New Quality Control Chart to Quickly Detect the Changes of Process Average

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Abstract

The objective of this article is to propose a new control chart—improved exponentially weighted moving average (IEWMA) control chart—to fast detect the mean shifts of process when quality characteristic data are normally distributed. This chart still has robust property even though its controls limits are created from data with outliers. The efficiency inspection of IEWMA control chart is managed 504 situations for the simulation data. Moreover, the four control charts, namely, exponentially weighted moving average (EWMA), robust exponentially weighted moving average (REWMA), median mean absolute deviation (MDMAD), and average control charts, are compared the performances with IEWMA control chart. All charts are constructed by using data set in two cases, i.e., the first case that data are not include outliers and the second case that data are composed of outliers. It is found that in the case of non-outliers in the data, the three charts—IEWMA, EWMA and REWMA control charts—tend to have the most capability for process average shift detection for all sample sizes and all levels of the process average changes. For the case of outliers in the data, the IEWMA control chart tends to have the most efficiency for all sample sizes, especially for the tiny process shifts from the target.

Keywords: control chart, exponentially weight, moving average, normal distribution, outliers, process mean

1. Introduction

Generally, qualities of the manufactured products in industrial production process are required to achieve or overcome the customer expectations. Moreover, the fundamental goals of production are to achieve the target mean of the process and to control the variation of product quality to be minimum level. The variation of industrial production process can be occurred all the time, no matter how the production process well designed or controlled. This variation is influenced by various factors that both controllable and uncontrollable factors, for example, it may be caused by using non-standard tools or using the wrong types of tools, the machine may deteriorate, the workers do not have production skills, the qualities of raw materials in the production process are lower qualities than specified levels, etc. These factors will affect the product quality. That is, if products in the production process have a small variation, it will result in the quality of each manufactured product being not much different quality. However, if products in the production process have a large variation, it will affect the product quality that does not achieve the product standards. In the industrial production process often encounter problems that products that do not meet the specified standards. One of the main reasons is the process average to be controlled has changed, but the process operators may be unaware of this change. Thus, if they allow production to continue resulting in non-standard products coming out of the process. These problems have the detrimental effect on the factory which may experience loss or damage the reputation of the factory when those inferior products are distributed to the consumers. The quality control chart is one of the most popular tools for statistical process control to warn the manufacturers

that the current production process has changed from a predetermined one. This makes it possible to quickly resolve quality issues when manufacturing malfunctions occur. If the chart is regularly analyzed in the production process, it helps to ensure consistent product quality and reduce variation in the manufacturing process. The idea of statistical control chart was first introduced by Shewhart (Montgomery, 2012), e.g., average control chart was devised under the assumption that data are normally distributed. This average control chart is generally utilized in quality control processes because it is easy to understand by the operators and it is not a complicated tool. However, several researches are shown that the average control chart is poor effective tool to detect abnormal process in cases of the small process mean shifts from the target (Abu-Shawiesh, 2009; Wang, 2009; Huang, Tai and Lu, 2014; Chew et al., 2015; Shafqat et al., 2020; Alevizakos, Chatterjee and Koukouvinos, 2021). After that, there are many fast response control charts that to be better than the average control chart for detection the slight mean shifts were proposed as follows: in 1959, the exponentially weighted moving average control chart or EWMA control chart was proposed by using the weighted mean of all past and current observations for monitoring the process (Roberts, 2000). Therefore, this chart is very effective for the slight process shifts (Lucas and Saccucci, 1990; Knoth, 2005; Khoo and Sim, 2006; Montgomery, 2012; Shamsuzzaman and Wu, 2012; Şentürk et al., 2014; Chakraborti and Graham, 2019; Hesamian, Akbari and Ranjbar, 2019; Mitra, Lee and Chakraborti, 2019). In 1982, cumulative sum control chart was developed by using the property of fast initial response (FIR) and this research was found that it perform well at the outset of the anomalous process (Lucas and Crosier, 1982). Later, the same FIR feature was used to develop EWMA control chart and it has a good performance in detecting of process mean

