SOLUÇÃO ANALÍTICA DA EQUAÇÃO DO SCHRÖDINGER PARA O POTENCIAL DE YUKAWA COM MASSA VARIÁVEL EM COORDENADA TOROIDAL USANDO MECÂNICA QUÂNTICA SUPERSIMÉTRICA

ANALYTICAL SOLUTION OF SCHRÖDINGER EQUATION FOR YUKAWA POTENTIAL WITH VARIABLE MASS IN TOROIDAL COORDINATE USING SUPERSYMMETRIC QUANTUM MECHANICS

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RESUMO

A equação de Schrödinger em uma coordenada toroidal foi proposta na física teórica para obter as informações e o comportamento do sistema de partículas. Foi resolvido recentemente, no caso de uma partícula escalar carregada interagindo com um campo magnético uniforme, um campo elétrico uniforme e uma carga neutra restringida à superfície. A metodologia usada no referido trabalho foi resolver a equação de Schrödinger usando uma abordagem descrita no tratado Whittaker-Watson que lida com um problema de valor próprio infinito-dimensional e com valores particulares específicos do campo aplicado para o problema de função própria, enquanto no problema da mecânica quântica um tinha um problema de autovalor generalizado de dimensão infinita. Este estudo teve como objetivo obter o valor próprio de energia não-relativística e a função de onda radial da equação de Schrödinger sob a influência do potencial de Yukawa. O método da Mecânica Quântica Supersimétrica (SUSY QM) foi usado como base para abordar o objetivo principal deste artigo, estudar o problema de uma partícula com massa variável em coordenadas toroidais. O super potencial adequado foi usado para lidar com a forma hiperbólica do potencial efetivo e os espectros de energia foram calculados para diferentes números quânticos, profundidade potencial e parâmetros potenciais. A equação da função de onda radial para o solo e o estado excitado foi obtida. Os resultados mostraram que o valor crescente dos números quânticos fez com que os espectros de energia do sistema aumentassem para o valor mais alto quando o número quântico era igual ao parâmetro potencial, o que significa que o valor energético mais eficaz foi produzido e depois diminuiu. Enquanto o valor da energia não dependia da alteração do parâmetro potencial. Essa propriedade poderia ser usada para produzir essa equação como uma aplicação dos resultados anteriores. A função própria de Schrödinger foi usada como ponto de partida para resolver a outra equação no mesmo cenário e potencial geométrico.

Palavras-chave: *Valor próprio da energia, espectros de energia, função de onda no estado fundamental*

ABSTRACT

Schrodinger equation on a toroidal coordinate was proposed in theoretical physics to get the information and the behavior of the system of particle. It was solved just recently in case of a charged scalar particle interacting with a uniform magnetic field, a uniform electric field, and a neutral charge constrained to the surface. The methodology used in the referred work was to solve the Schrodinger equation using an approach outlined in the Whittaker-Watson treatise, which deals with an infinite-dimensional eigenvalue problem and specific particular values of the applied field for eigenfunction problem. In contrast, in the quantum mechanical problem, one had an infinite-dimensional generalized eigenvalue problem. This study aimed to obtain the non-relativistic energy eigenvalue and the radial wave function of the Schrodinger equation under the influence of Yukawa potential. The Supersymmetric Quantum Mechanics (SUSY QM) method was used as a basis to tackle the primary objective of this paper to study the problem of a particle with variable mass in toroidal coordinate. The proper super potential was used to deal with the hyperbolic form of effective potential, and the energy spectra were calculated for different quantum numbers, potential depth, and potential parameters. The radial wave function equation for ground and excited state were obtained. The results showed that the increasing value of the quantum numbers caused the energy spectra of the system to increase to the highest value when the quantum number was equal to the potential parameter, which means the most effective energy value was produced, then it was decreased

afterward. While the energy value did not depend on the change of the potential parameter. This property could be used to produce this equation as an application of the previous results, the Schrödinger eigenfunction was used as the starting points to solve the other equation in the same geometrical setting and potential.

