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A stabilized finite element method for calculating balance velocities in ice sheets

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Abstract

We present a numerical method for calculating vertically averaged velocity fields using a mass conservation approach, commonly known as balance velocities. This allows for an unstructured grid, is not dependent of a heuristic flow routing algorithm, and is both parallelizable and efficient. We apply the method to calculating depth-averaged velocities of the Greenland Ice Sheet.

1 Introduction

Balance velocities are useful in evaluating the dynamics of ice-sheets, as a means to fill missing velocity data (e.g. Joughin et al., 2010), and as an additional point of comparison for data-derived and modelled velocities (Bamber et al., 2000). Stemming from a statement of mass conservation, balance velocity provides an intuitive means for understanding the distribution of flux within an ice sheet. It has often provided estimates of velocity with superior fidelity to data than even advanced ice sheet models, while relying on fewer assumptions. It also gives us the means to assess the distance from equilibrium of an extant ice sheet. Heretofore, balance velocity has been calculated by applying discrete routing algorithms to spatially distribute flux. These have traditionally been drawn from the hydrological literature (e.g. Tarboton, 1997; Budd and Warner, 1996). To leading order, hydrological routing and glaciological routing are similar; flow directions in both cases are governed by driving stresses, which are determined by surface slope. In overland routing of liquid water, this method is appropriate. However, in glacial ice the flow direction is also determined by longitudinal stresses (and to a lesser extent, vertical resistive stresses), and neglecting these terms yields an over-convergent pattern. This emphasis on local slopes also tends to exacerbate grid dependence, causing the same routing algorithm to produce markedly different velocity fields for different grid resolutions (LeBrocq et al., 2006). Algorithms overcome this by using a spatially averaged slope rather than purely local slope, with smoothing lengths and the shape of the averaging filter derived heuristically (Testut et al., 2003) or from theoretical results of parameterizing longitudinal stresses (Kamb and Echelmeyer, 1986).

The aim of this paper is to show how balance velocity can be accomplished by solving a partial differential equation for the conservation of mass using finite elements rather than discrete flow routing algorithms. This approach confers a number of advantages. First, non-standard boundary conditions can easily be applied. For example, when balance velocity is used as a method for covering gaps in interferometric velocities (as in Joughin et al., 2010), velocity can be specified at arbitrary locations, such as the edges of InSAR coverage. This is made possible by the use of an unstructured grid which allows nodal points to be placed arbitrarily. An unstructured grid also allows for enhanced resolution in regions of special interest, analogous to the mesh refinement used by contemporary next-generation ice sheet models (Larour et al., 2012; Seddik et al., 2012; Brinkerhoff and Johnson, 2013), or to simply scale grid size by ice thickness. This approach also makes the incorporation of longitudinal stress gradients straightforward by parameterizing longitudinal stresses by solving an additional linear system. To these ends, we present the governing equations and the method of their numerical solution with finite elements. We apply this method to the Greenland Ice Sheet and show that this approach yields quality and grid-independent balance velocity fields.

2 Continuum formulation

For an incompressible fluid, conservation of mass is stated as

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

where $\boldsymbol{u}$ is the three dimensional fluid velocity field, with kinematic boundary conditions on the surface S and bed B

$$\frac{\partial S}{\partial t} - \boldsymbol{u} \cdot \boldsymbol{n}|_S = \dot{a} \tag{2}$$

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and

$$\frac{\partial B}{\partial t} - \boldsymbol{u} \cdot \boldsymbol{n}|_B = m_{\mathrm{b}}, \tag{3}$$

respectively. Vertically integrating Eq. (1), applying Leibniz rule, and substitution of Eqs. (2) and (3) yields a vertically averaged statement for conservation of mass, commonly called the continuity equation

$$\frac{\partial H}{\partial t} + \nabla_{\|} \cdot \overline{\boldsymbol{u}}_{\|} H = \dot{a} - m_{\mathrm{b}}, \tag{4}$$

