Domination in Some Operations on Interval-valued Fuzzy Graphs

Ahmed N. Shain
Department of Mathematics
Faculty of Education and Sciences
Sheba Region University, Marib, Yemen

Mahiuob M.Q. Shubatah
Department of Mathematics
Faculty of Sciences and Education
AL-Baydha University, AL-Baydha, Yemen

Saqr H. Al-Emrany
Department of Mathematics
Faculty of Education and Sciences
Sheba Region University, Marib, Yemen

Nojood A. Al-Khadari
Department of Mathematics
Faculty of Education and Sciences
Sheba Region University, Marib, Yemen

ABSTRACT

In this paper, the domination number γ of some operations on interval-valued fuzzy graphs was introduced and established investigated for some operations on bipolar fuzzy graphs such as union, join, Cartesian product, strong product and composition.

General Terms

AMS Mathematics Classification (2010), 03E72, 68R10, 68R05.

Keywords

Interval-valued fuzzy graph, domination number

1. INTRODUCTION

Fuzzy graph is one of the application tool in the field of mathematics, which allow the users to describe the relationship between any notions easily.

Zadeh [15] was introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [14] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [9] and he developed the structure of fuzzy graphs. Recently, Akram and Dudek have studied several properties and operations on interval-valued fuzzy graphs [1]. Some important works in fuzzy graph theory can be found in [4, 5, 6, 7, 8, 10, 13]. Somasundaram and Somasundaran [11, 12] introduced and investigated the concept of domination in fuzzy graphs, while P. Debnath [3] introduced and investigated the concept of domination in interval-valued fuzzy graphs. Since the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences engineering, statistic, graph theory, artificial intelligence, signal processing, multi-agent sys-

tems, decision making and automate theory and also the domination in fuzzy graphs, this is what motivated us to submit this work. In this paper, we introduced and establish the domination number γ of some operation on interval-valued fuzzy graphs, such as union, join, Cartesian product and composition.

2. PRELIMINARIES

We review in this section, some basic definitions related to interval-valued fuzzy graphs and domination in fuzzy graph.

An interval-valued fuzzy graph of a graph $G^* = (V, E)$ is a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V , $B = [\rho_B^-, \rho_B^+]$ is an interval-valued fuzzy relation on V , such that $\mu_A^-(x) \leq \mu_A^+(x); \forall x \in V$ and $\rho_B^-(x, y) \leq \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\rho_B^+(x, y) \leq \min\{\mu_A^+(x), \mu_A^+(y)\}; \forall (x, y) \in E$. In an interval-valued fuzzy graph G , when $\rho^-(u, v) = \rho^+(u, v) = 0$ for some u and v , then there is no edge between u and v . Otherwise there exists an edge between u and v . Let $G = (A, B)$ be an interval-valued fuzzy graph. Then the cardinality of interval-valued fuzzy graph G is defined as

$$|G| = \sum_{u \in V} \frac{1 + \mu^+(u) - \mu^-(u)}{2} + \sum_{(u, v) \in E} \frac{1 + \rho^+(u, v) - \rho^-(u, v)}{2}.$$

The vertex cardinality of an interval-valued fuzzy graph G is defined by

$$|V| = \sum_{u \in V} \frac{1 + \mu^+(u) - \mu^-(u)}{2}.$$

For all $u \in V$ is called the order of an interval-valued fuzzy graph and is denoted by $p(G)$. The edge cardinality of an interval-valued fuzzy graph G is defined by