Keywords: Energy eigenvalue, energy spectra, ground-state wave function

1. INTRODUCTION:

There had been increasing interest in theoretical physics to obtain the exact solution of the Schrodinger equation in spacetimes with non-spherical topology and negative cosmological constant (Krisch and Glass, 2003). The main purposes of this interest were to obtain the energy eigenvalue and the wave function. It was important to understand because it contained all the information about the quantum system. However, SUSY QM allowed one to determine eigenvalues and eigenstates of known analytically solvable potentials using algebra operator formalism without ever had to solve the Schrödinger differential equation by standard series method (Cari *et al.*, 2014). The other methods have been used to solve the Schrodinger equation, such as the Hypergeometric method (Dianawati *et al.*, 2018), Nikiforov-Uvarov method (Onate *et al.*, 2018), and Asymptotic Iteration Method (Elviyanti *et al.*, 2018).

The method of separation of variables in various coordinate systems was a classic approach to find the exact solutions of the Schrodinger equation that had been thoroughly studied. One such set of coordinates was the toroidal system, but it will be argued that some of the usefulness of this coordinate system had been hidden. It was because, while the usual way of separating the variables was appropriate for some situations, there was another way that more suited to a certain class of problems, in particular, some interesting problems in electrostatics (Andrews, 2006)

The toroidal coordinates (Moon and Spencer, 1971) of any point were given by the intersection of a torus, a sphere with its center on the axis of the torus (the z-axis), and an azimuthal half-plane (terminated by the z-axis). A toroid with a circular cross-section was a torus, so a torus was a toroid, and tori were toroids (Lucht, 2016). The radius and center of the sphere were determined by the spherical coordinate μ , the major and minor radii of the torus were given by the toroidal coordinate η , and the particular half-plane was specified by its azimuthal angle ϕ .

The solutions for the Schrodinger equation with a position-dependent mass lie at the center of

interest of the professional public because it helped us to understand the behavior of quantum particles in the cases in which their mass varies spatially (Bednarik and Cervenka, 2017). For instance, the variable mass Schrodinger equation played an essential role in the study of the electronic properties of inhomogeneous crystals, quantum dots, and quantum liquids (Golubov and Mazin, 1997). A suitable choice of a mass position function of the exponential-like form had been devised (Meyur, *et al.*, 2014). However, the application of variable mass in the Schrodinger equation was needed to be studied in a significant way.

Energy eigenvalue and wave function of non-relativistic particle have been solved using Schrodinger equation in the presence of many potentials such as Eckart potential (Goudarzi and Vahidi, 2011), Poschl-Teller (Suparmi *et al.*, 2012), Hylleraas (Suparmi *et al.*, 2017), and ring-shaped pseudo-harmonic oscillatory (Oliveira and Schmidh, 2019). The energy eigenvalue and wave function also could be used to determine other properties and further information of the system such as thermodynamics properties (Adebimpe *et al.*, 2019), entropy information (Faniandari *et al.*, 2019; Ma'arif *et al.*, 2019) and optical properties (Falkovsky, 2008).

The hyperbolic form of Yukawa potential was put into the formulation of the radial part of the Schrodinger equation in toroidal coordinates. The supersymmetric quantum mechanics method and variable mass were used. Therefore, this study aimed to obtain the energy eigenvalue, the un-normalized ground state wave function, and the exciting wave function of the system.

2. MATERIALS AND METHODS:

2.1. Schrödinger Equation with Variable Mass

The Laplacian in this coordinate was given by:

$$\nabla^2 \Psi = \frac{1}{h_\mu^3} \left[\frac{1}{\sinh \mu} \frac{\partial}{\partial \mu} \left(h_\mu \sinh \mu \frac{\partial \Psi}{\partial \mu} \right) + \frac{\partial}{\partial \eta} \left(h_\eta \frac{\partial \Psi}{\partial \eta} \right) + \frac{h_\mu}{\sinh^2 \mu} \frac{\partial^2 \Psi}{\partial \phi^2} \right] \quad (\text{Eq. 1})$$

where

$$\Psi = \sqrt{\cosh \mu - \cos \eta} F(\mu, \eta, \phi) \quad (\text{Eq. 2})$$

$$h_\mu = h_\theta = \frac{a}{\cosh \mu - \cos \eta} \quad (\text{Eq. 3a})$$

$$h_\phi = \frac{a \sinh \mu}{\cosh \mu - \cos \eta} \quad (\text{Eq. 3b})$$

After applying the general wave function and scaling factors, the Laplacian could be simplified, so the Schrodinger equation with energy eigenvalue and potential was defined as

