. Generalization helps one combine their experience and knowledge to solve problems in new conditions. Tall (2014) tried to encourage students in generalization by using the application of everyday issues.

Some of the results of previous studies showed that students used various strategies in predicting and generalizing patterns. The results of a study conducted by Radford (2018) showed several strategies that students carried out in solving problems, namely a strategy of naïve induction (trial and error), generalization strategies with arithmetic, and generalizations with algebra (factual, conceptual, and symbolic). The study of El Mouhayar (2018) shows that students frequently use recursive strategies and functional strategies to generalize patterns. Recursive strategies pointed the characteristic difference between pairs of consecutive terms and repetitively added the constant from term to term to extend the pattern. The functional strategy was relating parts of the pattern to the figural step number.

2.3. Algebra in Indonesian Classroom

In Indonesia, algebra does not yet formally exist in the elementary school curriculum. The Indonesian mathematics curriculum requires students to write, simplify, and substitute algebraic expressions only in secondary school. Meanwhile, in elementary school, more emphasis is given to natural numbers and fractions, geometry and measurement, and statistics. On the other hand, the Indonesian educational stakeholders are not familiar with the introductory algebra lesson. Hence, the curriculum still separates algebra and arithmetic so that secondary school students should learn algebra in a formal way (Minister of Education and Culture of Indonesia, 2016).

As a consequence of the above condition, most students rigidly learn algebra and hardly see the relationship between algebra and their previous learning in another mathematical

domain. Unlike in Indonesia, in some countries such as Singapore, South Korea, and China, algebraic thinking is implicitly inserted into the elementary school curriculum (Ferrucci *et al.*, 2004). Demonty *et al.* (2018) stressed the importance of algebraic thinking concepts to teach primary and secondary schools continuously.

2.4. Theoretical Framework

Kaput (1999) defines generalization as: “deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures, and the relationship across and among them” (p. 136).

El Mouhayar (2018) combined previous research on pattern generalization strategies students used to develop several category frameworks to describe pattern generalization strategies. In this framework, student’s strategies in pattern generalization are (1) counting from a drawing, (2) recursive, (3) chunking, (4) whole-object, and (5) functional. In the counting from a picture, student counts the elements of a particular figural term in a pattern. In the recursive strategy, the student points out the characteristic difference between pairs of consecutive terms and repetitively adds the constant from term to term to extend the pattern. Students multiply the common difference between two terms in the pattern by the number of steps in the chunking strategy and adding the result to the starting figural term. In the whole-object strategy, students identifying the value of a term by using multiples of a previous term. Students find a general term or a term in a specific figural step in the functional strategy without the need to find the preceding figural steps.

2.5. Research Questions

The goal was to investigate the strategies that students use to examine a linear geometric growing pattern. To accomplish the goal, there was an attempt to answer the following research questions. What strategies are used by students to generalize linear geometric growing pattern? and How do they generalize linear geometric growing pattern using these strategies?

3. MATERIALS AND METHODS:

3.1. Cross-Age Study

It was employed a type of cross-sectional study to investigate the students’ strategies used in generalizing patterns. The cross-sectional study was used to produce a “snapshot” of the population at one particular time (Cohen, L., Manion, L., and Morrison, 2012). In education, a cross-sectional study is used to observe changes in the physical and mental development and knowledge of students drawn from representative age levels. One type of cross-sectional study is a cross-age study. Morg and Yörük (2006) stated that the cross-age study could quickly point out the understanding levels of students from different age groups.

3.2 Participants and Instrument

The participant of this study is 156 students (average ages 11-14) from grades 5 to 8 in two private schools, which are located in Semarang city area in the Central Java province Indonesia. Every student declares to agree to participate in this study. All procedures performed in this study regarding samples taken involving human participants were in line with the local institutional of Semarang city education department as the stakeholder and/ or national research committee with the 1964 Helsinki declaration and its later amendments or comparable ethical standards. No formal consent is required for this type of study. These private schools serve students coming from a middle socio-economic status as judged by school tuition levels. The details participants’ information are shown in Table 1.