$$|E| = \sum_{(u, v) \in E} \frac{1 + \rho^+(u, v) - \rho^-(u, v)}{2}.$$

For all $(u, v) \in E$ is called the size of an interval-valued fuzzy graph and is denoted by $q(G)$. An edge $e = (x, y)$ of an interval-valued fuzzy graph is called effective edge if $\rho^+(x, y) = \min\{\mu^+(x), \mu^+(y)\}$ and $\rho^-(x, y) = \min\{\mu^-(x), \mu^-(y)\}$. The degree of a vertex can be generalized in different ways for an interval-valued fuzzy graph $G = (A, B)$. The effective degree of a vertex v in an interval-valued fuzzy graph, $G = (A, B)$ is defined to be summation of the weights of the effective edges incident at v and it is denoted by $d_E(v)$. The minimum effective edges degree of G is $\delta_E(G) = \min\{d_E(v) | v \in V\}$. The maximum effective degree of G is $\Delta_E(G) = \max\{d_E(v) | v \in V\}$. A vertex u of an interval-valued fuzzy graph G is said to be an isolated vertex if $\rho^-(uv) < \min\{\mu^-(u), \mu^-(v)\}$ and $\rho^+(uv) < \min\{\mu^+(u), \mu^+(v)\}$ for all $v \in V - \{u\}$ such that there is an edge between u and v , i.e., $N(u) = \phi$. A set S of vertices of an interval-valued fuzzy graph G is said to be independent if $\rho^-(uv) < \min\{\mu^-(u), \mu^-(v)\}$ and $\rho^+(uv) < \min\{\mu^+(u), \mu^+(v)\}$ for all $u, v \in S$. An interval-valued fuzzy graph, $G = (A, B)$ is said to be Complete interval-valued fuzzy graph if $\rho^-(v_i, v_j) = \min\{\mu^-(v_i), \mu^-(v_j)\}$, $\rho^+(u, v) = \min\{\mu^+(u), \mu^+(v)\}$, for all $u, v \in V$ and denoted by K_p . The complement of an interval-valued fuzzy graph, $G = (A, B)$ is an interval-valued fuzzy graph, $\bar{G} = (\bar{A}, \bar{B})$ where

$\bar{\rho}^-(u, v) = \min\{\mu^-(u), \mu^-(v)\} - \rho^-(u, v)$ and $\bar{\rho}^+(u, v) = \min\{\mu^+(u), \mu^+(v)\} - \rho^+(u, v)$ for all $u, v \in G$. An interval-valued fuzzy graph $G = (A, B)$ of a graph $G^* = (V, E)$ is said to be bipartite if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\rho^-(xy) = 0$ and $\rho^+(xy) = 0$ if $x, y \in V_1$ or $x, y \in V_2$. Further if $\rho^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\rho^+(xy) = \min\{\mu_A^+(x), \mu_A^+(y)\}$ for all $x \in V_1$ and $y \in V_2$. Then G is called a complete bipartite fuzzy graph is denoted by $K_{\mu_A^-, \mu_A^+}$,

where μ_A^- and μ_A^+ are restrictions of μ_A^- and μ_A^+ on V_1 and V_2 respectively. An edge $e = xy$ of an interval-valued fuzzy graph G is called an effective edge if $\rho^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ and $\rho^+(xy) = \min\{\mu_A^+(x), \mu_A^+(y)\}$. In this case, the vertex x is called a neighbor of y and conversely. $N(x) = \{y \in V : y \text{ is a neighbour of } x\}$ is called the neighborhood of x . A subset $(S \subseteq V)$ of $V(G)$ is called dominating set in interval-valued fuzzy graph G , if for every $v \in V - S$ there exists $u \in S$ Such that (u, v) is strong edge. A dominating set S of G is called minimal dominating set of G if $S - \{u\}$ is not dominating set for every $u \in S$. A minimal dominating set S , with $|S| = \gamma(G)$ is denoted by γ -set.

3. MAIN RESULTS

DEFINITION 1. [1] Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graph on A_1, A_2 , respectively with $A_1 \cap A_2 = \phi$. The union of G_1 and G_2 is the interval-valued fuzzy graph $G_1 \cup G_2$ on $A_1 \cup A_2$ and defined as $G = G_1 \cup G_2 = \{(\mu_{A_1}^- \cup \mu_{A_2}^-), (\mu_{A_1}^+ \cup \mu_{A_2}^+), (\rho_{B_1}^- \cup \rho_{B_2}^-), (\rho_{B_1}^+ \cup \rho_{B_2}^+)\}$. Where

$$(\mu_{A_1}^- \cup \mu_{A_2}^-)(u) = \begin{cases} \mu_{A_1}^- & \text{if } u \in A_1 \\ \mu_{A_2}^- & \text{if } u \in A_2 \end{cases}$$

$$(\mu_{A_1}^+ \cup \mu_{A_2}^+)(u) = \begin{cases} \mu_{A_1}^+ & \text{if } u \in A_1 \\ \mu_{A_2}^+ & \text{if } u \in A_2 \end{cases}$$

$$(\rho_{B_1}^- \cup \rho_{B_2}^-)(uv) = \begin{cases} \rho_{B_1}^- & \text{if } uv \in B_1 \\ \rho_{B_2}^- & \text{if } uv \in B_2 \end{cases}$$

$$(\rho_{B_1}^+ \cup \rho_{B_2}^+)(vu) = \begin{cases} \rho_{B_1}^+ & \text{if } uv \in B_1 \\ \rho_{B_2}^+ & \text{if } uv \in B_2 \end{cases}.$$

In the flowing we given γ of the union of every tow interval-valued fuzzy graphs G_1 and G_2 .