$$\left\{ \frac{(\cosh \mu - \cos \eta)^2}{a^2} \left[\frac{1}{\sinh \mu} \frac{\partial}{\partial \mu} \left(\sinh \mu \frac{\partial F}{\partial \mu} \right) + \frac{\partial^2 F}{\partial \eta^2} \right] + \frac{1}{\sinh^2 \mu} \frac{\partial^2 F}{\partial \phi^2} + \frac{1}{4} F \right\} - \frac{2m}{\hbar^2} VF(\mu, \eta, \phi) = -\frac{2m}{\hbar^2} EF(\mu, \eta, \phi) \quad (\text{Eq. 4})$$

The variable mass was proposed as

$$m \approx \frac{m_0 (\cosh \mu - \cos \eta)^2}{a^2} \quad (\text{Eq. 5})$$

Figure 1. Plot for mass function m with the variation of η , $m_0 = 1$, $a = 1$

Figure 2. Plot for mass function m with the variation of μ , $m_0 = 1$, $a = 1$

The non-central potential was applied to Eq. (4) which expressed by

$$V(\mu, \eta, \phi) = V(\mu) + V(\eta) + \frac{V(\phi)}{\sinh^2 \mu} \quad (\text{Eq. 6})$$

Generally, the Schrodinger equation could be solved by reducing the system of particles to the second-order differential equation. For the wave function

$$F = M(\mu)H(\eta)P(\phi) \quad (\text{Eq. 7})$$

The angular parts were

$$\frac{\partial^2 H}{\partial \eta^2} - \frac{2m_0}{\hbar^2} V(\eta)H = \lambda_1 H \quad (\text{Eq. 8})$$

$$\frac{\partial^2 P}{\partial \phi^2} - \frac{2m_0}{\hbar^2} V(\phi)P = \lambda_2 P \quad (\text{Eq. 9})$$

and the radial part was

$$\begin{aligned} & -\frac{1}{\sinh \mu} \frac{\partial}{\partial \mu} \left(\sinh \mu \frac{\partial M}{\partial \mu} \right) + \frac{2m_0}{\hbar^2} V(\mu)M \\ & -\lambda_1 M - \left(\frac{1}{4} + \frac{2m_0 E}{\hbar^2} \right) M = \frac{\lambda_2}{\sinh^2 \mu} M \end{aligned} \quad (\text{Eq. 10})$$

The solution was simplified to be

$$\left\{ -\frac{\hbar^2}{2m_0} \frac{\partial^2 K}{\partial \mu^2} + \frac{\hbar^2}{2m_0} \frac{-\lambda_2 - \frac{1}{4}}{\sinh^2 \mu} K \right\} \quad (\text{Eq. 11})$$

$$+V(\mu)K - \lambda_1 \frac{\hbar^2}{2m_0} K - EK = 0$$

where $M = \frac{K}{\sqrt{\sinh \mu}}$.

2.2. Supersymmetric Quantum Mechanics Method

Quantum mechanics, in its development, had brought the concept of symmetry as a foundation for thinking to simplify physical systems. The concept of symmetry mathematically means the invariant nature of a system when subject to a particular transformation. Closed algebra under the anti-commutation relationship was introduced in the supersymmetry system. The Supersymmetry system in quantum mechanics was defined as a system consisting of a supercharge operator Q_i which was commutative towards Hamiltonian Supersymmetry (H_{SS}) (Witten, 1981).

$$[Q_i, H_{SS}] = 0 \text{ and } i = 1, 2, \dots, N \quad (\text{Eq. 12})$$

where N was the number of generators and meets the anti-commutation relationship below

$$\{Q_i, Q_j\} = \delta_{ij} H_{SS} \quad (\text{Eq. 13})$$

Hamiltonian Supersymmetry (H_{SS}) was defined as the sum of squares of the supercharge operator (Q_i):

$$H_{SS} = 2Q_1^2 = 2Q_2^2 \quad (\text{Eq. 14})$$

The simplest SUSY QM system, where $N = 2$ with supercharge operators Q_1 and Q_2 could be associated with spin 1/2 particles moving in a straight line. Q_1 and Q_2 in this simple system were defined as follows:

$$Q_1 = \frac{1}{\sqrt{2}} \left(\sigma_1 \frac{p}{\sqrt{2m}} + \sigma_2 \phi(x) \right) \quad (\text{Eq. 15a})$$

$$Q_2 = \frac{1}{\sqrt{2}} \left(\sigma_2 \frac{p}{\sqrt{2m}} - \sigma_1 \phi(x) \right) \quad (\text{Eq. 15b})$$

where σ_1 and σ_2 were matrices from Pauli (Suparmi, 2011).