The students were given a task about a linear geometric growing pattern formed of a growing number of squares problem (Figure 1) adapted from previous studies by El Mouhayar (2018). The participants were requested (a) to find the answer for small step number; (b) to find the answer for large step number, either by writing a correct mathematics rule; (c) to generalize a rule for n th step number, either by writing a correct mathematics rule. The task used for every student was the same in the form of a linear growing number of the square problem, as shown in Figure 1. It aimed to be able to compare the strategies used by upper primary school and upper secondary students in solving the problem of pattern generalization.

3.3 Data Analysis

The data were analyzed with a mixed quantitative-qualitative descriptive approach. First, frequencies and percentages were determined for each of the frequently used strategies. Second, a

cross-tabulation of strategy use by generalization type by grade level was done to explore differences in strategy use variation across grades level. A bar chart also represented the trends in students' use of strategies across the grade level. Third, Student's interview was analyzed qualitatively to understand differences in their strategies to generalize the pattern.

Semi-structured interviews were conducted to focus on the participant's general ability to solve the task and articulate a general mathematics rule. Each participant was interviewed in about 30 minutes. During the interview, participants were asked to solve the task and explain their answers and how they found those answers.

Data collected from written answers and interviews are classified according to the category of strategy in pattern generalization compiled by El Mouhayar (2018) by referring to the strategy of generalizing patterns from the results of previous studies (Lannin *et al.*, 2006; Mouhayar and Jurdak, 2014; Rivera, 2010; Stacey, 1989), as shown in Table 2. An additional category called "incorrect" is used for wrong answers without using any strategy and non-responded problems.

4. RESULTS AND DISCUSSION:

4.1. Strategies of the Students Used in Generalizing the Linear Growing Pattern

The generational activity problem is related to a linear growing pattern formed of an increasing number of square problems (Figure 1). The questions are: (1) finding how many squares would the number of the square pattern have on step 7 and explain how to obtain the answer, (2) finding how many squares would the number of the square pattern have on step 50 and explain how to obtain the answer, (3) If someone gives any step number, finding the number of squares on that step and explain how to obtain the answer. Furthermore, after the data were analyzed, the differences in the strategies obtained for each grade level students of the school are presented in Table 3 with percentages.

Table 3 shows the differences between upper primary school students and lower secondary school students in generalizing linear growing patterns. As can be seen from Table 3, The students used four strategies to solve the problem. In problem 1, the recursive category is the most frequently used strategy (27.6%) among the students, followed by the strategy (26.3%), chunking strategies (19.9%), and counting from

drawing strategies (5.1%), respectively. However, the students did not use the whole-object strategy at all. A total of 21.2 % of the respondents gave the wrong answer without using any strategy or did not respond to the problem. In problem 2, the students used four strategies to generalize the pattern. The functional strategy was the most frequently used category (26.3%) among the students, followed by the recursive strategy (18.6%), chunking strategy (18.6%), and counting from drawing (3.2%), respectively. However, the students did not use counting from drawing strategies at problem 2. Besides, (30.1%) of the students gave the wrong answers or did not respond. In problem 3, the students used four strategies to generalize the pattern. The functional strategy was the most frequently used category (21.8%) among the students, followed by the recursive strategy (19.2%), chunking strategy (18.6%), and counting from drawing (3.2%), respectively. However, the students did not use counting from drawing strategies at 1b and 1c problems. Besides, there were (21.0%) of the students in lower secondary school gave wrong answers or did not respond at all.

The results indicated that students used different pattern generalization strategies. The student generally preferred recursive, chunking, and functional strategy in each generalization task, whereas few used counting from drawing strategies to generalize patterns. For problem 1, The trend of variation of the counting from drawing strategy increased across grades 5-6, and it decreased across grades 6-8. The trend of variation of the recursive strategy decreased across grades 5-8. The trend of variation of the chunking strategy unchanged across grades 5-6, and then it increased across grades 6-8. The variation of the functional strategy increased across grades 5-6, decreased across grades 6-7 and then increased again across grades 7-8. Figure 2 shows the trends of variations in the frequency of usage of the strategies for the problems 1.