THEOREM 2. Let G_1 and G_2 be two vertices disjoint interval-valued fuzzy graphs, then

$$\gamma(G_1 \cup G_2) = \gamma(G_1) + \gamma(G_2).$$

PROOF. Let S_1 and S_2 be tow γ_1 -set and γ_2 of G_1 and G_2 respectively. Since G_1 and G_2 are vertices disjoint. Then $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$. Therefore, $S_1 \cup S_2$, is dominating set of $(G_1 \cup G_2)$. Hence,

$$\gamma(G_1 \cup G_2) = |S_1 \cup S_2| = \gamma(G_1) + \gamma(G_2).$$

□

The following example discusses in detail the result in the above Theorem.

EXAMPLE 1. Consider the interval-valued fuzzy graphs G_2 and G_1 given in the Figures 3.1a and 3.1b, respectively, such that all edges of G_1 and G_2 are strong edges.

Fig 3.1a

Fig 3.1b.

Fig 3.1c : $(G_1 \cup G_2)$

The union of G_1 and G_2 is shown in Figure (3.1c,) we see that $S_1 = \{a\}$ is a γ -set of G_1 , $S_2 = \{d\}$ is a γ -set of G_2 and $\gamma_1 = 0.65, \gamma_2 = 0.55$. Now, $S = S_1 \cup S_2$ is a γ -set of $G_1 \cup G_2$. Hence $\gamma(G_1 \cup G_2) = |S_1 \cup S_2| = |S_1 + S_2| = 0.65 + 0.55 = 1.2 = \gamma_1 + \gamma_2$.

DEFINITION 3. [1] The join $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ of two produce interval-valued fuzzy graphs G_1 and G_2 is defined as follows:

$$\begin{aligned}(\mu_{A_1^-} + \mu_{A_2^-})(u) &= (\mu_{A_1^-} \cup \mu_{A_2^-})(u), \\(\mu_{A_1^+} + \mu_{A_2^+})(u) &= (\mu_{A_1^+} \cup \mu_{A_2^+})(u) \text{ if } u \in V_1 \cup V_2. \\(\rho_{B_1^-} + \rho_{B_2^-})(uv) &= (\rho_{B_1^-} \cup \rho_{B_2^-})(uv), \\(\rho_{B_1^+} + \rho_{B_2^+})(uv) &= (\rho_{B_1^+} \cup \rho_{B_2^+})(uv) \text{ if } uv \in E_1 \cap E_2. \\(\rho_{B_1^-} + \rho_{B_2^-})(uv) &= \min(\mu_{A_1^-}(u), \mu_{A_2^-}(v)), \\(\rho_{B_1^+} + \rho_{B_2^+})(uv) &= \min(\mu_{A_1^+}(u), \mu_{A_2^+}(v)) \text{ if } uv \in E, \\ \text{where } E &\text{ is the set of all edges joining the vertices of } V_1 \text{ and } V_2.\end{aligned}$$

The flowing Theorem given γ of $G_1 + G_2$.

THEOREM 4. Let G_1 and G_2 be any two interval-valued fuzzy graphs, then

$$\gamma(G_1 + G_2) = \min\{\gamma(G_1), \gamma(G_2)\}.$$

PROOF. First we prove both S_1 and S_2 are dominating sets of $G_1 + G_2$. By definition of join of tow interval-valued fuzzy graphs G_1 and G_2 . Then for-all each vertex $v \in V(G_1)$ and $u \in V(G_2)$, we have $uv \in G_1 \cup G_2$ is a strong edge. Since $S_1 \subseteq V_1$. Then $\forall u \in V(G_1 \cup G_2) \exists u \in S_1$, such that uv is a strong edge. Therefore, S_1 is dominating set of $G_1 \cup G_2$. Similarly S_2 is dominating set of $G_1 \cup G_2$. Hence $\gamma(G_1 \cup G_2) = \min\{\gamma(G_1), \gamma(G_2)\}$. If $y \in V(G_1)$ obviously is dominating by a $x \in S_1$, since S_1 is the dominating set of G_1 , if $y \in G_2$, now we prove $x \in S_1$ dominates y . Since $x \in G_1$ and $y \in G_2$, then $(\rho_{B_1^-} + \rho_{B_2^-})(xy) = \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y))$, $(\rho_{B_1^+} + \rho_{B_2^+})(xy) = \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y))$, clearly x dominates y . Therefore, S_1 is the dominating set of $G_1 + G_2$. We assume S_1 is not minimum dominating set of $G_1 + G_2$, there exist $x \in S_1$ such that $S_1 - x$ is a dominating set of $G_1 + G_2$. Therefore $x \in G_1 + G_2$, there exist $y \in S_1$ such that $(\rho_{B_1^-} + \rho_{B_2^-})(xy) = \min(\mu_{A_1^-}(x), \mu_{A_2^-}(y))$, $(\rho_{B_1^+} + \rho_{B_2^+})(xy) = \min(\mu_{A_1^+}(x), \mu_{A_2^+}(y))$, this implies $S_1 - x$ is the dominating set of G_1 , this is contradicting to is S_1 is minimum. Similarly: we prove S_2 is a minimum dominating set of $G_1 + G_2$. Let S be the minimum dominating set of $G_1 + G_2$. Then $|S| = |S_1 + S_2|$ Hence