H_{SS} was Hamiltonian supersymmetry, $p = -i\hbar \frac{d^2}{dx^2}$ was linear momentum (bosonic

momentum), x was bosonic coordinate, and $\phi(x)$ was bosonic superpotential. Hamiltonian Supersymmetry matrix could be shown as:

$$H_{SS} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (\text{Eq. 16})$$

where

$$H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} \quad (\text{Eq. 17a})$$

$$H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \phi^2(x) + \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} \quad (\text{Eq. 17b})$$

Eq. (12) caused the results in the generation of degenerated energy H_- and H_+ , which were SUSY partners of the Fermionic Hamiltonian (descendant) and Bosonic (ascending), both of which were also written as H_{SS} . The standard Schrödinger equation was stated in the Hamiltonian SUSY as follows

$$H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x) \quad (\text{Eq. 18a})$$

$$H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+(x) \quad (\text{Eq. 18b})$$

where $V_-(x)$ and $V_+(x)$ were called as supersymmetry potential couple, $\phi(x)$ was superpotential, while $\phi'(x)$ was the first derivative of $\phi(x)$.

Lowering and raising Hamiltonian could be factorized from the equation (18a) and (18b) and respectively were stated as follows:

$$A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \quad (\text{Eq. 19a})$$

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \quad (\text{Eq. 19b})$$

A^+ was raising operator and A was lowering operator (Rodrigues, 2002).

Based on the character of lowering operator (A), if (A) was operated on the ground state wave function $\psi_0^{(-)}$, it will equal to zero (Cooper *et al.*, 1995).

$$A\psi_0^{(-)} = \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \right) \psi_0^{(-)} = 0 \quad (\text{Eq. 20})$$

From Eq. (20), then the ground state wave function was determined as

$$\psi_0^{(-)}(x) = N \exp \left[-\frac{\sqrt{2m}}{\hbar} \int_{-x}^x \phi(x) dx \right] \quad (\text{Eq. 21})$$

with N was the normalization factor.

3. RESULTS AND DISCUSSION:

The radial part of the Schrodinger equation with energy eigenvalue and potential was expressed as

$$-\frac{\hbar^2}{2m_0} \frac{\partial^2 K}{\partial \mu^2} + \frac{\hbar^2}{2m_0} \left[-\frac{1}{4 \sinh^2 \mu} - \frac{\lambda_2}{\sinh^2 \mu} \right] K + V(\mu)K = \varepsilon' K \quad (\text{Eq. 22})$$

$$\text{with } \varepsilon' = \lambda_1 \frac{\hbar^2}{2m_0} + E.$$

Yukawa potential was used and it's general form was (Yukawa, 1935).

$$V(\rho) = -V_0 \frac{\hbar^2}{2m_0} \left(\frac{e^{-\xi \rho}}{\rho} \right) \quad (\text{Eq. 23})$$

By set $\xi \rho = \mu$

$$V(\mu) = -V_0 \xi \frac{\hbar^2}{2m_0} \left(\frac{e^{-\mu}}{\mu} \right) \quad (\text{Eq. 24})$$

The following approximation for centrifugal terms was applied (Greene and Aldrich, 1976; Qiang and Dong, 2007; Jia, *et al.*, 2009; Dong, *et al.*, 2013).

$$\frac{1}{\mu^2} \approx 4\xi^2 \frac{e^{-2\xi\mu}}{(1 - e^{-2\xi\mu})^2} \quad (\text{Eq. 25})$$

$$\frac{1}{\mu} \approx \frac{2\xi e^{\mu}}{e^{2\mu} - 1} \quad (\text{Eq. 26})$$

Then Eq. (24) was modified as

$$V(\mu) = -\frac{\hbar^2}{2m_0} V_0 \xi^2 (\coth \mu - 1) \quad (\text{Eq. 27})$$