For problem 2, The trend of variation of the counting from drawing strategy decreased across grades 5-8. The trend of variation of the recursive strategy decreased across grades 5-6, increased across grades 6-7, and then it decreased across grades 7-8. The trend of variation of the chunking strategy increased across grades 5-7, and then it decreased across grades 7-8. The variation of the functional strategy increased across grades 5-6, decreased across grades 6-7, and then increased again across grades 7-8. Figure 3 shows the trends of variations in the frequency of usage of

the strategies for the problems 2.

For problem 3, The trend of variation of the counting from drawing strategy decreased across grades 5-8. The trend of variation of the recursive strategy decreased across grades 5-6, increased across grades 6-7, and decreased across grades 7-8. The trend of variation of the chunking strategy increased across grades 5-7, and then it decreased across grades 7-8. The trend of variation of the functional strategy unchanged across grades 5-6, and then it increased across grades 6-8. Figure 4 shows the trends of variations in the frequency of usage of the strategies for the problems 3.

4.2 How the Student Generalize a Linear Growing Figural Pattern

After doing the test, the students were interviewed to recheck the understanding of the researcher toward the student's written work. The interview was given to the fourth focus students, Fina (grade 5), Andi (grade 6), Lia (grade 7) and Ziki (grade 8). Figure 5 shows a sample from Fina (grade 5) using counting from drawing strategy. She tried by drawing step 7 and counting its squares without seeing the structure and the relationship between the step and the number square. Fina said if I counted the squares until step 7, I would get 17 squares. During the interview, the researcher asked what will happen with the number squares of step 50. She still uses the same strategy with drawing until step 50 to get the answer, but this strategy makes her tired if she has to draw and count the square until step 50 or more.

On the one hand, Andi (grade 6) used a recursive strategy by looking at the common differences between pairs of consecutive steps and adding the constant from step to step to get the answer. Based on Figure 6, it can be observed that Andi use a recursive strategy preferred to move from visual pattern to symbol representation by using numbers. Becker and Rivera (2006) reported that students generally tend to change visual patterns into rows of numbers. Hence, they adopt an approach numeric by finding the common difference of visual pattern in each step. Andi found the number of squares in step 7 by repeatedly adding common difference 2 from step 4 until step 7. When the researcher asked him to find the number of squares in the far step, he got difficult if he has to add 2 until any step. For example:

[1] Researcher: How you get to step 7?

[2] Andi: Look at this. I found the number of squares in step 7 by

repeatedly adding 2 from step 4 until step 7.

[3] Researcher: Why do you add 2 in each step?

[4] Andi: Here 5, 7, 9, 11, ... the difference in every step is 2.

So I add every step by 2.

[5] Researcher: How about step 50? Are you still using the same strategy?

[6] Andi: Yes, I keep adding until step 50 to get the answer. So the number of squares in step 50 is 103

[7] Researcher: How about this problem? (*pointed to the third question*) If I ask you, how about step 100, 1000, or any?

[8] Andi: I can add until any step 2, but it would be complicated.

(*Student 2 was thinking and realized to find another strategy*).

On the other hand, as shown in Figure 7, Lia observed the pattern and recognized multiplicative and additive relationships in several pattern steps. In this first problem, she used chunking strategies to solve the problem. She also described a chunking rule for the n th term. "I multiplied the number of any step minus 4 by two, and added the number of step 4. For example, to get the number of step 50, I multiply the number 50 minus 4 by two and I add the 11 of step 4. So the number of step 50 is 103." Furthermore, Ziki (grade 8) completed the problem using a functional strategy. He tried to observe the structure of the pattern and the relationship between the step. He related the number of steps to the number of above squares and bottom squares with gave an invisible marked in steps, as shown in Figure 8.