changes for various situations. This control chart is known as the fast initial response exponentially weighted moving average control chart or FIR_EWMA control chart (Lucas and Saccucci, 1990). Furthermore, the adjusted fast initial response value for the FIR_EWMA control chart was introduced and compared its performance with EWMA control chart (Steiner, 1999). This was found that the FIR_EWMA control chart was more effective than the EWMA control chart for detecting the change in process mean from the target at all shifts of process mean. Further, the FIR_EWMA control chart was more effective in detecting variations of the process mean than running sum control chart (Reynolds, 1971; Rigdon and Champ, 1997) and the combined Shewhart-cumulative sum control chart (Lucas, 1982) for all levels of process mean shifts. In 2006, the robust exponentially weighted moving average control chart or REWMA control chart for process average was proposed (Khoo and Sim, 2006). The process standard deviation for construction of the REWMA control chart was estimated by using the quartile range. Then, the results of this study was found that the capability of REWMA control chart performs well under case of data contain outliers (Khoo and Sim, 2006). In 2009, the MDMAD control chart (Abu-Shawiesh, 2009) was proposed by estimating the mean and standard deviation of process with the sample mean and sample median absolute deviation (Hampel, 1974; Rousseeuw and Croux, 1993), respectively. Moreover, the MDMAD control chart has a good performance for outliers in the data collection or data are non-normally distributed. Generally, the production process is often out of control, but commonly used control charts may be slow to detect process changes which causes waste to occur in the process. Especially, when the sample data with outliers are used to create the control limits. This reason, a new control chart, namely IEWMA control chart, is proposed by using the property of fast initial response and estimating the process mean and process standard deviation with the two robust estimators (Sinsomboonthong, Abu-Shawiesh and Kibria, 2020), namely, sample median and sample median absolute deviation. This proposed control chart is derived for case of the data were normally distributed with outlier occurred. In addition, the performance comparison in term of sensitivity to detect the process mean shifts of the proposed and four control charts—EWMA, REWMA, MDMAD and average control charts—are investigated for 504 data scenarios by simulation study.

2. Materials And Methods

In this paper, the efficiency comparison of five control charts for process mean are studied as follows:

2.1 Shewhart Average Control Chart

Shewhart average control chart or average control chart (Montgomery, 2012) is a familiar tool for quality control process. The procedures of control chart construction as follows: let X be a quality characteristic in the process. Suppose a random variable X is normally distributed with the mean μ and variance σ^2 . Let $X_1, X_2, \dots, X_n$ be the random samples of size n that are taken from this process. Hence, the sample mean $(\bar{X})$ is normally distributed with mean $\mu_{\bar{X}} = \mu$ and variance $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$. In case of the process average and the

process standard deviation are known, the upper control limit $(UCL_{\bar{X}})$, center line $(CL_{\bar{X}})$ and lower control limit $(LCL_{\bar{X}})$ of the average control chart can be determined in equation (1).

$$\begin{aligned} UCL_{\bar{X}} &= \mu + L_1 \frac{\sigma}{\sqrt{n}} = \mu_{\bar{X}} + L_1 \sigma_{\bar{X}} \\ CL_{\bar{X}} &= \mu_{\bar{X}} = \mu \quad \text{and} \\ LCL_{\bar{X}} &= \mu_{\bar{X}} - L_1 \sigma_{\bar{X}} = \mu - L_1 \frac{\sigma}{\sqrt{n}} \end{aligned} \quad (1)$$

L_1 is a constant that defines the control limit width corresponding to the control region $(1-\alpha)$ and a required average run length until a false alarm for the in-control process (ARL0) for the average control chart. For example, if a required ARL0 approximates 370 or the control region $(1-\alpha)$ equals 0.9973, then the usual three sigma control limits will be

constructed and the constant L_1 is equal to 3. In practice, the mean and standard deviation of the process are usually unknown. Then, the upper control limit $(UCL_{\bar{X}})$, center line $(CL_{\bar{X}})$ and lower control limit $(LCL_{\bar{X}})$ for the average control chart can be estimated in equation (2).

$$\begin{aligned} UCL_{\bar{X}} &= \bar{\mu} + L_1 \frac{\bar{\sigma}}{\sqrt{n}}, \\ CL_{\bar{X}} &= \bar{\mu} = \text{and } \frac{1}{g} \sum_{i=1}^g \bar{X}_i = \bar{\bar{X}} \\ LCL_{\bar{X}} &= \bar{\mu} - L_1 \frac{\bar{\sigma}}{\sqrt{n}} \end{aligned} \quad (2)$$

where $\bar{X}_i$ is the sample mean for subgroup i . The $\bar{\mu} = \frac{\bar{S}}{c_4}$ estimator of the process standard deviation is given by

$c_4 = \sqrt{\frac{2}{n-1}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$ or c_4 are the constants as shown in Table 1, $\bar{S} = \frac{1}{g} \sum_{i=1}^g S_i$ and S_i is the sample standard deviation for subgroup i .

n	c_4	d_2^Q	b_n	J_n
2	0.7979	1.1284	1.1960	1.0597
3	0.8862	1.6926	1.4950	1.0815
4	0.9213	0.5940	1.3630	0.8539
5	0.9400	0.9900	1.2060	0.6758
6	0.9515	1.2835	1.2000	0.6138
7	0.9594	1.5147	1.1400	0.5399
8	0.9650	0.9456	1.1290	0.5001
9	0.9693	1.1439	1.1070	0.4624
10	0.9727	1.3121	1.0870	0.4307
11	0.9754	1.4577	1.0784	0.4074
12	0.9776	1.0737	1.0714	0.3875
13	0.9794	1.2057	1.0656	0.3703
14	0.9810	1.3235	1.0606	0.3552
15	0.9823	1.4298	1.0563	0.3417
16	0.9835	1.1400	1.0526	0.3297
17	0.9845	1.2389	1.0494	0.3189
18	0.9854	1.3269	1.0465	0.3091
19	0.9862	1.4132	1.0440	0.3001
20	0.9869	1.1806	1.0417	0.2919