Eq. (27) was inserted into Eq. (22)

$$\begin{aligned} & -\frac{\hbar^2}{2m_0} \frac{\partial^2 K}{\partial \mu^2} + \frac{\hbar^2}{2m_0} \left[\frac{v(v-1)}{\sinh^2 \mu} - V_0 \xi^2 \coth \mu + V_0 \xi^2 \right] K \\ & = \left(\lambda_1 \frac{\hbar^2}{2m_0} + E \right) K \end{aligned} \quad (\text{Eq. 28})$$

$$\text{where } -\frac{1}{4} - \lambda_2 = v(v-1).$$

This was the second-order differential equation of the Schrodinger equation. The shape invariant potential was used, so the hypothetical superpotential function in supersymmetric quantum mechanics was

$$\Phi(\mu) = A \coth \mu + B \quad (\text{Eq. 29})$$

to satisfy the Riccati equation below

$$\Phi^2(\mu) - \frac{\hbar}{\sqrt{2m_0}} \Phi'(\mu) = V_{\text{eff}} - \varepsilon \quad (\text{Eq. 30})$$

$$\left(A^2 + \frac{\hbar}{\sqrt{2m_0}} A \right) \text{csch}^2 \mu + 2AB \coth \mu + A^2 + B^2 \quad (\text{Eq. 31})$$

$$= \frac{\hbar^2}{2m_0} (v(v-1) \text{csch}^2 \mu - V_0 \xi^2 \coth \mu + V_0 \xi^2) - \varepsilon$$

where ε was the lowest energy eigenvalue. Comparing the left and the right side of Eq. (31), the solutions were determined.

$$A = \frac{\hbar}{\sqrt{2m_0}} v \quad (\text{Eq. 32})$$

$$B = -\frac{\hbar}{\sqrt{2m_0}} \frac{V_0 \xi^2}{2v} \quad (\text{Eq. 33})$$

$$\varepsilon = \frac{\hbar^2}{2m_0} \left(V_0 \xi^2 - \frac{(V_0 \xi^2)^2}{4v^2} - v^2 \right) \quad (\text{Eq. 34})$$

With Eq. (29), Eq. (32), Eq. (33), Eq. (18a) and Eq. (18b), the two partner potentials which have an invariant form were constructed as follows:

$$V_-(a_0, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{(v^2 + v) \text{csch}^2 \mu - V_0 \xi^2 \coth \mu}{\left(\frac{(V_0 \xi^2)^2}{4v^2} + v^2 \right)} \right) \quad (\text{Eq. 35})$$

$$V_+(a_0, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{(v^2 - v) \text{csch}^2 \mu - V_0 \xi^2 \coth \mu}{\left(\frac{(V_0 \xi^2)^2}{4v^2} + v^2 \right)} \right) \quad (\text{Eq. 36})$$

Then, a new set of the parameter was defined as $a_1 = v - 1$ from the old set of parameter $a_0 = v$.

$$V_-(a_1, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{(v^2 - v) \operatorname{csch}^2 \mu - V_0 \xi^2 \coth \mu}{+ \frac{(V_0 \xi^2)^2}{4(v-1)^2} + (v-1)^2} \right) \quad (\text{Eq. 37})$$

$$V_+(a_1, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{((v-1)^2 - (v-1)) \operatorname{csch}^2 \mu}{-V_0 \xi^2 \coth \mu + \frac{(V_0 \xi^2)^2}{4(v-1)^2} + (v-1)^2} \right) \quad (\text{Eq. 38})$$

In term of the parameter of the problem, the following relations were obtained:

$$R(a_1) = V_+(a_0, \mu) - V_-(a_1, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{(V_0 \xi^2)^2}{4v^2} - \frac{(V_0 \xi^2)^2}{4(v-1)^2} + v^2 - (v-1)^2 \right) \quad (\text{Eq. 39})$$

$$R(a_2) = V_+(a_1, \mu) - V_-(a_2, \mu) = \frac{\hbar^2}{2m_0} \left(\frac{(V_0 \xi^2)^2}{4(v-1)^2} - \frac{(V_0 \xi^2)^2}{4(v-2)^2} + (v-1)^2 - (v-2)^2 \right) \quad (\text{Eq. 40})$$

