Graphical Approach to Index and Retrieve Standard Young Tableaux

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ABSTRACT

The Standard Young Tableaux are used to label the basis vectors of the standard or Young Yamanouchi basis of the symmetric group. There is a one - to - one correspondence between the eigen values of the Complete Set of Commuting Operators-I (CSCO-I) of Symmetric group of degree n (S_n) and the Standard Young Tableaux (SYT). In this paper, a new graphical method to index and retrieve the standard Young tableau for a partition of degree n in S_n is presented. This method is illustrated with an example using a partition of degree 6 in symmetric group S_6 .

Keywords:

Symmetric group; Standard Young Tableaux; indexing; retrieving; CSCO-I

1. INTRODUCTION

In [3,6], the Young tableaux provide a convenient method of determining the weight vectors of a given irreducible representation of the classical group and the corresponding weight multiplicities. Stanley [1], motivated by the work of Patera and Sharp [2], demonstrated that the standard Young tableaux of classical group have an important role to play in determining the character generator of that classical group. Combinatorial items like graphs and trees in Graph theory, partitions in set theory, Standard Young Tableaux (SYT) and Semi Standard Young Tableaux (SSYT) in group theory occur naturally in Science and Engineering. In many situations, we need to index and retrieve these combinatorial items. Wilf [4] developed an algorithm to rank and unrank combinatorial items. It is also known that there is a one - to - one correspondence between the eigen values of the Complete Set of Commuting Operators-I (CSCO-I) of Symmetric group of degree n (S_n) and the SYT. In this paper, we show how this result and the general framework given by Wilf [4] can be adapted to index and retrieve SYT. A new graphical method is introduced to index and retrieve the SYT for a partition of degree n in symmetric group using branching rule.

2. PRELIMINARIES AND BRANCHING LAW

Definition 1. A Partition of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ satisfying $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$.

For instance, the number 5 has seven partitions : (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).

Definition 2. A Young diagram or Ferrers diagram is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing.

The Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is the one that has k rows, and λ_i boxes on the i^{th} row.

For instance, the Young diagrams corresponding to the partitions of 5 are

(5) (4,1) (3,2) (3,1,1) (2,2,1) (2,1,1,1) (1,1,1,1,1)

Definition 3. A (Young) tableau t of shape λ , is obtained by filling in the boxes of a young diagram of λ with 1, 2, 3, ..., n , with each number occurring exactly once. In this case, we say that t is a λ -tableau.

Definition 4. A standard (Young) tableau (SYT) is a young tableau whose the entries are increasing across each row and each column.

The permutation group has important applications in physics for many particle systems. The importance lies in the fact that a system of identical particles has permutation symmetry and that there exist many deep and delicate inter relations between the group and the unitary group. The representation theory of the permutation group is well established through the effect of Young, Frobenius, and Yamanouchi and other. This representation theory has many advantages, for example it gives the branching law for reducing the irreducible representation of S_n , into those of S_{n-1} , the intuitive and elegant way of labeling the irreducible way representation and the irreducible bases by the Young diagrams and the Young tableaux, etc.

Branching law:

An irreducible representation $[v]$ of S_n is a general reducible with respect to its subgroup S_{n-1} , $[v] \downarrow S_{n-1} = \sum_{v'} \oplus D^{[v']} (S_{n-1})$, where the Young diagram $[v']$

result from removing one box in all possible way in the young diagram $[v]$, and each irreducible representation $[v']$ occurs only once. For example the irreducible representation $[4 \ 3 \ 1]$ of S_8 contains each of the irreducible representation $[4 \ 3]$, $[4 \ 2 \ 1]$ and $[3 \ 3 \ 1]$ once.

$$[4 \ 3 \ 1] \rightarrow [4 \ 3] \oplus [4 \ 2 \ 1] \oplus [3 \ 3 \ 1]$$

Dimensions of $[4 \ 3 \ 1]$, $[4 \ 3]$, $[4 \ 2 \ 1]$ and $[3 \ 3 \ 1]$ are 70, 14, 35 and 21 respectively.

Dimension (dim) of $[4 \ 3 \ 1] = \text{dim of } [4 \ 3] + \text{dim of } [4 \ 2 \ 1] + \text{dim of } [3 \ 3 \ 1]$

In general, we have

$$[v_1 \ v_2 \ \dots v_i \ \dots v_n] = \sum_{i=n}^1 [v_1 \ v_2 \ \dots v_{i-1} \ \dots v_n] \quad (1)$$

The summation in Eq.(1) indicates that $v_i - 1 \geq 0$ and $v_i - 1 > v_{i+1}$ in order that $[v']$ is a young diagram. The dimensions of the both sides of Eq.(1) must be equal; therefore

$$h[v_1 \ v_2 \ \dots v_i \ \dots v_n] = \sum_{i=n}^1 h[v_1 \ v_2 \ \dots v_{i-1} \ \dots v_n] \quad (2)$$

Eq.(1) is referred to as the branching law for the permutation group. The

$$S_n \supset S_{n-1} \supset S_{n-2} \dots \supset S_2 \supset S_1$$

irreducible basis is the standard basis of the permutation group. It is also called the Young-Yamanouchi basis from the branching law [Eq.(1)] and the fact that the group S_2 is commutative, we know that the group chain

$$S_n \supset S_{n-1} \supset S_{n-2} \dots \supset S_2 \supset S_1$$

is canonical (the last group is redundant).