To obtain the answer in step 4, the number of above squares is 4, the number of bottom squares is 4. Then he added 3 (the different of each step). He also described a functional rule for the n th term. A common response of Ziki is shown below.

[1] Researcher: How did you get the number of squares in step 7?

[2] Ziki: This is 4 and 4 in step 4 then I added 3, this is 3 and 3 then I added 3, this is 2 and 2 then I added 3, and this is 1 and

1 in step 1 then I added 3 (gave an invisible marked in step 4 until 1). So to obtain the answer in step 7 was $7 + 7 + 3$,

[3] Researcher: How about step 50?

[4] Ziki : $50 + 50 + 3$

[5] Researcher: If I ask you to find the number of squares in any step, how do you find it?

[6] Ziki: The number of steps + the number of steps, and then I added 3.

[7] Researcher: If you use n to represent any step, how do you calculate?

[8] Ziki: I got the formula for any step is $n + n + 3$

The first result indicated that students used different pattern generalization strategies. The student generally preferred recursive, chunking, and functional strategy in each generalization task, whereas few used counting from drawing strategies to generalize pattern. According to research Mouhayar and Jurdak (2014), grades 4-11 students frequently use recursive and functional strategies in each generalizing pattern. Their study supports the results of our study of the strategies students use in generalizing patterns. On the other hand, the chunking strategy was frequently used in lower secondary school students (grades 7-8) and a few of the students used counting from drawing strategy. The explanation for this result may be due to the student's ability to observe the pattern and recognize the pattern's structure and the relationship between the step.

The second result of this study is that the trend of variation of the recursive strategy decreased, whereas, the trend of variation of the chunking strategy and the functional strategy increased across grades 5-8 for the problems, as the demands of the problem changed from constructing a step-by-step answer to finding a general formula. The explanation of this result may be due to the nature of the problem (type of generalization) and that of the strategy being used to generalize the pattern. If students can find general formulas from patterns, then they apply strategies according to their choices, but if they cannot find general formulas, then they tend to use recursive strategies to find a near step (problem 1). On the other hand, to find a far step (problem 2 and 3) the student does not use recursive because they know it is difficult for them. Compared to study

on students' strategies across generalization types, this result supports findings of research studies that suggest that while the recursive strategy is more frequently used near step generalizations, the chunking and functional strategy is more frequently used in far step generalizations. This study's results are in line with findings from the research of Mouhayar and Jurdak (2014). In general, students are inclined to use recursive strategies to get near steps while they prefer to look for general formulas and use chunking and functional strategies to get distant steps.

Another explanation for this result may be due to experience and age of the students. The present study showed that with age and experience the strategy choices of the students become more adaptive with increasingly adaptive choices of the more effective strategies. This explanation is supported with findings from the previous research of Siegler (2000). This study focused on variation of strategy to show that experience enables students to modify their strategies and ways of reasoning in a way they show a higher dependency on the more advanced strategies that exist in their repertoire (in this case the functional strategy) and lower dependency on less advanced strategy (the recursive strategy in this case). According to the Indonesian curriculum, the students in this research were indirectly exposed to pattern generalization through their exposure to algebraic concepts such as variables, equations, algebraic expressions in grade 7 and to functions and their representations starting from grade level 8. Consequently, the experience of the students to these pattern-related topics may explain the decrease of the recursive strategy and the increase of use of the chunking and functional strategy across grade level.

The third result of this study was the recursive and chunking strategy student preferred to move from visual pattern to symbol representation by using numbers. Becker and Rivera (2006) reported that students generally tend to change visual patterns into rows of numbers. Hence, they adopt an approach numeric by finding the common difference of visual pattern in each step. According to research from Akkan and Çakiroğlu (2012), students are more successful in solving patterns in the form of rows of numbers rather than visual patterns.