$$\gamma(G_1 + G_2) = \min\{\gamma(G_1), \gamma(G_2)\}.$$

□

The following example discusses in detail the result in the above Theorem.

EXAMPLE 2. Consider the interval-valued fuzzy graphs G_1 and G_2 given in the Figures 3.2a and 3.2b respectively, such that

all edges in G_1 and G_2 are effective.

Fig 3.2a

Fig 3.2b

The join of G_1 and G_2 given in Figure (3.2b) we can verify that $S_1 = \{u_2\}$ is a γ -set of G_1 , $S_2 = \{v_1, v_2\}$ is a γ -set of G_2 and $S = S_1$ or $S = S_2$ is a dominating set of $G_1 + G_2$. Thus $\gamma(G_1 + G_2) = \min\{\gamma(G_1), \gamma(G_2)\} = 0.55$

DEFINITION 5. [1] Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs on A_1, A_2 , respectively with $A_1 \cap A_2 = \phi$. The Cartesian product of G_1 and G_2 defined by $G_1 \times G_2$ is the interval-valued fuzzy graph $G = G_1 \times G_2$ on $A_1 \times A_2$ and defined by $G = G_1 \times G_2 = \{(\mu_{A_1^-} \times \mu_{A_2^-}, \mu_{A_1^+} \times \mu_{A_2^+}), (\rho_{B_1^-} \times \rho_{B_2^-}, \rho_{B_1^+} \times \rho_{B_2^+})\}$, such that

$$\begin{aligned} &\left\{ \begin{aligned} (\mu_{A_1^-} \times \mu_{A_2^-})(xy) &= \min\{\mu_{A_1^-}(x), \mu_{A_2^-}(y)\} \\ (\mu_{A_1^+} \times \mu_{A_2^+})(xy) &= \min\{\mu_{A_1^+}(x), \mu_{A_2^+}(y)\} \end{aligned} \right. \quad \forall x, y \in A_1 \times A_2 \\ &\left\{ \begin{aligned} (\rho_{B_1^-} \times \rho_{B_2^-})((x_1y_1)(x_2y_2)) &= \min\{\mu_{A_1^-}(x_1), \rho_{B_2^-}(y_1y_2)\} \\ (\rho_{B_1^+} \times \rho_{B_2^+})((x_1y_1)(x_2y_2)) &= \min\{\mu_{A_1^+}(x_1), \rho_{B_2^+}(y_1y_2)\} \end{aligned} \right. \quad \forall x \in A_1, y_1, y_2 \in A_2 \\ &\left\{ \begin{aligned} (\rho_{B_1^-} \times \rho_{B_2^-})((x_1y)(x_2y)) &= \min\{\rho_{B_1^-}(x_1x_2), \mu_{A_2^-}(y)\} \\ (\rho_{B_1^+} \times \rho_{B_2^+})((x_1y)(x_2y)) &= \min\{\rho_{B_1^+}(x_1x_2), \mu_{A_2^+}(y)\} \end{aligned} \right. \quad \forall y \in A_2, x_1, x_2 \in A_1 \end{aligned}$$

The flowing Theorem given γ for the cartisian product of any tow interval-valued fuzzy graphs G_1 and G_2 .