Table 1: Factor for constructing the variable control charts

2.2 Exponentially Weighted Moving Average Control Chart

Exponentially weighted moving average control chart or EWMA control chart is a good alternative to the Shewhart average control chart for detecting small shifts (Roberts, 2000). For construction of the control limits of EWMA control chart, the assumptions about quality characteristic distribution are the same details as mentioned in section [2.1]. The exponentially weighted moving average at any time t or any subgroup t can be defined in equation (3).

$$G_t = \lambda \bar{X}_t + (1-\lambda)G_{t-1} \quad (3)$$

where $0 < \lambda \leq 1$ is a constant and the starting value is the process mean or process target, that is, $G_0 = \mu_0$ where μ_0 is the average of process quality characteristic to be controlled. The upper control limit (UCL), center line (CL) and lower control limit (LCL) for EWMA control chart can be estimated and written in equation (4).

$$\begin{aligned} UCL &= \bar{\bar{X}} + L_2 \sigma \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \\ CL &= \bar{\bar{X}} = \frac{1}{g} \sum_{t=1}^g \bar{X}_t \quad \text{and} \\ LCL &= \bar{\bar{X}} - L_2 \sigma \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \end{aligned} \quad (4)$$

where L_2 is a constant that defines the control limit width

corresponding to the control region $(1-\alpha)$ and a required ARL_0 for EWMA control chart, and $\bar{X}_t$ is the sample mean for subgroup t . The estimator of the process standard deviation is

$$\sigma = \frac{\bar{S}}{c_4}$$

also given by

2.3 Robust Exponentially Weighted Moving Average Control Chart

The robust exponentially weighted moving average control chart or REWMA control chart was introduced in 2006 (Khoo and Sim, 2006). This control chart developed by using the

exponentially weighted moving average $G_t = \lambda \bar{X}_t + (1-\lambda)G_{t-1}$ as same as the construction of EWMA control chart. The process standard deviation of this chart was estimated by using the quartile range. Let $IQR = X_{(s)} - X_{(r)}$ be the interquartile range where $X_{(r)}$

and $X_{(s)}$ are denoted statistics of orders $r = \left[\frac{n}{4}\right] + 1$ and

$s = n - r + 1$, respectively. The symbol $\left[\frac{n}{4}\right]$ is denoted the

greatest integer that is not greater than $\frac{n}{4}$. Rocke demonstrated that the mathematical expectation of IQR is given by $E(IQR) = d_2^Q \sigma_X$ where d_2^Q is the constant that can be

studied in the researches of (Rocke, 1992; Khoo and Sim, 2006). In this paper, these constants are shown in Table 1. The upper control limit (UCL), center line (CL) and lower control limit (LCL) for REWMA control chart can be estimated and illustrated in equation (5).

$$\begin{aligned}
 UCL &= \bar{\bar{X}} + L_3 \frac{\overline{IQR}}{d_2^Q} \sqrt{\frac{\lambda}{n(2-\lambda)}}, \\
 CL &= \bar{\bar{X}} = \frac{1}{g} \sum_{t=1}^g \bar{X}_t \quad \text{and} \\
 LCL &= \bar{\bar{X}} - L_3 \frac{\overline{IQR}}{d_2^Q} \sqrt{\frac{\lambda}{n(2-\lambda)}} \\
 \text{where } \overline{IQR} &= \frac{1}{g} \sum_{t=1}^g IQR_t \quad \text{is the average of the subgroup}
 \end{aligned}
 \tag{5}$$

interquartile ranges, and L_3 is a constant that defines the control limit width corresponding to the control region ($1-\alpha$) and a required ARL0 for REWMA control chart.

2.4 Median Mean Absolute Deviation Control Chart

Median mean absolute deviation control chart or MDMAD control chart was proposed in 2009 (Abu-Shawiesh, 2009). This control chart was constructed based on the robust estimators. Suppose that the sample median is represented by symbol MD . The sample median absolute deviation or MAD can be written in the formula of equation (6).

$$MAD = 1.4826 \times MD \left\{ \left| X_j - MD \right| \right\} \quad \text{for } j = 1, 2, \dots, n
 \tag{6}$$

Then, the estimation of upper control limit (UCL), center line (CL) and lower control limit (LCL) for MDMAD control chart can be written in equation (7).

$$\begin{aligned}
 UCL &= \overline{MD} + L_4 \frac{1.253b_n \overline{MAD}}{\sqrt{n}}, \\
 CL &= \overline{MD} = \frac{1}{g} \sum_{i=1}^g MD_i \quad \text{and} \\
 LCL &= \overline{MD} - L_4 \frac{1.253b_n \overline{MAD}}{\sqrt{n}}
 \end{aligned}
 \tag{7}$$

where $\overline{MAD}$ is the average of the subgroup sample median absolute deviation, b_n is the constant that can be studied in the researches of (Rousseeuw and Croux, 1993;

Abu-Shawiesh, 2009), and L_4 is a constant that defines the control limit width corresponding to the control region ($1-\alpha$) and a required ARL0 for MDMAD control chart.