5. CONCLUSIONS:

The student generally preferred recursive, chunking, and functional strategy in each

generalization task, whereas few used counting from drawing strategies to generalize pattern. The trend of variation of the recursive strategy decreased, whereas, the trend of variation of the chunking strategy and the functional strategy increased across grades 5-8 for the problems, as the demands of the problem changed from constructing a step-by-step answer to finding a general formula. The student who used the recursive and chunking strategy preferred to change visual patterns into rows of numbers. Hence, they adopt a numeric approach by finding the common difference of visual pattern in each step. The student chose a recursive strategy or functional strategy to find near the step, while they prefer to use chunking and functional strategy to look for distant step.

Some students are not successful in making correct generalizations because they have very little conceptual understanding of the pattern. Therefore, the teacher must use various types of patterns and strategies to enrich student knowledge and also focus on the ability of students to understand the structure and relationships between terms to develop students' algebraic thinking. Involving students in various patterns generalization will increase their variety of more effective strategies and their skill to choose between these strategies. Future research should focus on generalizing strategies toward the generalization of linear visual pattern problem and generalizing non-linear visual pattern, which also encourages the students to think about more advanced strategy.

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Table 1. Participant Information

Grade	Male	Female	Total
5	15	19	34
6	19	22	41
7	20	18	38
8	18	25	43
Total	72	84	156

Table 2. A framework of Strategy In Pattern Generalization

Strategy	Description
Counting from a draw	Counting the elements of a particular figural term in a pattern
Recursive	Pointing the common difference between pairs of consecutive terms and repetitively adding the constant from term to term to extend the pattern
Chunking	Multiply the common difference between two terms in the pattern by the number of steps and adding the result to starting figural term
Whole-object	Identify the value of a term by using multiples of a previous term.
Functional	Students find a general term or a term in a specific figural step without finding the preceding figural steps.

Table 3. Strategies Used for The Linear Geometric Growing Pattern of Squares

			Grade Level				Total
			5	6	7	8	
Problem 1	Incorrect	Count	6	12	7	8	33
		% within Grade Level	17.6%	29.3%	18.4%	18.6%	21.2%
	Counting from drawing strategy	Count	3	4	1	0	8
		% within Grade Level	8.8%	9.8%	2.6%	0.0%	5.1%
	Recursive strategy	Count	8	9	8	18	43
		% within Grade Level	23.5%	22.0%	21.1%	41.9%	27.6%
	Chunking strategy	Count	4	4	11	12	31
		% within Grade Level	11.8%	9.8%	28.9%	27.9%	19.9%
	Functional strategy	Count	13	12	11	5	41
		% within Grade Level	38.2%	29.3%	28.9%	11.6%	26.3%
Total		Count	34	41	38	43	156
		% within Grade Level	100.0%	100.0%	100.0%	100.0%	100%
Problem 2	Incorrect	Count	11	18	9	9	47
		% within Grade Level	32.4%	43.9%	23.7%	20.9%	30.1%
	Counting from drawing strategy	Count	3	2	0	0	5
		% within Grade Level	8.8%	4.9%	0.0%	0.0%	3.2%
	Recursive strategy	Count	12	8	9	5	34
		% within Grade Level	35.3%	19.5%	23.7%	11.6%	21.8%
	Chunking strategy	Count	3	4	12	10	29
		% within Grade Level	8.8%	9.8%	31.6%	23.3%	18.6%
	Functional strategy	Count	5	9	8	19	41
		% within Grade Level	14.7%	22.0%	21.1%	44.2%	26.3%
Total		Count	34	41	38	43	156
		% within Grade Level	100.0%	100.0%	100.0%	100.0%	100 %
Problem 3	Incorrect	Count	11	25	10	12	58
		% within Grade Level	32.4%	61.0%	26.3%	27.9%	37.2%
	Counting from drawing strategy	Count	3	2	0	0	5
		% within Grade Level	8.8%	4.9%	0.0%	0.0%	3.2%
	Recursive strategy	Count	12	5	8	5	30
		% within Grade Level	35.3%	12.2%	21.1%	11.6%	19.2%
	Chunking strategy	Count	3	4	12	10	29
		% within Grade Level	8.8%	9.8%	31.6%	23.3%	18.6%
	Functional strategy	Count	5	5	8	16	34
		% within Grade Level	14.7%	12.2%	21.1%	37.2%	21.8%
Total		Count	34	41	38	43	156
		% within Grade Level	100.0%	100.0%	100.0%	100.0%	100%

1. What is the number of squares in step 7? show how you obtained your answer?
2. What is the number of squares in step 50? Show how you obtained your answer?
3. If someone give any step number, what is the number of squares on that step and explain how to obtain the answer?