THEOREM 6. Let S_1 and S_2 be γ -set of interval-valued fuzzy graphs G_1 and G_2 , respectively. Then

$$\gamma(G_1 \times G_2) = \min\{|S_1 \times V_2|, |V_1 \times S_2|\}.$$

PROOF. First we prove that $S_1 \times V_2$ is a dominating set of $(G_1 \times G_2)$. Let $(u_1, u_2) \notin S_1 \times V_2$. Hence $u_1 \notin S_1$. Since S_1 is dominating set of G_1 . Then there exist a vertex $v_1 \in S_1$, such that $\forall u_1 \in V_1 - S_1$ $\begin{cases} \rho_B^-(u_1, v_1) = \min\{\mu_A^-(u_1), \mu_A^-(v_1)\} \\ \rho_B^+(u_1, v_1) = \min\{\mu_A^+(u_1), \mu_A^+(v_1)\} \end{cases}$ $(v_1, u_2) \in S_1 \times V_2$,

$$\rho_B^-(u_1, u_2), (v_1, u_2)) = \min\{\rho_{B_1}^-(u_1 v_1), \mu_{A_2}^-(u_2)\}$$

$$= \min\{\mu_{A_1}^-(u_1) \wedge \mu_{A_1}^-(v_1), \mu_{A_2}^-(u_2)\} = \min\{\mu_{A_1}^-(u_1) \wedge \mu_{A_1}^-(v_1) \wedge \mu_{A_2}^-(u_2)\}$$

$$= \min\{\mu_{A_1}^-(u_1) \wedge \mu_{A_2}^-(u_2)\}, \min\{\mu_{A_2}^-(u_2) \wedge \mu_{A_1}^-(v_1)\} = \rho_B^-(u_1, u_2) \wedge \rho_B^-(v_1, u_2)$$

$$\rho_B^+((u_1, u_2), (v_1, u_2)) = \min\{\rho_{B_1}^+(u_1 v_1), \mu_{A_2}^+(u_2)\}$$

$$= \min\{\mu_{A_1}^+(u_1) \wedge \mu_{A_1}^+(v_1), \mu_{A_2}^+(u_2)\} = \min\{\mu_{A_1}^+(u_1) \wedge \mu_{A_1}^+(v_1) \wedge \mu_{A_2}^+(u_2)\}$$

$$= \min\{\mu_{A_1}^+(u_1) \wedge \mu_{A_2}^+(u_2)\}, \min\{\mu_{A_2}^+(u_2) \wedge \mu_{A_1}^+(v_1)\} = \rho_B^+((u_1, u_2) \wedge \rho_B^+(v_1, u_2)$$

This implies every $(u_1, u_2) \notin S_1 \times V_2$, there exist a vertex $(v_1, u_2) \in S_1 \times V_2$, such that $(u_1, u_2)(v_1, u_2)$ is a strong edge. Therefore $S_1 \times V_2$ is dominating set of $G_1 \times G_2$. Similarly; $V_1 \times S_2$ is dominating set of $G_1 \times G_2$. Next we prove that $V_1 \times S_2$ and $S_1 \times V_2$ are a minimal dominating sets. Suppose $S_1 \times V_2$ is not minimal dominating set of $G_1 \times G_2$. Then there exist a vertex $(x_1, u_2) \in S_1 \times V_2$, such that $\{(S_1 \times V_2) - (x_1, u_2)\}$ is a dominating set of $G_1 \times G_2$. Let $(u_1, u_2) \in S_1 \times V_2$, and (u_1, u_2) dominates (x_1, u_2) and since $S_1 \times V_2$ is dominating set of $G_1 \times G_2$. Then we get u_1 dominates $x_1 \in S_1$. Clearly $S_1 - \{x_1\}$ is a dominating set of G_1 , which a contradiction to our assumption S_1 is a minimal dominating set of G_1 . Therefore $S_1 \times V_2$ is minimal dominating set of $G_1 \times G_2$. Similarly; $V_1 \times S_2$ is minimal dominating set of $G_1 \times G_2$. Hence

$$\gamma(G_1 \times G_2) = \min\{|S_1 \times V_2|, |V_1 \times S_2|\}.$$

□

The following example discusses in detail the result in the above Theorem.

EXAMPLE 3. Consider the interval-valued fuzzy graphs G_2 and G_1 given in the Figures 3.3a and 3.3b.

Fig 3.3a

Fig 3.3b.

Fig 3.3c.