2.5 The Proposed Control Chart

This section, the proposed method is called improved exponentially weighted moving average (IEWMA) control chart which is estimated using normal distributed data with outliers. This control chart was developed from the fast initial response exponentially weighted moving average (FIR_EWMA) control chart (Lucas and Saccucci, 1990). By considering the adjusted fast initial response feature or FIRadj values (Steiner, 1999) were presented by Steiner as shown in equation (8).

$$FIR_{adj} = 1 - (1-f)^{1+a(t-1)}
 \tag{8}$$

$$a = \frac{1}{19} \left(\frac{-2}{\log_{10}(1-f)} - 1 \right)$$

where $0 < f \leq 1$ and the constant

In this research, the constant f is set at 0.5 and the past researches (Lucas and Saccucci, 1990; Steiner, 1999; Knoth, 2005) were found that this value is the optimal constant for obvious initial detection of process abnormalities. This also makes the control limits to be narrow. Therefore, the constant $a = 0.3$ will be obtained. Let X be a quality attribute in the process. Assume a random variable X is normally distributed

with the mean μ and variance σ^2 . Let $X_1, X_2, \dots, X_n$ be the random samples of size n that are taken from this process. The exponentially weighted moving average at any time t can

be defined as $G_t = \lambda \bar{X}_t + (1-\lambda)G_{t-1}$ where $0 < \lambda \leq 1$ are the constants and the starting value is the process mean or process target, that is, $G_0 = \mu$.

In case of the process mean and the process standard deviation are known, the upper control limit (UCL), center line (CL) and lower control limit (LCL) of the FIR_EWMA control chart can be determined in equation (9).

$$\begin{aligned}
 UCL &= \mu + L\sigma \left\{ \left(1 - (1-f)^{1+a(t-1)} \right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t} \right) \right\}, \\
 CL &= \mu \quad \text{and} \\
 LCL &= \mu - L\sigma \left\{ \left(1 - (1-f)^{1+a(t-1)} \right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t} \right) \right\}
 \end{aligned}
 \tag{9}$$

where L is a constant that defines the control limit width corresponding to the control region ($1-\alpha$) and a required ARL0 for FIR_EWMA control chart. When data are contaminated with outliers, the sample mean and the sample standard deviation are not robust estimators of the process mean and the process standard deviation, respectively. Therefore, in this study, the mean and standard deviation of the process are estimated by using the two robust estimator, i.e., the sample median and the corrected sample median absolute deviation (Hampel, 1974;

Rousseeuw and Croux, 1993), respectively. Let MD be the sample median of the random samples $X_1, X_2, \dots, X_n$ which can be calculated in equation (10).

$$MD = \begin{cases} X_{\left(\frac{n+1}{2}\right)} & \text{for } n \text{ is odd values} \\ \frac{1}{2} \left(X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)} \right) & \text{for } n \text{ is even values} \end{cases} \quad (10)$$

where the symbol $X_{(m)}$ is denoted the m^{th} order statistics. From the study of (Rousseeuw and Croux, 1993), it is found that an unbiased estimator of the process standard deviation σ is given by the corrected sample median absolute

deviation or $b_n MAD$ where b_n is the correction term that give the mathematical expectation in the form of $E(b_n MAD) = \sigma$ to be true. For $n \leq 20$, the factor b_n were demonstrated by (Abu-Shawiesh, 2009) and these values are shown in Table 1. For $n > 20$, the factor b_n can be computed

$b_n = \frac{n}{n-0.8}$ (Rousseeuw and Croux, 1993). In 1974, Hampel first published the sample median absolute deviation or $MAD = h \times MD \left\{ \left| X_j - MD \right| \right\}$ (Hampel, 1974). In case of data are taken from normal distributed data, the constant $h = 1.4826$ (Rousseeuw and Croux, 1993) will give consistency estimator of σ . Hence, the optimal sample median absolute deviation can be written in equation (11).

$$MAD = 1.4826 \times MD \left\{ \left| X_j - MD \right| \right\} \quad \text{for } j = 1, 2, \dots, n \quad (11)$$

Definition

Let $X_1, X_2, \dots, X_n$ be the random samples of size n that are drawn from a population with continuous distribution function $f(x)$ and its cumulative distribution function is differentiable and has derivative $f'(x) = F'(x)$. For strictly increasing continuous function $F(x)$, the p^{th} quantile of distribution is denoted as ξ_p or $F(\xi_p) = p$ where $0 < p < 1$. Further, the p^{th} sample quantile is signified by symbol $\$p = F_n^{-1}(p)$, and F_n is the empirical cumulative distribution function.