A Young tableau is an arrangement of the numbers 1, 2, 3, ..., n in a Young diagram in which numbers increase as one moves to the right and as one goes down. A Young tableau is denoted by $Y_m^{[v]}$, where m is the index of the tableau. From a given Young tableau $Y_m^{[v]}$, one obtains another Young tableau $Y_{m'}^{[v']}$ involving n-1 numbers by removing the box containing the numbers n; by removing the box with numbers (n-1) one obtains yet another Young tableau $Y_{m''}^{[v'']}$ and so on.

3. CSCO-I OF SYMMETRIC GROUP

Class operators:

A sum of all elements belonging to the same class is called class operator. Suppose the i^{th} class g_i elements $a_1, a_2, a_3, \dots, a_{g_i}$, the class operator C_i is

$$C_i = \sum_{l=1}^{g_i} a_l^{(i)}, \quad i=1, 2, \dots, n$$

Properties:

1. The class operators commute with any elements of the group.
2. The class operators commute with each other
3. The class operators are closed under group multiplication.

Because of the above three properties, the class operator plays a decisive role in the new approach to group representation theory.

Class space:

The n-dimensional vector space spanned by the n-class operators C_i of a group is called the class space.

For Example, the three class operators of symmetric group of degree 3(S_3) are

$$C_1 = e, \quad C_2 = (1\ 2) + (1\ 3) + (2\ 3) \text{ and } C_3 = (1\ 2\ 3) + (1\ 3\ 2)$$

The first kind of complete set of commuting operators of group G (CSCO-I) : If a set of operators $C = \{C_{i_1}, C_{i_2}, \dots, C_{i_l}\}$ selected out of the class operators is a CSCO of the class space, then C is called a CSCO of the first kind of the group or CSCO-I

Note:

1. The set of n class operators ($C_1, C_2, \dots, C_n$) is necessarily a CSCO of G
2. Any class operator C_i of a group G is a function of the CSCO of G.
3. Any eigen vector of the CSCO of group G is necessarily a simultaneous eigen vector of all the n class operators of group G.
4. Different CSCO's of G are equivalent.

Multiplication table of the class operators of S_3 are presented in Table –I. Using the Table – I, the natural representation of the class operators of S_3 are

$$D(C_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(C_2) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}, D(C_3) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Table-I: Multiplication table of the class operators of S

	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	$3(C_1 + C_3)$	$2C_2$
C_3	C_3	$2C_2$	$2C_1 + C_3$

4. A NEW GRAPHICAL METHOD TO INDEX AND RETRIEVE THE SYT

We apply the general frame work outlined in section 2 of Wilf's paper [4] for indexing and retrieving of combinational items.

Problem statement:

- a) Indexing Problem :

Given a SYT in the SYT ($[\lambda], n$), find the index 'r' such that $0 \leq r \leq t-1$, where 't' is the number of SYT in the set ($[\lambda], n$).

- b) Retrieving Problem:

Given an integer 'r': ($0 \leq r \leq t-1$). Find the corresponding SYT in the SYT ($[\lambda], n$).

Index and Retrieving of SYT:

Steps for Indexing SYT

Step 1: Construct the Graph 'G' to get the code word for the given SYT

Step 2: For each of the vertices in the codeword, compute the function $f(e) = \sum b(e')$ (all $e' < e$ in the level.)

Step 3: The index of the given SYT = $\sum f(e)$ (for all 'e' of the codeword).

Steps for Retrieving SYT

Given an index I of the SYT, get the SYT

Step 1: From the first row in the graph, find the vertex e_1 , where $f(e'_1) < I < f(e_1)$ and e'_1 is the previous node of e_1 .

Step 2: Find $I' = I - f(e')$.

Step 3: Repeat step 1 for I' in second row and get e_2 . Repeat this step till we reach the terminal node.

As an illustration, let us take $[\lambda] = [3 \ 2 \ 1]$, $n=6$. We identify this vertex ' v ' of directed graph as explained in section 2 of ref. [4]. To apply the general framework by Wilf, we first need to identify the branching rules for SYT.

SYT's of symmetric group are equivalent to CSCO-I of symmetric group [5].