From Figures (3.3a) and (3.3b), we can verify that $S_1 = \{d\}$ is a γ -set of G_1 and $S_2 = \{a\}$ is a γ -set of G_2 . Also we can verify that from Figure (3.3c), $(V_1 \times S_2)$ and $(V_2 \times S_1)$ are dominating sets of $G_1 \times G_2$. Hence

$$\gamma(G_1 \times G_2) = \min\{|S_1 \times V_2|, |V_1 \times S_2|\} = \min\{|\{ad, ae\}|, |\{ad, bd, cd\}|\}$$

DEFINITION 7. [1] Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs on A_1, A_2 , respectively with $A_1 \cap A_2 = \phi$. The composition of G_1 and G_2 defined by $G_1 \circ G_2$ is the interval-valued fuzzy graph $G = G_1 \circ G_2$ on $A_1 \circ A_2$ and defined by $G = G_1 \circ G_2 = \{(\mu_{A_1}^- \circ \mu_{A_2}^-, \mu_{A_1}^+ \circ \mu_{A_2}^+), (\rho_{B_1}^- \circ \rho_{B_2}^-, \rho_{B_1}^+ \circ \rho_{B_2}^+)\}$, such that

$$\begin{cases} (\mu_{A_1}^- \circ \mu_{A_2}^-)(xy) = \min\{\mu_{A_1}^-(x), \mu_{A_2}^-(y)\} \\ ((\mu_{A_1}^+ \circ \mu_{A_2}^+)(xy) = \min\{\mu_{A_1}^+(x), \mu_{A_2}^+(y)\} \end{cases} \quad \forall x, y \in A_1 \times A_2$$

$$\begin{cases} (\rho_{B_1}^- \circ \rho_{B_2}^-)((xy_1)(xy_2)) = \min\{\mu_{A_1}^-(x), \rho_{B_2}^-(y_1 y_2)\} \\ (\rho_{B_1}^+ \circ \rho_{B_2}^+)((xy_1)(xy_2)) = \min\{\mu_{A_1}^+(x), \rho_{B_2}^+(y_1 y_2)\} \end{cases} \quad \forall x \in A_1, y_1 y_2 \in A_2$$

$$\begin{cases} (\rho_{B_1}^- \circ \rho_{B_2}^-)((x_1 y)(x_2 y)) = \min\{\rho_{B_1}^-(x_1 x_2), \mu_{A_2}^-(y)\} \\ (\rho_{B_1}^+ \circ \rho_{B_2}^+)((x_1 y)(x_2 y)) = \min\{\rho_{B_1}^+(x_1 x_2), \mu_{A_2}^+(y)\} \end{cases} \quad \forall y \in A_2, x_1 x_2 \in A_1$$

$$\begin{cases} (\rho_{B_1^-} \circ \rho_{B_1^-})((x_1 y_1)(x_2 y_2)) = \min\{\rho_{B_1^-}(x_1 x_2), \mu_{A_2^-}(y_1 \wedge \mu_{A_2^-}(y_2))\} \\ (\rho_{B_1^+} \circ \rho_{B_2^+})((x_1 y_1)(x_2 y_2)) = \min\{\rho_{B_1^+}(x_1 x_2), \mu_{A_2^+}(y_1) \wedge \mu_{A_2^+}(y_2)\} \end{cases} \quad \forall (y_1 y_2) \in A_2, (x_1 x_2) \in B_1.$$

Such that $(y_1 \neq y_2)$.

THEOREM 8. Let S_1 and S_2 be γ -set of interval-valued fuzzy graphs G_1 and G_2 , respectively. Then

$$\gamma(G_1 \circ G_2) = |S_1 \times S_2|.$$

PROOF. Let $(u, v) \in S_1 \times S_2$. Then we have three cases:

Case 1: $u \notin S_1$ and $v \in S_2$. Let $u \notin S_1$. Then there exist $u_1 \in S_1$, such that u_1 dominates u . Hence

$$\begin{cases} \rho_{B_1^-}(u_1, u) = \min\{\mu_{A_1^-}(u_1), \mu_{A_2^-}(u)\} \\ \rho_{B_1^+}(u_1, u) = \min\{\mu_{A_1^+}(u_1), \mu_{A_2^+}(u)\} \end{cases} \quad \text{Now } (u_1, v) \in S_1 \times S_2$$

$$\rho_{B_1^-}((u_1, u), (v, u_1)) = \rho_{B_1^-}(u_1, u) \wedge \mu_{A_2^-}(v) = \min\{\rho_{B_1^-}(u_1, u), \mu_{A_2^-}(v)\}$$