Theorem

Let $X_1, X_2, \dots, X_n$ be the random samples of the large

sizes n which are taken from infinite normally distributed population with the probability density function $f(x; \mu, \sigma^2)$ or $f(x)$. The symbol $\xi_{0.5}$ is denoted as the population median. If $f(\xi_{0.5}) \neq 0$ and $f(x)$ is continuous in some neighborhood of $x = \xi_{0.5}$, then the sample median MD is asymptotically normally distributed with the mean and standard deviation are given in the forms of equation (12) and equation (13), respectively.

$$\mu_{MD} = \xi_{0.5} \quad (12)$$

$$\sigma_{MD} = 1.2533 \frac{\sigma}{\sqrt{n}} \quad (13)$$

Proof

Let $X_1, X_2, \dots, X_n$ be independent and identically random variables which are sampled from a normal population. Refer to Theorem of the asymptotic distribution of sample quantiles that was studied by (Walker, 1968; CRAMÉR, 1999), it is found that

$\sqrt{n}(\$p - \xi_p)$ converges in law to normal distribution with mean and variance are equal to 0 and $\frac{p(1-p)}{f^2(\xi_p)}$, respectively when $n \rightarrow \infty$. The mean and variance of sample quantile of order p can be demonstrated as follows:

$$\begin{aligned} E\left[\sqrt{n}(\$p - \xi_p)\right] &= E\left[\sqrt{n}\$p - \sqrt{n}\xi_p\right] = 0 \\ &= \sqrt{n}E[\$p] - \sqrt{n}\xi_p = 0 \end{aligned}$$

That is, the mean of $\$p$ can be resolved in a form of equation (14).

$$\mu_{\$p} = E[\$p] = \xi_p \quad (14)$$

Next, to prove the variance of the p^{th} sample quantile.

$$\begin{aligned} V\left[\sqrt{n}(\$p - \xi_p)\right] &= V\left[\sqrt{n}\$p - \sqrt{n}\xi_p\right] = \frac{p(1-p)}{f^2(\xi_p)} \\ &= nV[\$p] = \frac{p(1-p)}{f^2(\xi_p)} \end{aligned}$$

That is, the variance of $\$p$ can be simplified in the formula of equation (15).

$$\sigma_{\xi_p}^2 = V\left[\frac{\$}{\xi_p}\right] = \frac{p(1-p)}{nf^2(\xi_p)} \quad (15)$$

Therefore, the $\frac{\$}{\xi_p}$ also converges in law to normal distribution with mean ξ_p and variance $\frac{p(1-p)}{nf^2(\xi_p)}$ when $n \rightarrow \infty$.

From Definition, if $p = 0.5$ then the sample quantile of order 0.5, denoted as $\frac{\$}{\xi_{0.5}} = F_n^{-1}(0.5)$. This quantile will become the sample median (MD) or $\frac{\$}{\xi_{0.5}} = F_n^{-1}(0.5) = MD$. Additionally, the random samples $X_1, X_2, \dots, X_n$ are taken from a population with normal distribution which is symmetric curve, hence the population mean is equal to the population median, that is, $\xi_{0.5} = \mu$.

From equation (14), it is obvious evidence that the mean of MD is equal to the population median and this can be written in the form of equation (16).

$$\mu_{\frac{\$}{\xi_{0.5}}} = \mu_{MD} = \xi_{0.5} \quad (16)$$

Next, the proof of variance for sample median are as follows:

$$\begin{aligned} f(\xi_{0.5}) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\xi_{0.5} - \mu)^2} \\ \text{consider,} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(\mu - \mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \end{aligned} \quad (17)$$

For $p = 0.5$, equation (17) is substituted into the equation (15), then the variance of sample median is illustrated as equation (18).

$$\sigma_{\frac{\$}{\xi_{0.5}}}^2 = \sigma_{MD}^2 = \frac{0.5(1-0.5)}{n\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^2} = \frac{1}{4n\left(\frac{1}{2\pi\sigma^2}\right)} = \frac{\pi\sigma^2}{2n} \quad (18)$$

Therefore, $\pi = 3.14159$ is substituted in equations (18) and the standard deviations of sample median is written as

$$\sigma_{MD} = 1.2533 \frac{\sigma}{\sqrt{n}}$$

The following procedures are illustrated the construction of control limits for IEWMA control chart by using the robust estimators to scale and location parameters. Then, considering the control limits from equation (9), the two parameters— μ and σ —are substituted with the mean and standard deviation of sample median (or μ_{MD} and σ_{MD}) and this equation can be written in the form of equation (19).