with surface mass balance $\dot{a}$, basal melt m_{b}, and thickness H. $\nabla_{\|}\cdot$ is the divergence operator in the two horizontal directions, and $\overline{\boldsymbol{u}}_{\|} = [\overline{u}, \overline{v}]$ is the vertically averaged horizontal velocity vector. We henceforth drop the parallel bars, and assume that all vectors and operators work on the horizontal plane. This equation is well known to ice sheet modellers as the prognostic equation for evolving the geometry of an ice sheet. In this case, we assume and estimate of $\partial_t H$, and group it with the other source terms, yielding

$$\nabla \cdot \overline{\boldsymbol{u}} H = F \tag{5}$$

where $F = \dot{a} - m_{\mathrm{b}} - \partial_t H$. Equation (5) is often used to calculate H (Morlighem et al., 2010; Johnson et al., 2012). Here, we assume that H is known, and instead use Eq. (5) to calculate $\overline{\boldsymbol{u}}$. As stated, the system is underdetermined, with only one equation for both velocity components. For closure, we restate the problem in terms of flow direction $\boldsymbol{N}$ and speed $\overline{U} = \|\boldsymbol{u}\|_2$, such that

$$\boldsymbol{N}\overline{U} = \overline{\boldsymbol{u}}, \|\boldsymbol{N}\|_2 = 1. \tag{6}$$

This gives the scalar equation for unknown $\overline{U}$

$$\nabla \cdot \boldsymbol{N} H \overline{U} = F. \tag{7}$$

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Flow direction is specified as the solution to the problems

$$\boldsymbol{\tau}_\mathrm{s} = \nabla \cdot (lH)^2 \nabla \boldsymbol{\tau}_\mathrm{s} - \boldsymbol{\tau}_\mathrm{d} \tag{8}$$

and

$$\boldsymbol{N} = \frac{\boldsymbol{\tau}_\mathrm{s}}{\|\boldsymbol{\tau}_\mathrm{s}\|_2}. \tag{9}$$

The solution to Eq. (8) is equivalent to the application of a Gaussian average of variable length scale lH to the driving stress $\boldsymbol{\tau}_\mathrm{d}$ of the type suggested by Kamb and Echelmeyer (1986). Theoretical work typically expresses stress coupling length scales in terms of ice thicknesses, hence the notation lH; l is the number of ice thicknesses over which longitudinal coupling should act. Flow direction $\boldsymbol{N}$ is then proportional to the smoothed driving stress $\boldsymbol{\tau}_\mathrm{s}$ with unit normalization. In the case where the boundary of the computational domain corresponds to the complete boundary of an ice mass (balance velocity for all of Greenland, say), no boundary condition need be specified, as the solution is implicitly defined to be zero at the ice divide due to the problem geometry. When considering a partial domain, a Dirichlet condition must be specified once per flowline.

3 Dicretization and stabilization

Equations (5), (8), and (9) are closed, and can be used to calculate balance velocity. We use the finite element method in order to discretize the governing equations. Equation (8), is symmetric, and can be discretized with standard Galerkin methods (e.g. Zienkiewicz and Taylor, 2000). It's weak form is

$$\int_\Omega \boldsymbol{\tau}_\mathrm{s} \cdot \boldsymbol{\phi} + \nabla \boldsymbol{\phi} \cdot (lH)^2 \nabla \boldsymbol{\tau}_\mathrm{s}\, \mathrm{d}\Omega = -\int_\Omega \boldsymbol{\tau}_\mathrm{d} \cdot \boldsymbol{\phi}\, \mathrm{d}\Omega, \tag{10}$$

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where $\boldsymbol{\phi}$ is a vector valued test function. Equation (9) can be calculated from Eq. (8) and does not require discretization. Equation (5) is hyperbolic and requires stabilization in order to suppress spurious oscillations. We use the Streamline Upwind Petrov–Galerkin (SUPG) method as a stabilization technique (Brooks and Hughes, 1982). SUPG have been used with success for the continuity equation in the ice sheet modelling context extensively (Morlighem et al., 2010; Larour et al., 2012). This case differs from previous work in that we are here attempting solve for velocity rather than thickness. This means that velocity and thickness switch roles in the stabilization scheme; $\overline{U}$ is advected by the pseudo-velocity $\boldsymbol{N}H$. The SUPG weak form is

$$\int_\Omega (\lambda + \tau \nabla \cdot \boldsymbol{N} H \lambda)(\nabla \cdot \boldsymbol{N} H \overline{U} - F)\,\mathrm{d}\Omega = 0 \tag{11}$$

where λ is a bilinear basis function, τ is a mesh dependent stabilization parameter given by

$$\tau = \frac{h}{2\|\boldsymbol{N}H\|_2}, \tag{12}$$

and h is the element circumradius. The inclusion of this unusual stabilization term is key to achieving meaningful numerical solutions; without it, the solutions are plagued by non-physical oscillations. This instability is likely the reason that this approach has not been seen in the literature previously.