Figure 1. A Linear growing number of square problems.

Figure 2. Trends of variation of the strategies across grades 5-8 for problems 1.

Figure 3. Trends of variation of the strategies across grades 5-8 for the problems 2.

Figure 4. Trends of variation of the strategies across grades 5-8 for problems 3

Figure 5. Fina used counting from drawing strategy to solving part an of the task

Diagram showing the first five steps of a square pattern:

- Tahap 1: 5 squares
- Tahap 2: 7 squares
- Tahap 3: 9 squares
- Tahap 4: 11 squares
- Tahap 5: 13 squares

a) Berapa jumlah persegi ditahap 7? Jelaskan bagaimana kamu mendapatkan jawaban tersebut?

17 persegi

Tahap 6: $13 + 2 = 15$
Tahap 7: $15 + 2 = 17$

Translation:
a) What is the number of squares in step 7? show how you obtained your answer? **17 squares**
Step 6: $13 + 2 = 15$
Step 7: $15 + 2 = 17$

b) Berapa jumlah persegi ditahap 50? Jelaskan bagaimana kamu mendapatkan jawaban tersebut?

103 persegi

8: $17 + 2 = 19$	12: $25 + 2 = 27$	16: $33 + 2 = 35$
9: $19 + 2 = 21$	13: $27 + 2 = 29$	20: $35 + 2 = 37$
10: $21 + 2 = 23$	14: $29 + 2 = 31$	24: $37 + 2 = 39$
11: $23 + 2 = 25$	15: $31 + 2 = 33$	28: $39 + 2 = 41$

36: $63 + 12 = 75$
40: $75 + 8 = 83$
50: $83 + 20 = 103$

Translation:
b) What is the number of squares in step 50? show how you obtained your answer? **103 squares.**

Figure 6. Andi used recursive strategy to solve part a and b of the task

a) Berapa jumlah persegi ditahap 7? Jelaskan bagaimana kamu mendapatkan jawaban tersebut?

Jawab: $2 \times 3 = 6$ $6 + 11 = 17$

Saya kalikan jumlah Tahap dengan 2 dan kemudian menambahkan dengan Tahap 4

Translation:
a) What is the number of squares in step 5? show how you obtained your answer?
I multiply the number step by 2 and I add by the number of step 4.

b) Berapa jumlah persegi ditahap 50? Jelaskan bagaimana kamu mendapatkan jawaban tersebut?

Jawab: $2 \times 46 = 92$ $92 + 11 = 103$

UNTUK mendapatkan jumlah persegi ke-50, saya kalikan dengan jumlah Tahap 2 dan menambahkan dengan Tahap 4

Translation:
b) What is the number of squares in step 5? show how you obtained your answer?
To get the number of step 50, I multiply the number of step by 2 and I add by the number of 4.

c) Jika ditanya tahap sembarang, jelaskan bagaimana kamu menemukan jumlah persegi ditahap tersebut? jawab: Untuk mendapatkan jumlah persegi ditahap sembarang, saya kalikan jumlah Tahap 2 dan menambahkan dengan Tahap 4

Translation:
c) If someone gives any step number, finding the number of squares on that step and explain how to obtain the answer.
To get the number of any steps, I multiply the number of step by 2 and I add by the number of 4.

Figure 7. Lia used a chunking strategy to solve part a, b, and c of the task

Figure 8. *The illustration of Ziki's invisible marks in step 4.*