$$UCL = \mu_{MD} + L\sigma_{MD} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\}$$

$$CL = \mu_{MD}$$

$$LCL = \mu_{MD} - L\sigma_{MD} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\}$$

(19)

From equation (12) and equation (13) of the above Theorem,

$$\mu_{MD} = \xi_{0.5} \text{ and } \sigma_{MD} = 1.2533 \frac{\sigma}{\sqrt{n}}. \text{ Then, the center line (CL) for IEWMA control chart can be estimated by using the}$$

sample median. That is, $CL = \hat{\mu}_{MD} = \frac{\$}{\xi_{0.5}}$. Additionally, the collected data consist of g subgroups with the sample size in each subgroup being n , then the CL can be estimated by using the average of sample medians for g subgroups as illustrated in equation (20).

$$CL = \overline{MD} = \frac{1}{g} \sum_{t=1}^g MD_t \quad (20)$$

where MD_t is the sample median of subgroup t for $t = 1, 2, \dots, g$. Moreover, the upper and lower control limits of IEWMA control chart that are illustrated in equation (19) can be estimated as follows:

$$\begin{aligned} UCL &= \hat{\mu}_{MD} + L\hat{\sigma}_{MD} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\} \\ &= \hat{\mu}_{MD} + L \frac{1.2533\hat{\sigma}}{\sqrt{n}} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\} \\ &= \frac{\$}{\xi_{0.5}} + L \frac{1.2533\hat{\sigma}}{\sqrt{n}} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\} \end{aligned}$$

$$\frac{\$}{\xi_{0.5}} = \overline{MD} \text{ and } \hat{\sigma} = b_n \overline{MAD}$$

where $\frac{\$}{\xi_{0.5}} = \overline{MD}$ and $\hat{\sigma} = b_n \overline{MAD}$. Then, the robust estimated upper control limit of IEWMA control chart is given

$$\overline{MD} + LJ_n \overline{MAD} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\}.$$

In the same way, the robust estimated lower control limit of IEWMA control chart can be estimated as follows:

$$LCL =$$

$$\overline{MD} - LJ_n \overline{MAD} \left\{ \left(1 - (1-f)^{1+a(t-1)}\right) \sqrt{\frac{\lambda}{n(2-\lambda)}} \left(1 - (1-\lambda)^{2t}\right) \right\}.$$

where $\overline{MAD} = \frac{1}{g} \sum_{t=1}^g MAD_t$ and MAD_t is the sample median absolute deviation of subgroup t for

$t = 1, 2, \dots, g$. The values of constant J_n depend on the sample size n . For $n \leq 20$, these values are provided in Table

1. For $n > 20$, the values of J_n can be calculated as in equation (21).

$$J_n = \frac{1.2533b_n}{\sqrt{n}} = \frac{1.2533\left(\frac{n}{n-0.8}\right)}{\sqrt{n}} = \frac{1.2533\sqrt{n}}{(n-0.8)} \quad (21)$$

The plotted statistical values in the IEWMA control chart are the exponentially weighted moving averages at any time t . The initial value of the exponentially weighted moving average is

$$G_0 = \bar{\bar{X}} = \frac{1}{g} \sum_{t=1}^g \bar{X}_t$$

given by

3. Results of a Simulation Study

In this section, the efficiency comparison in term of sensitivity to detect changes in process mean from the target of the proposed control chart and the four control charts—EWMA, REWMA, MDMAD and average control charts—are studied via the simulation data. The control limits of five control charts were constructed by using data set in two cases, i.e., the first case that data not contain outliers and the second case that data contain outliers. The population data for the in-control process are generated to have the normal distribution with mean $\mu_0 = 10$ and variance $\sigma^2 = 4$. Then, the samples of size n equals 5, 10, 15, 20 are randomly taken from these populations and repeated 10,000 times for each sample size and set the subgroup size (g) equals 30. For each sample size, the random sample data are constructed to contaminate with 5% outliers of the total amount of the collected data in each replication. Then, these simulated data are used to create the control limits of five control charts. In addition, the control limits for three control charts—IEWMA, EWMA and REWMA control charts—are

created by using the weighted constant (λ) with different values of 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 and 0.9. In this simulation study, the control limits of the five control charts are constructed in case of the in-control average run length (ARL0) equals 370. After that these generated control charts are used to compare the performance when the processes are out-of-control for different levels of the mean shifts from the target, that is, the studied out-of-control processes averages equal

$\mu_1 = \mu_0 + \delta\sigma$ where $\delta = 0.4, 0.6, 0.8, 1.0, 1.5, 2.0$. The efficiency comparison of five control charts are investigated by considering the out-of-control average run lengths or ARL1. If any control chart give the smallest ARL1, then it is sensitivity to detect abnormal process. The results of this study to compare the efficiency in term of sensitivity for process mean shifts detection of five control charts that are divided into 2 cases: the first case, five control charts were created from the sample data without outliers. In the second case, the five control charts are constructed from the sample data that are contaminated with 5% outliers of the total collected data.