$$= \min\{\mu_{A_1^-}(u_1) \wedge \mu_{A_1^-}(v_1), \mu_{A_2^-}(u_2)\} = \mu_{A_1^-}(u_1) \wedge \mu_{A_2^-}(v) \wedge \mu_{A_2^-}(v) \wedge \mu_{A_1^-}(u_1) = (\mu_{A_2^-}(v), \mu_{A_2^-}(v_1), \mu_{A_1^+}(u) \wedge \mu_{A_1^+}(u_1))$$

$$= (\mu_{A_1^-} \circ \mu_{A_2^-})^-(u, v) \wedge (\mu_{A_1^-} \mu_{A_2^-})^-(u_1, v)$$

$$= (\mu_{A_1} \circ \mu_{A_2})^-(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^-(u_1, v)$$

and

$$\rho_{B_1^+}((u_1, u), (v, u_1)) = \rho_{B_1^+}(u_1, u) \wedge \mu_{A_2^+}(v) = \min\{\rho_{B_1^+}(u_1, u), \mu_{A_2^+}(v)\}$$

$$= \min\{\mu_{A_1^+}(u_1) \wedge \mu_{A_1^+}(v_1), \mu_{A_2^+}(u_2)\} = \mu_{A_1^+}(u_1) \wedge \mu_{A_2^+}(v) \wedge \mu_{A_2^+}(v) \wedge \mu_{A_1^+}(u_1)$$

$$= (\mu_{A_1^+} \circ \mu_{A_2^+})^+(u, v) \wedge (\mu_{A_1^+} \circ \mu_{A_2^+})^+(u_1, v)$$

$$= (\mu_{A_1} \circ \mu_{A_2})^+(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^+(u_1, v).$$

Hence (u_1, v) dominates (u, v) .

Case 2: $u \in S_1$ and $v \notin S_2$. Let $v_2 \in S_2$, such

$$\text{that } \begin{cases} \rho_{B_2^-}(v_2, v) = \min\{\mu_{A_1^-}(v_2), \mu_{A_2^-}(v)\} \\ \rho_{B_2^+}(v_2, v) = \min\{\mu_{A_1^+}(v_2), \mu_{A_2^+}(v)\} \end{cases} \quad \text{Now}$$

$(u, v_2) \in S_1 \times S_2$

$$\rho_{B_2^-}((v_2, v), (v, u)) = \rho_{B_2^-}(v_2, v) \wedge \mu_{A_2^-}(u)$$

$$= \min\{\mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v), \mu_{A_2^-}(v_2)\} = \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v) \wedge \mu_{A_1^-}(u) \wedge \mu_{A_2^-}(v_2)$$

$$= (\mu_{A_1^-} \circ \mu_{A_2^-})^-(u, v) \wedge (\mu_{A_1^-} \circ \mu_{A_2^-})^-(u, v_2)$$

$$= (\mu_{A_1} \circ \mu_{A_2})^-(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^-(u, v_2).$$

$$\rho_{B_2^+}((v_2, v), (v, u)) = \rho_{B_2^+}(v_2, v) \wedge \mu_{A_2^+}(u)$$

$$= \min\{\mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v), \mu_{A_2^+}(v_2)\} = \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v) \wedge \mu_{A_1^+}(u) \wedge \mu_{A_2^+}(v_2)$$

$$= (\mu_{A_1^+} \circ \mu_{A_2^+})^+(u, v) \wedge (\mu_{A_1^+} \circ \mu_{A_2^+})^+(u, v_2)$$

$$= (\mu_{A_1} \circ \mu_{A_2})^+(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^+(u, v_2).$$

Case 3: $u \notin S_1$ and $v \notin S_2$. Since S_1 and S_2 are minimal dominating sets of G_1 and G_2 , respectively. Therefore there exist $u_1 \in S_1$ and $v_1 \in S_2$, such that

$$\rho_{B^-}(u, u_1) = \min(\mu_{A_1^-}(u), \mu_{A_2^-}(u_1)), \rho_{B^+}(u, u_1) = \min(\mu_{A_1^+}(u), \mu_{A_2^+}(u_1))$$