The energy eigenvalue was determined as:

$$E_n^{(-)} = \sum_{k=1}^n R(a_k) = \frac{\hbar^2}{2m_0} \left(\frac{(V_0 \xi^2)^2}{4v^2} - \frac{(V_0 \xi^2)^2}{4(v-n)^2} + v^2 - (v-n)^2 \right) \quad (\text{Eq. 41})$$

This gave the energy equation as

$$E_n = E_n^{(-)} + E_0 = \frac{\hbar^2}{2m_0} \left(V_0 \xi^2 - \left(\frac{V_0 \xi^2}{2(v-n)} \right)^2 - (v-n)^2 \right) \quad (\text{Eq. 42})$$

where $\hbar$ was Planck constant, m_0 was the mass of the particle, V_0 was the potential depth, ξ was the potential width, v was the parameter of Yukawa potential, and n was the quantum number.

From Eq. (42), the energy spectra were determined from the energy eigenvalue with the variation of the quantum number, potential depth, and parameter of Yukawa potential in Table 1.

In Figure 3, when $v=1$ and $v=2$ the graph was tended to tilt to the left, while $v=4$ and $v=5$ the graph was tended to tilt to the right. It was

symmetrical at the center when $v=n=3$. Table 1 and Figure 3 showed that the increasing value of the quantum number (n) causing the energy value to increase until it reached the highest value when $n = v$. After that, the energy value was decreased for the system in a toroidal coordinate. It also had a symmetric curve and when $n = v$ it became the axis of symmetry.

Figure 3. Energy spectra of Yukawa potential in toroidal coordinate ($V_0 = 0$, $\hbar = 1$, $m_0 = 1$, $\xi = 0,01$)

For the increasing value of the potential parameter v , the energy both increased and decreased. So, practically the potential parameter did not have a direct impact on the value of the energy.

To obtain the un-normalized wave function, the superpotential was defined again as stated in Eq. (29).

$$\Phi(\mu) = \frac{\hbar}{\sqrt{2m_0}} \left(v \coth \mu - \frac{V_0 \xi^2}{2v} \right) \quad (\text{Eq. 43})$$

The relation of shape invariance in SUSY QM was used and the raising and lowering operators were obtained from Eq. (43).

$$A^\dagger = -\frac{\hbar}{\sqrt{2m_0}} \frac{d}{d\mu} + \frac{\hbar}{\sqrt{2m_0}} \left(v \coth \mu - \frac{V_0 \xi^2}{2v} \right) \quad (\text{Eq. 44})$$

$$A = \frac{\hbar}{\sqrt{2m_0}} \frac{d}{d\mu} + \frac{\hbar}{\sqrt{2m_0}} \left(v \coth \mu - \frac{V_0 \xi^2}{2v} \right) \quad (\text{Eq. 45})$$

So, the ground state wave function $\Psi_0^{(-)}$ was obtained

$$\left\{ \frac{\hbar}{\sqrt{2m_0}} \frac{d}{d\mu} + \Phi(\mu) \right\} \Psi_0^{(-)} = 0 \quad (\text{Eq. 46})$$

$$\Psi_0^{(-)} = \frac{CV_0 \xi^2 \mu}{2\nu} \sinh^{-\nu} \mu \quad (\text{Eq. 47})$$

The raising operator in Eq. (44) was operated to the ground state wave function $\Psi_0^{(-)}$ in Eq. (47) and the un-normalized excited wave function was determined.

$$\Psi_1^{(-)}(x; a_0) = \Psi_0^{(-)} \frac{\hbar}{\sqrt{2m_0}} \frac{\nu}{\nu+1} \left(\frac{\sinh \mu}{\nu \coth \mu - \frac{V_0 \xi^2}{2\nu} - \frac{1}{\mu}} + (v+1) \cosh \mu \right) \quad (\text{Eq. 48})$$

4. CONCLUSIONS:

The analytical solution of the Schrodinger equation using Yukawa potential in the form of hyperbolic function in the toroidal coordinate was obtained. The energy spectra and wave function were solved with the Supersymmetric Quantum Mechanics method (SUSY QM). The energy spectra were reported numerically based on the energy eigenvalue equation. The un-normalized ground state wave function and the excited state wave function was determined by operating the lowering and raising operator.