4 Application to the Greenland Ice Sheet

We apply this balance velocity approach to the Greenland Ice Sheet. We used the 1 km gridded GLAS/ICESat data set (DiMarzio et al., 2007) for surface elevations and a bed DEM from Bamber et al. (2001) for bed elevations. Annual average surface mass balance rates are derived from RACMO (Ettema et al., 2009). We assume that

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basal melt is small compared to surface mass balance, and neglect it. We also assume that the $\partial_t H$ is negligible, or that the ice sheet is in balance. This is doubtless an incorrect assumption in some regions of the ice sheet, but although estimates for this field exist (e.g. Pritchard et al., 2009), it is not yet possible to determine what proportion of this signal is a result of ice dynamics, as opposed to other mechanisms such as firn densification that should not be included here.

4.1 Grid dependence

In order to assess the degree of grid dependence exhibited by this solution method, we start with a very coarse mesh, with an element circumradius of $h = 32H$ and calculate balance velocity over progressively finer meshes, essentially halving the element size at each iteration, down to an element circumradius of $h = H$ or 500 m, whichever is greater. We do this for smoothing lengths $l \in \{0, 4, 10, 15\}$. The difference between the coarse solution and progressively finer solutions is shown in Fig. 1. We see that for smoothing lengths of $l \in \{4, 10, 15\}$ the norm of the difference between the refined and unrefined solutions stops changing with increasing refinement. When $l = 0$, the solution continues to change as the mesh becomes more refined. This indicates that incorporating a parameterization of longitudinal stress in flow routing can overcome the tendency for the flow field to overconverge, even for very finely resolved meshes.

4.2 Flow direction smoothing radius

Theoretical results from Kamb and Echelmeyer (1986) suggest that the value of l for an ice sheet should fall between 4 and 10 ice thicknesses (although this range is based on temperate ice). Previous studies of longitudinal coupling lengths for ice sheets typically indicate a value of l at the high end of this range (LeBrocq et al., 2006; Fricker et al., 2000), and often even higher (Testut et al., 2003; Joughin et al., 1997), in order to achieve heuristically good results. Identifying the optimal longitudinal coupling length is also complicated by the fact that l should almost certainly be spatially variable.

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Nevertheless, we present balance velocities for $l \in \{4, 10, 15\}$, for a mesh size of $h = H$, which based on results from the previous paragraph should be a sufficiently small mesh size such that any smoothing of the flow is due to longitudinal coupling rather than a lack of mesh detail. Figure 2 gives the balance velocity for the GrIS at these length scales and mesh sizes, as well as the observed surface velocity. $l = 4$ produces an obviously overconvergent flow field, as evidenced by the abundance of discrete and overly narrow ice streams. $l = 10$ produces a better result, and we can see that most of the main flow features of the ice sheet are captured. Kangerdlugssuaq and Jakobshavn Isbrae are both robustly present and have a similar shape and extent to the measured velocity fields (although since these results are depth-averaged, while observations are of surface velocities, so a quantitative comparison is not strictly possible). The north-east ice stream, while apparent, is less significant than indicated by observations. At $l = 15$, features begin to wash out, most notably the characteristic multi-pronged ice streams of Kangerdlugssuaq glacier.

5 Conclusions

We presented a novel numerical method for calculating the balance velocity of an ice sheet using the finite element method. This approach is an advance over classical routing techniques because it is not dependent on a heuristic routing algorithm and relies solely on a continuum conservation law and a theoretically motivated parameterization of flow directions. An unstructured grid easily allows for variable spatial resolutions. This method is made possible by two specific insights. First, flow directions that include longitudinal stresses can be calculated by applying a sptially variable diffusion operator to the driving stress. Second, the balance velocity equations can be viewed as an advection equation with a pseudo-velocity field specified by thickness and flow direction, with velocity as the advected quantity. This problem is unstable. We use the Streamline Upwind Petrov–Galerkin method to make it tractable.