Let $(u_2, v_2) \in S_1 \times S_2$. Then

$$\rho_{B^-}((u, v), (u_1, v_2)) = \min(\rho_{A_2^-}(v), \mu_{A_2^-}(v_1), \rho_{B_1^-}(u, u_1)) = (\mu_{A_2^-}(v), \mu_{A_2^-}(v_1), \mu_{A_1^+}(u) \wedge \mu_{A_1^+}(u_1))$$

$$= (\mu_{A_1} \circ \mu_{A_2})^-(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^-(u_1, v_2) = \min(\mu_{A_1^-}(u, v), \mu_{A_1^-}(u_1, v_2)).$$

$$\rho_{B^+}((u, v), (u_1, v_2)) = \min(\mu_{A_2^+}(v), \mu_{A_2^+}(v_1), \mu_{A_1^+}(u) \wedge \mu_{A_1^+}(u_1))$$

$$= (\mu_{A_1} \circ \mu_{A_2})^+(u, v) \wedge (\mu_{A_1} \circ \mu_{A_2})^+(u_1, v_2)$$

$$= \min(\mu_{A_1^+}(u, v), \mu_{A_1^+}(u_1, v_2)).$$

Therefore, (u_1, v_2) dominates (u, v) in $G_1 \circ G_2$, this implies $S_1 \times S_2$ is a dominating set of $G_1 \circ G_2$.

Now, we prove $S_1 \times S_2$ is minimal. Let $(x, y) \in S_1 \times S_2$, $x \in S_1$ and $y \in S_2$ and since S_1 and S_2 are minimal dominating sets of G_1 and G_2 , respectively.

Therefore, $S_1 - \{x\}$ and $S_2 - \{y\}$ are not a dominating sets of G_1 and G_2 , respectively and then $(S_1 \times S_2) - (x \times y)$ is not dominating set of $G_1 \circ G_2$, this implies that $S_1 \times S_2$ minimal dominating set of $G_1 \circ G_2$. $\square$

The flowing Example explain the results in above Theorem.

EXAMPLE 4. For the interval-valued fuzzy graphs G_2 and G_1 given in the Figures 3.4a and 3.4b, respectively.

Fig 3.4a

Fig 3.4b.

We can verify that $S_1 = \{a\}$ is γ -set of G_1 and $S_2 = \{d\}$ is γ -set of G_2

- [14] L. A. Zadeh, "Fuzzy Sets", Information and Control, 8, 1965, 338-353.
[15] L. A. Zadeh, The concept of a linguistic and application to approximate reasoning I, Inform.Sci. 8 (1975) 149-249.

Fig 3.4c : $G_1 \circ G_2$

We can verify that $S_1 \times S_2 = \{ad\}$ is a γ -set of $G_1 \circ G_2$ Hence
 $\gamma_{G_1 \circ G_2} = |S_1 \times S_2| = |ad| = 0.55$

4. REFERENCES

- [1] M. Akram, Interval-valued fuzzy line graphs, Neural Computing and Applications, 21(1), (2012), 256-267.
- [2] M. Akram and W. A. Dudek, Interval-valued fuzzy graphs, Compute. Math. Appl. 61(2), (2011), 289-299.
- [3] P. Debnath, "Domination in interval-valued fuzzy graphs", Annals of Fuzzy Mathematics and Informatics, 6(2), (2013), 363-370.
- [4] F. Haray, "Graph theory", Addison Wesley, Third printing, October, (1972).
- [5] A. Kaufman, Introduction a La Theorie des Sous-ensembles Flous, Paris, Masson et cie Editeurs, (1973).
- [6] J. N. Mordeson and C. S. Peng, Operations on fuzzy graphs, Inform. Sci. 79 (1994) 159-170.
- [7] M. Naga Maruthi Kumari and R. Chandrasekhar, "Operations on interval-valued fuzzy graphs", IJARSE; 4(4),(2015), 618-627.
- [8] T. Pramanik, Sovan Samanta and Mahdumangai Pal, "interval-valued fuzzy graphs", International Journal of Fuzzy logic and Intelligent Systems, 20 (4), (2020), 316-323.
- [9] A. Rosenfeld, Fuzzy graphs, In: L. A. Zadeh, K. S. Fu and M. Shimura, (Eds). "Fuzzy Sets and their Applications", Academic Press, 1975, New York.
- [10] H. Rashmanlou, Sovan Samanta and M. Pal, "Some results on interval-valued fuzzy graphs", International Journal of Computer Science and Electronics Engineering , 3(3), (2015), 205-211.
- [11] A. Somasundaram, S. Somasundaran, "Domination in fuzzy graphs-I", Pattern Recognition Letters, 19, (1998), 787-791.
- [12] A. Somasundaram, "Domination in fuzzy graphs-II", Journal of Fuzzy Mathematics, 20, (2000), 281-289.
- [13] D.Umameswari and P.Thangaraj, "Domination operation on bipolar fuzzy graphs", International Journal of Computational Research and Development, 2,(2017), 56-65.