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6. REFERENCES:

1. Adebimpe, O., Onate, C. A., Salawu, S. O., Abolanriwa, A., and Lukman, A. F. (2019). Eigensolutions, scattering phase shift and thermodynamic properties of Hulthén-Yukawa potential. *Results in Physics*, 14, 102409. doi: 10.1016/j.rinp.2019.102409
2. Andrews, M. (2006). Alternative separation of Laplace's equation in toroidal coordinates and its application to electrostatics. *Journal of electrostatics*, 64(10), 664-672. doi: 10.1.1.205.5658
3. Bednarik, M., and Cervenka, M. (2017). The exact solution of the Schrödinger equation with a polynomially spatially varying mass. *Journal of Mathematical Physics*, 58(7), 072103. doi: 10.1063/1.4993194
4. Cari, C., Suparmi, S., and Saregar, A. (2016). Solution of the Schrödinger Equation for Trigonometric Scarf Plus Poschl-Teller Non-Central Potential Using Supersymmetry Quantum Mechanics. *INDONESIAN JOURNAL OF APPLIED PHYSICS*, 4(01), 1-13. Retrieved from https://www.researchgate.net/profile/Antomi_Saregar2/publication/303036225
5. Cooper, F., Khare, A., and Sukhatme, U. (1995). Supersymmetry and quantum mechanics. *Physics Reports*, 251(5-6), 267-385. doi: 10.1142/4687
6. Suparmi, A., Dianawati, D. A., and Cari, C. (2019). Solution of Klein-Gordon equation for fraction potential with q-deformed of radial momentum using Romanovski polynomial. In *AIP Conference Proceedings* (Vol. 2202, No. 1, p. 020037). AIP Publishing LLC. doi: 10.1063/1.5054568
7. Dong, S., Sun, G. H., and Dong, S. H. (2013). Arbitrary l-wave solutions of the Schrödinger equation for the screen coulomb potential. *International Journal of Modern Physics E*, 22(06), 1350036. doi: 10.1142/S0218301313500365
8. Elviyanti, I. L., Pratiwi, B. N., Suparmi, A., and Cari, C. (2018). The Application of Minimal Length in Klein-Gordon Equation with Hulthen Potential Using Asymptotic Iteration Method. *Advances in Mathematical Physics*, 2018, 9658679. doi: 10.1155/2018/9658679
9. Falkovsky, L. A. (2008). Optical properties of graphene. In *Journal of Physics: conference series* (Vol. 129, No. 1, p. 012004). IOP Publishing. doi: 10.1088/1742-6596/129/1/012004
10. Faniandari, S., Ma'arif, M., Suparmi, A., and Cari, C. (2019, December). Rényi entropy of ground state eigen function of non-relativistic particle in the presence of DRSCO potential and cosmic string framework. In *AIP Conference Proceedings* (Vol. 2202, No. 1, p. 020031). AIP Publishing LLC. doi: 10.1063/1.5141644