We applied this method to the Greenland Ice Sheet. Balance velocities were calculated over a number of different mesh resolutions, and we found that for given sufficient longitudinal coupling distances, the solution shows grid independence. We also showed the balance velocity field calculated for theoretically justifiable smoothing lengths on detailed meshes. The resulting balance velocity compare favorably with a measured velocity field.

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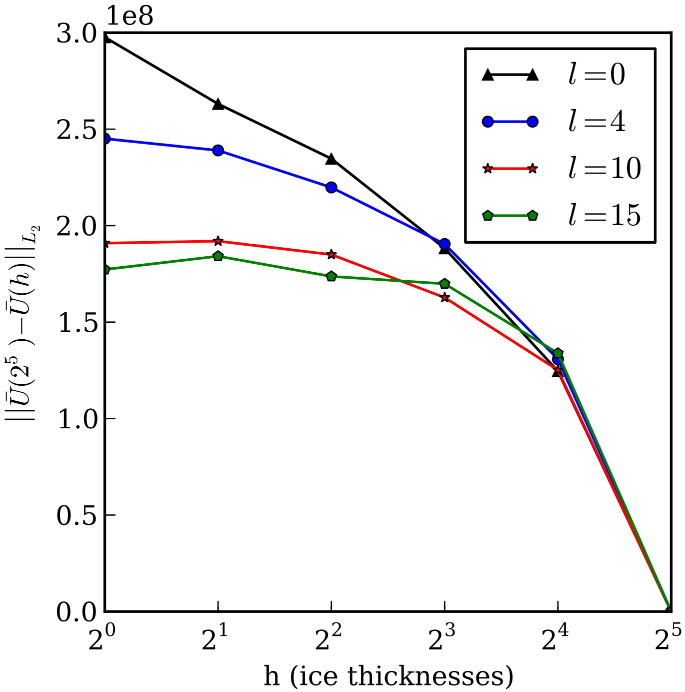

Figure 1. Residual between balance velocity solution at a coarse and progressively finer length scales for $l \in \{0, 4, 10, 15\}$.

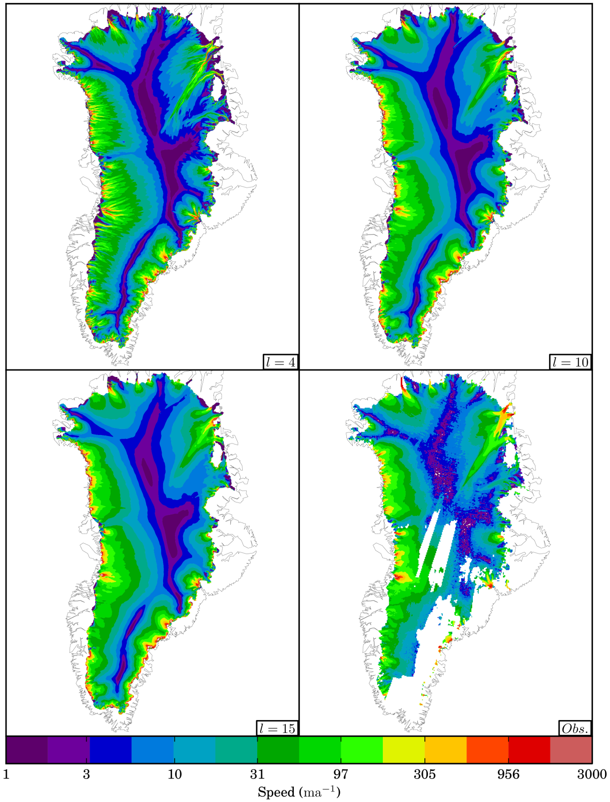

Figure 2. Balance velocity solution for a mesh size of $h = H$ and $l \in \{4, 10, 15\}$ as well as InSAR surface velocities.

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