Before considering the efficiency comparison of five control charts, we will consider the optimal weighted constant or optimal value of λ that makes the three exponentially weighted moving average control charts—IEWMA, REWMA and EWMA control charts—have the best sensitivity to detect process mean shift from the target. In the case of creating control charts from sample data without outliers, Figure 1 to Figure 4 are shown that

0.1 is the optimal value of λ for construction these charts whatever the sample size will be. It means that the construction of three exponentially weighted moving average control charts by using λ equals 0.1, these charts give the smallest ARL1.

Additionally, if the values of λ increase, the performances of these three charts will tend to slowly detect process abnormality for the small shifts from the target. However, for the large shifts

from the target ($\delta = 1.5, 2.0$), all values of λ can be used because the performances of three charts are not different, especially for the large sample sizes ($n = 10, 15, 20$) in each subgroup.

Figure 1: The out- of-control ARL₁ of three control charts when calculating using the different values of λ and n equals 5 for data not contaminate outliers.

Figure 2: The out- of-control ARL_1 of three control charts when calculating using the different values of λ and n equals 10 for data not contaminate outliers.

Figure 3: The out- of-control ARL_1 of three control charts when calculating using the different values of λ and n equals 15 for data not contaminate outliers.

Figure 4: The out- of-control ARL_1 of three control charts when calculating using the different values of λ and n equals 20 for data not contaminate outliers.

As a result of Table 2, it is found that the three charts—IEWMA, EWMA and REWMA control charts (when using λ equals 0.1)—have the lowest ARL1, which is ARL1 mostly equals 1, for all levels of δ and all levels of sample sizes.

Furthermore, the MDMAD and average control charts tend to slowly detect the changes of process mean when the small process mean shifts from the target are occurred.

n	δ	Control charts				
		$\bar{X}$	IEWMA	EWMA	REWMA	MDMAD
5	0.4	57.5	1.2*	1.2*	1.2*	57.5
	0.6	23.0	1.0*	1.0*	1.0*	23.0
	0.8	9.6	1.0*	1.0*	1.0*	9.6
	1.0	4.8	1.0*	1.0*	1.0*	4.7
	1.5	1.6	1.0*	1.0*	1.0*	1.6
	2.0	1.1	1.0*	1.0*	1.0*	1.1
10	0.4	25.5	1.0*	1.0*	1.0*	25.5
	0.6	7.8	1.0*	1.0*	1.0*	7.8
	0.8	3.3	1.0*	1.0*	1.0*	3.3
	1.0	1.8	1.0*	1.0*	1.0*	1.8
	1.5	1.0*	1.0*	1.0*	1.0*	1.0*
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*
15	0.4	12.1	1.0*	1.0*	1.0*	15.1
	0.6	3.7	1.0*	1.0*	1.0*	4.3
	0.8	1.8	1.0*	1.0*	1.0*	2.0
	1.0	1.2	1.0*	1.0*	1.0*	1.3
	1.5	1.0*	1.0*	1.0*	1.0*	1.0*
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*
20	0.4	8.6	1.0*	1.0*	1.0*	8.6
	0.6	2.6	1.0*	1.0*	1.0*	2.6
	0.8	1.4	1.0*	1.0*	1.0*	1.4
	1.0	1.1	1.0*	1.0*	1.0*	1.1
	1.5	1.0*	1.0*	1.0*	1.0*	1.0*
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*

Note: * means that the control charts have the lowest ARL1 in that situation.

Table 2: ARL₁ of five control charts when calculating using λ equals 0.1 for data not contaminate outliers

Additionally, Figure 5 to Figure 8 show the results in case of control charts are created from the sample data set that are contaminated with 5% outliers of the total collected data. These Figures are also illustrated that the best value of constant λ that makes the three control charts perform well is the same as case of data are not contaminated with outliers. That is, λ equals 0.1 will make these charts have the smallest ARL1 for all sample sizes. Moreover, if the values of λ increase, the efficiencies of these three charts seem to gradually detect the changes of process mean for the small shifts. These simulation study results are shown that if the process are massive

changes, e.g. $\delta = 1.5, 2.0$, all values of λ are suitable features to estimate the control limits because the expressions in term of ARL1 for the three charts are not different, especially for the larger sample sizes ($n = 15, 20$) in each subgroup. Further, when using the values of λ are greater than 0.1, e.g., λ equals 0.9, it is found that the ARL1 of IEWMA control chart tends to be smaller than those of EWMA and REWMA control charts for all levels of sample sizes, especially when there is a small change in the average process from the target.

Figure 5: The out- of-control ARL1 of three control charts when calculating using the different values of λ and n equals 5 for data contaminate 5% outliers.

Figure 6: The out- of-control ARL1 of three control charts when calculating using the different values of λ and n equals 10 for data contaminate 5% outliers.