11. Golubov, A. A., and Mazin, I. I. (1997). Effect of magnetic and nonmagnetic impurities on highly anisotropic superconductivity. *Physical Review B*, 55(22), 15146. doi: 10.1103/PhysRevB.55.15146
12. Goudarzi, H., and Vahidi, V. (2011). Supersymmetric approach for Eckart potential using the NU method. *Adv. Studies Theor. Phys*, 5(10), 469-476. Retrieved from https://www.researchgate.net/profile/Hadi_Goudarzi2/publication/267386470
13. Greene, R. L., and Aldrich, C. (1976). Variational wave functions for a screened Coulomb potential. *Physical Review A*, 14(6), 2363. doi: 10.1103/PhysRevA.14.2363
14. Jia, C. S., Chen, T., and Cui, L. G. (2009). Approximate analytical solutions of the Dirac equation with the generalized Pöschl–Teller potential including the pseudo-centrifugal term. *Physics Letters A*, 373(18-19), 1621-1626. doi: 10.1016/j.physleta.2009.03.006
15. Krisch, J. P., and Glass, E. N. (2003). A space–time in toroidal coordinates. *Journal of Mathematical Physics*, 44(7), 3046-3058. doi: 10.1063/1.1580999
16. Lucht, P. (2016). *The Charged Bowl in Toroidal Coordinates*. Utah, UT: Rimrock Digital Technology.
17. Ma'arif, M., Faniandari, S., Suparmi, A., Cari, C., and Pambudi, N. A. (2019, December). Tsallis entropy of ground state classical cosmic string framework with DRSCO potential. In *AIP Conference Proceedings* (Vol. 2202, No. 1, p. 020032). AIP Publishing LLC. doi: 10.1063/1.5141645
18. Meyur, S., Maji, S., and Debnath, S. (2014). Analytical solution of the Schrödinger equation with spatially varying effective mass for generalised Hylleraas potential. *Advances in High Energy Physics*, 2014, 952597. doi: 10.1155/2014/952597
19. Moon, P., and Spencer, D. E. (1971). *Field Theory Handbook*. Berlin, Germany: Springer.
20. Oliveira, M. D., and Schmidt, A. G. (2019). Exact solutions of Schrödinger and Pauli equations for a charged particle on a sphere and interacting with non-central potentials. *Journal of Mathematical Physics*, 60(3), 032102. doi: 10.1063/1.5079798
21. Onate, C. A., Adebimpe, O., Lukman, A. F., Adama, I. J., Okoro, J. O., and Davids, E. O. (2018). Approximate eigensolutions of the attractive potential via parametric Nikiforov-Uvarov method. *Heliyon*, 4(11), e00977. doi: 10.1016/j.heliyon.2018.e00977
22. Qiang, W. C., and Dong, S. H. (2007). Analytical approximations to the solutions of the Manning–Rosen potential with centrifugal term. *Physics Letters A*, 368(1-2), 13-17. doi: 10.1016/j.physleta.2007.03.057
23. Rodrigues, R. D. L. (2002). *The Quantum Mechanics SUSY Algebra: an Introductory Review*. Cajazeiras, Brazil: Federal University of Campina Grande.
24. Suparmi, A., Cari, C., Handhika, J., Yanuarief, C., and Marini, H. (2012). Approximate Solution of Schrodinger Equation for Modified Poschl-Teller plus Trigonometric Rosen-Morse Non-Central Potentials in Terms of Finite Romanovski Polynomials. *IOSR Journal of Applied Physics (IOSR-JAP)*, 2(2), 43-51. Retrieved from <https://www.researchgate.net/publication/258655314>
25. Suparmi, A., Cari, C., Pratiwi, B. N., and Nugraha, D. A. (2017, March). Construction of solvable potential partner of Generalized Hylleraas potential in one-dimensional Schrodinger system. In *Journal of Physics: Conference Series* (Vol. 820, No. 1, p. 012023). IOP Publishing. Retrieved from <http://iopscience.iop.org/1742-6596/820/1/012023>
26. Suparmi, (2011). *Mekanika Kuantum II*. Surakarta, Indonesia: Jurusan Fisika FMIPA UNS.
27. Witten, E. (1981). Dynamical breaking of supersymmetry. *Nuclear Physics B*, 188(3), 513-554. doi: 10.1016/0550-3213
28. Yukawa, H. (1935). On the interaction of elementary particles. I. *Proceedings of the Physico-Mathematical Society of Japan*. 3rd Series, 17, 48-57. doi: 10.11429/ppmsj1919.17.0_48

Table 1. Energy Spectra of Yukawa Potential in Toroidal Coordinate ($\hbar = 1$, $m_0 = 1$, and $\xi = 0,01eV$)

ν	V_0	E				
		$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	1	0,00000050	-0,00499950	-0,01999950	-0,04499950	-0,07999950
	5	0,00000250	-0,00499750	-0,01999750	-0,04499750	-0,07999751
	10	0,00000500	-0,00499500	-0,01999501	-0,04499501	-0,07999502
2	1	-0,00499950	0,00000050	-0,00499950	-0,01999950	-0,04499950
	5	-0,00499850	0,00000150	-0,00499850	-0,01999850	-0,04499850
	10	-0,00499500	0,00000500	-0,00499500	-0,01999501	-0,04499501
3	1	-0,01999950	-0,00499950	0,00000050	-0,00499950	-0,01999950
	5	-0,01999750	-0,00499750	0,00000250	-0,00499750	-0,01999750
	10	-0,01999501	-0,00499500	0,00000500	-0,00499500	-0,01999501