Using the branching theorem [5], SYT's can be partitioned as follows.

1. SYT having this group is shown in Figure 1. Note that these are SYT([3,2,1],6)
2. The group that have symbol 6 only once are in Figure 2. We can see that the symbol 6 can be in the first or second or third row, the generic patterns for these are also shown in Figure 2.

The letter x is where we would have symbol '6'.

The corresponding SYT sets are as follows SYT ([3 2],5), SYT([3,1,1],5), SYT([2 2 1],5)

The sets are 'ordered'. The ordering is defined as the set SYT ($[\lambda]$, n_1) precedes the set SYT

Figure 1: Young Tableaux for [3 2 1]

Figure 2: General pattern having symbol '6' only once

Figure 3: General pattern having symbol '5' only once.

$([\tilde{\lambda}], n_1)$ if $\tilde{\lambda}_1 > \lambda_1$; when $\lambda_1 = \tilde{\lambda}_1$ then $\lambda_2 = \tilde{\lambda}_2$ etc. we shall refer to this definition as definition (1).

3. SYT's having the symbol 5 only once these cases along with their generic patterns are shown in Figure 3. The letter x is where we would have symbol '5'. The sets are 'ordered'. The ordering with definition (1).
4. SYT's having the symbol 4 only once this cases along with their generic patterns are shown in Figure 4. The letter x is where we would have symbol '4'. The corresponding SYT sets are as follows SYT ([3], 3), SYT([2,1],3), SYT([1 1 1],3)
5. SYT's having the symbol 3 only once this cases along with their generic patterns are shown in Figure 5. The letter x is where we would have symbol '3'. The corresponding SYT sets are as follows SYT([2],2), SYT([1 1],2)
6. SYT's having the symbol 2 only once this cases along with their generic patterns are shown in Figure 6. The letter x is where we would have symbol '2'.

The corresponding SYT set is as follows SYT([1],1)

The corresponding SYT sets are as follows SYT ([3 1], 4), SYT ([2 2], 4), SYT ([2 1 1], 4)

Figure 4 : General pattern having symbol '4' only once

Figure 5: General pattern having symbol '3' only once

Figure 6: General pattern having symbol '2' only once

The number of SYT's in each of these sets can be calculated using the equation in step 3. From the above analysis, we get the first level vertices v_2, v_3, v_4 the vertex v in the directed graph G (section 2 in reference [4])

Let $b(v)$ be the number of SYT's under v . This is given by the equation in step 3, noting that $v = \text{SYT's } ([3 \ 2 \ 1], 6)$.

The corresponding steps are given as follows:

Compute the product of hook lengths [5] for the partition $[3 \ 2 \ 1]$ =

$$\frac{6!}{5.3.3} = \frac{6.5.4.3.2.1}{5.3.3} = 16$$

Therefore $b(v)$ =the number of SYT's= 16.

Similarly $b(v_1), b(v_2) \dots$ are calculated. The result is shown in Table II. The directed graph G derived based on the discussion in section 2 of reference [4] is shown in Figure 7

(combined with Table III). We note that in the directed graph G in Figure 7. The vertex v is the combinatorial object of order v for SYT $([3 \ 2 \ 1], 6)$. Each path from vertex to the terminal vertex τ is called 'code word' as in reference [4]. Each edge in the code word connects a vertex at a certain level to another vertex in its immediate lower level. In the present example between the vertex v and terminal vertex τ , we have 5 levels in the graph in Figure 7, the vertices v_2, v_3, v_4 corresponding to the first level the vertices v_5, v_6, v_7 corresponding to the second level the vertices v_8, v_9, v_{10} corresponding to the third level the vertices v_{11}, v_{12} corresponding to the forth level. The vertex v_{13} corresponding to fifth level. The details of the connection between the vertices of different level is given in Table III.

Figure 7: Graphical representation

Table II Values of ν and $b(\nu)$

Level	1	2	2	2	3	3	3	4	4	4	5	5	6	
v	v ₁	v ₂	v ₃	v ₄	v ₅	v ₆	v ₇	v ₈	v ₉	v ₁₀	v ₁₁	v ₁₂	v ₁₃	τ
b(v)	16	5	6	5	3	2	3	1	2	1	1	1	1	0

In the graph G , the numbers under each vertex are the number of SYT's for that vertex or the number of 'code words' from the vertex. This number is obtained by using the equation in the text book [5]. We could also note that this number of 'code words' of a vertex is the sum of 'code words' of the 'final vertices' of the vertex.

For example from Table III, we note that the vertex $v_3 \rightarrow (v_5, v_7)$.

This implies that $b(v_3) = b(v_5) + b(v_7)$.