Figure 7: The out- of-control ARL1 of three control charts when calculating using the different values of λ and n equals 15 for data contaminate 5% outliers.

Figure 8: The out- of-control ARL_1 of three control charts when calculating using the different values of λ and n equals 20 for data contaminate 5% outliers.

From Table 3, it is found that the IEWMA control chart (for using λ equals 0.1) have the lowest ARL_1 for all levels of δ regardless of the sample size. Further, the three charts with using λ equals 0.1—IEWMA, EWMA and REWMA control charts—have the lowest ARL_1 (it means that ARL_1 equals 1) for

the values of δ that are greater than 0.6 and sample sizes in each subgroup equal 10, 15 and 20. Furthermore, the MDMAD and average control charts tend to slowly detect process abnormalities for the small shifts of process, these obviously found for the small sample sizes in each subgroup.

n	δ	Control charts				
		$\bar{X}$	IEWMA	EWMA	REWMA	MDMAD
5	0.4	1,428.6	2.8*	21.1	11.8	344.8
	0.6	500.0	1.1*	1.8	1.4	82.6
	0.8	123.5	1.0*	1.0*	1.0*	31.5
	1.0	46.1	1.0*	1.0*	1.0*	12.7
	1.5	5.3	1.0*	1.0*	1.0*	2.6
	2.0	1.7	1.0*	1.0*	1.0*	1.3
10	0.4	909.1	1.2*	5.2	3.0	83.3
	0.6	126.6	1.0*	1.0*	1.0*	19.4
	0.8	30.4	1.0*	1.0*	1.0*	6.4
	1.0	9.0	1.0*	1.0*	1.0*	2.9
	1.5	1.6	1.0*	1.0*	1.0*	1.1
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*
15	0.4	555.6	1.0*	3.2	1.9	34.8
	0.6	59.9	1.0*	1.0*	1.0*	7.9
	0.8	11.4	1.0*	1.0*	1.0*	2.9
	1.0	3.5	1.0*	1.0*	1.0*	1.5
	1.5	1.1	1.0*	1.0*	1.0*	1.0*
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*
20	0.4	344.8	1.0*	1.7	1.3	23.6
	0.6	34.4	1.0*	1.0*	1.0*	4.9
	0.8	6.3	1.0*	1.0*	1.0*	1.9
	1.0	2.2	1.0*	1.0*	1.0*	1.2
	1.5	1.0*	1.0*	1.0*	1.0*	1.0*
	2.0	1.0*	1.0*	1.0*	1.0*	1.0*

Note: * means that the control charts have the lowest ARL_1 in that situation.

Table 3: ARL_1 of five control charts when calculating using λ equals 0.1 for data contaminate 5% outliers

4. Discussion

In case of control charts are created from the sample data without outliers, the simulation results show that the performance of exponentially weighted moving average control chart is better than the Shewhart average control chart for all situations. This conforms to the researches of (Huang, Tai and Lu, 2014) and (Chew et al., 2015). Moreover, exponentially weighted moving average control chart seems to fast detect for a small change of process average in accordance with the research of (Shafqat et al., 2020), (Taboran, Sukparungsee and Areepong, 2021), (Alevizakos, Chatterjee and Koukouvinos, 2021) and (Saeed and Abu-Shawiesh, 2021). This research is also found that REWMA control chart, which is constructed from the sample data that are contaminated with outliers, have a good efficiency in detection of process anomalies when process mean changes from the target as mention by (Khoo and Sim, 2006). Even though, the proposed control chart is estimated from data with outliers, it will rapid detection when process mean has slightly change from the target because the proposed control limits are derived by using two robust properties, i.e., the fast initial response feature (Lucas and Crosier, 1982; Lucas and Saccucci, 1990) and the robust point estimators of location and scale parameters (Hampel, 1974; Rousseeuw and Croux, 1993; Sinsomboonthong, Abu-Shawiesh and Kibria, 2020). These properties make the performance of IEWMA control chart superior to others.

5. Conclusion

In practice, the sample data often contains some abnormal high or low quality products when they are taken in the process. These effect to construct the traditional control charts. As a result, the created control chart is not effective at detecting changes in process averages or it slowly detect the change of process mean. Therefore, the proposed control chart or IEWMA control chart was developed as a tool to help process operators quickly recognize that a process is out of control. This will enable the process controller to inform technicians or those involved in the production process to resolve problems in a timely manner. When a factory has a good process control tools, these will help to reduce the amount of non-standard products. In case of IEWMA control chart is constructed from the sample data that are not contaminated with outliers, its performance to detect about process mean shifts from the target tends to be close to exponentially weighted moving average control chart and REWMA control chart for all levels of the process mean shifts regardless of the sample size. Additionally, if control limits of IEWMA control chart are constructed from the sample data that contaminate with 5% outliers of the total amount of the collected data, it also performs the most efficiency for process mean shifts detection for all sample sizes, especially for the small shifts from the process mean.

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