The SYT corresponding to the ‘code word’ is $v \rightarrow v_2 \rightarrow v_5 \rightarrow v_9 \rightarrow v_{12} \rightarrow v_{13} \rightarrow \tau$ is the figure 8. In the graph, each ‘code word’ uniquely represents a SYT for $([3\ 2\ 1], n=6)$. As an illustration the ‘code word’ $v \rightarrow v_2 \rightarrow v_5 \rightarrow v_9 \rightarrow v_{12} \rightarrow v_{13} \rightarrow \tau$ corresponds to the SYTs shown in figure 8. In the graph ‘G’, we depict the pair $(v, b(v))$ inside a node. The vertex is identified by the number and corresponding $b(v)$ appears just under it. The table-II gives the values of $b(v)$, for each vertex ‘ v ’ of the graph ‘G’

Table III: Interconnection of nodes

$v_1 \rightarrow (v_2, v_3, v_4)$	$v_2 \rightarrow (v_5, v_6)$	$v_3 \rightarrow (v_5, v_7)$	$v_4 \rightarrow (v_6, v_7)$	$v_5 \rightarrow (v_8, v_9)$	$v_6 \rightarrow (v_9)$
$v_7 \rightarrow (v_9, v_{10})$	$v_8 \rightarrow (v_{11})$	$v_9 \rightarrow (v_{11}, v_{12})$	$v_{10} \rightarrow (v_{12})$	$v_{11} \rightarrow (v_{13})$	$v_{12} \rightarrow (v_{13})$

$$f(v_2) = 5 \quad f(v_5) = 3 \quad f(v_9) = 2 \quad f(v_{12}) = 1 \\ f(v_{13}) = 1$$

Hence, indexing = $5+3+2+1+1 = 12$.

Retrieving for the SYT, shown in figure 8.

Let us try to get the SYT corresponding to the index $I = 12$.

Step 1: For the first row, we have $f(v_2) = 5$, $f(v_2) = 11$,
 $f(v_2) = 16$

Step 2: $I' = I - f(e')$, and the first part of the codeword is
 $v_2 \rightarrow v_5$

We take v_5 as root. Use relation $v_5 \rightarrow (v_8, v_9)$.

Repeating steps 1 and 2, we get the following:

$$v_5 \rightarrow v_9, v_9 \rightarrow v_{12}, v_{12} \rightarrow v_{13}.$$

Hence, the codeword corresponding to the index = 12 is
 $v \rightarrow v_2 \rightarrow v_5 \rightarrow v_9 \rightarrow v_{12} \rightarrow v_{13} \rightarrow \tau$. This corresponds to
the SYT in figure 8.

6. CONCLUSION

In this paper, we have proposed a method to index and retrieve SYTs. This has been achieved by using the general framework of Wilf and the properties of SYT. A new graphical method to index and retrieve the Standard Young Tableau for a partition of degree n in S_n has presented. This method has been illustrated with an example using a partition of degree 6 in symmetric group S_6 .

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5. NUMERICAL EXAMPLE FOR INDEXING AND RETRIEVING OF SYT

Indexing for the SYT, shown in Figure-8 as discussed earlier, the codeword for the SYT is

$$v \rightarrow v_2 \rightarrow v_5 \rightarrow v_9 \rightarrow v_{12} \rightarrow v_{13} \rightarrow \tau$$

Table-3: Description of nodes v_1 to v_{13}

Description of nodes for first level

$$v_1 = \text{SYT}([3, 2, 1], 6)$$

Description of nodes for second level

$$v_2 = \text{SYT}([3, 2], 5) \quad v_3 = \text{SYT}([3, 1, 1], 5)$$

$$v_4 = \text{SYT}([2, 2, 1], 5)$$

Description of nodes for third level

$$v_5 = \text{SYT}([3, 1], 4) \quad v_6 = \text{SYT}([2, 2], 4)$$

$$v_7 = \text{SYT}([2, 1, 1], 4)$$

Description of nodes for fourth level

$$v_8 = \text{SYT}([3], 3) \quad v_9 = \text{SYT}([2, 1], 3)$$

$$v_{10} = \text{SYT}([1, 1, 1], 3)$$

Description of nodes for fifth level

$$v_{11} = \text{SYT}([2], 2) \quad v_{12} = \text{SYT}([1, 1], 2)$$

Description of nodes for sixth level

$$v_{13} = \text{SYT}([1], 1)$$

1	3	4
2	5	
6		

Figure 8: The SYT of G

(Steps for ranking) and their entries in table-III this gives us

$$f(v_2) = b(v_5) + b(v_6) \quad f(v_5) = b(v_8) + b(v_9)$$

$$f(v_9) = b(v_{11}) + b(v_{12}) \quad f(v_{12}) = b(v_{12})$$

$$f(v_{13}) = b(v_{13})$$

Using the values of $b()$ in table II, we get