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Research Article

The multicolor star-critical Gallai-Ramsey number for a path of order 5

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Abstract

In this paper, the t-color star-critical Gallai-Ramsey number for a path of order 5 is determined. It is proved that $t+1$ edges are both necessary and sufficient to add between a vertex and a critical coloring for the t-color Gallai-Ramsey number for P_5 in order to guarantee the existence of a monochromatic subgraph isomorphic to P_5. The proof depends on a well-known structural result for Gallai colorings as well as a general lower bound due to Faudree, Gould, Jacobson, and Magnant.

Keywords: Ramsey numbers; Gallai colorings; critical colorings.

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1. Introduction

Let P_n denote a path of order n. This paper focuses on the evaluation of the t-color star-critical Gallai-Ramsey number for P_5. Before discussing the main results, we must review the relevant background and definitions. Denote the complete graph of order p by K_p and the complete bipartite graph with partite sets having cardinalities m and n by $K_{m,n}$. When $m=1$, the resulting complete bipartite graph $K_{1,n}$ is called a *star* and the vertex in the partite set with cardinality 1 is called the *center vertex* for the star.

For $t \geq 2$, define a *t-coloring* of a graph $G=(V(G),E(G))$ to be a map $f: E(G) \longrightarrow \{1,2,\ldots,t\}$. The *$t$-color Ramsey number* $r^t(G)$ is the least natural number p such that every t-coloring of the edges of K_p contains a monochromatic subgraph isomorphic to G. A *critical coloring for* $r^t(G)$ is a t-coloring of $K_{r^t(G)-1}$ that lacks a monochromatic copy of G. In 1967, Gerencsér and Gyárfás [5] proved that

$$r^2(P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1, \quad \text{for all } n \geq 2,$$

where P_n is a path of order n.

The star-critical Ramsey number serves as a refinement of the concept of a Ramsey number. In order to define it, let $K_n \sqcup K_{1,k}$ be the graph formed by taking the union of a vertex v with the graph K_n and joining v with edges to exactly k vertices in the K_n $(1 \leq k \leq n)$. The *star-critical Ramsey number* $r_*^t(G)$ is then defined to be the least k such that every t-coloring of $K_{r^t(G)-1} \sqcup K_{1,k}$ contains a monochromatic copy of G. These numbers were first defined by Hook in her dissertation [7] (see also [8] and Section 2.1 of [1]), where she showed that

$$r_*^2(P_n) = \left\lceil \frac{n}{2} \right\rceil, \quad \text{for all } n \geq 2.$$

A t-coloring f of a graph G is called a *Gallai t-coloring* of G if it does not contain any rainbow triangles (see [3,9]). That is, $|f(xy), f(yz), f(xz)| \leq 2$ for all distinct $x,y,z \in V(G)$. The *t-color Gallai-Ramsey number* $gr^t(G)$ is then defined to be the least natural number p such that every Gallai t-coloring of K_p contains a monochromatic copy of G. Note that when $t=2$, $r^2(G)=gr^2(G)$. A *critical coloring for* $gr^t(G)$ is a Gallai t-coloring of $K_{gr^t(G)-1}$ that lacks a monochromatic copy of G. The *star-critical Gallai-Ramsey number* $gr_*^t(G)$ is the least k such that every Gallai t-coloring of $K_{gr^t(G)-1} \sqcup K_{1,k}$ contains a monochromatic copy of G. The following structure theorem for Gallai colorings can be found in [6] and is a reinterpretation of a classic result of Gallai [4]. It is the basis of many upper bound results for Gallai-Ramsey numbers.

Theorem 1.1. *Every Gallai-colored complete graph can be formed by replacing the vertices of a 2-colored complete graph of order at least two with Gallai-colored complete graphs.*

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In such a coloring, the 2-colored complete graph is called the *base graph* while the Gallai-colored complete graphs that replace the vertices in the base graph are called *blocks*. The partition of the vertex set of the full Gallai-colored complete graph into subsets that correspond with the blocks is referred to as a *Gallai partition*.

In [2], Faudree, Gould, Jacobson, and Magnant showed that for all $t \geq 3$,

$$gr^t(P_4) = t + 3 \quad \text{and} \quad gr^t(P_5) = t + 4.$$

The lower bounds for both of these Gallai-Ramsey numbers came from a general construction which implies that if G is a connected graph of order n, then

$$gr^t(G) \geq n + (c(G) - 1)(t - 1). \tag{1}$$

Here, $c(G)$ is the *edge cover number*, defined to be the minimum cardinality of a subset $C \subseteq V(G)$ such that every edge in $E(G)$ is incident with some element in C. In 2022, Su and Liu [10] determined the t-color star-critical Gallai-Ramsey number for P_4:

$$gr^t_*(P_4) = t, \quad \text{for all } t \geq 3.$$

The focus of this paper is the evaluation

$$gr^t_*(P_5) = t + 1.$$

2. Main results

Before considering the evaluation of $gr^t_*(P_5)$, we prove an important property of the the critical colorings of $gr^t(P_5)$. The following lemma depends heavily on Theorem 1.1.

Lemma 2.1. *If $t \geq 2$, then every critical coloring for $gr^t(P_5)$ contains a vertex that is incident with edges that are all in the same color class.*

Proof. When $t = 2$, this lemma follows from the critical colorings of $r(P_5, P_5)$ described in Proposition 2.6 of [8], all of which contain a vertex incident with edges that are all in the same color class (also, see Theorem 2.1 of [1]). For $t \geq 3$, consider a Gallai t-coloring of K_{t+3} that lacks a monochromatic P_5. By Theorem 1.1, this complete graph can be formed by replacing the vertices of a 2-colored complete graph of order at least two with Gallai-colored complete graphs. Let $\mathcal{B}$ be a base graph of minimum order $q \geq 2$ among all possible Gallai partitions and denote the vertex sets that correspond with the vertices in $\mathcal{B}$ by $X_1, X_2, \ldots, X_q$. The Ramsey number $r^2(P_5) = 6$ (see [5]) implies that $q \leq 5$. Also, Lemma 3.1 of [9] implies that $q \neq 3$ since q is chosen to be minimal and the case of $q = 3$ can be reduced to the case in which $q = 2$. The values $q = 2, 4, 5$ must be considered separately.

<u>Case 1.</u> Assume that $q = 2$. By the pigeonhole principle, either $|X_1| \geq 3$ or $|X_2| \geq 3$ since $t + 3 \geq 6$. If both blocks have order at least two, then without loss of generality, assume that $x_1, y_1, z_1 \in X_1$ and $x_2, y_2 \in X_2$. Then $x_1x_2y_1y_2z_1$ is a monochromatic P_5, contradicting the assumption that we are considering a critical coloring. It follows that one block has only a single vertex and that vertex is incident with edges that are all in the same color class.

<u>Case 2.</u> Assume that $q = 4$. If some block, say X_1, satisfies $|X_1| \geq 3$, then suppose that $x_1, y_1, z_1 \in X_1$ and $x_i \in X_i$ for each i such that $2 \leq i \leq 4$. By the pigeonhole principle, at least two of X_2, X_3, X_4 must join to X_1 via edges of the same color. Without loss of generality, suppose that X_2 and X_3 both join to X_1 via red edges (see Figure 1). Then $x_1x_2y_1x_3z_1$ is a red P_5, contradicting the assumption that we are considering a critical coloring.

If no X_i contains at least three vertices, then at least two X_i must contain exactly two vertices and $t \leq 5$. Assume that $|X_1| = |X_2| = 2$, $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$, $x_3 \in X_3$, $x_4 \in X_4$, and all edges joining X_1 and X_2 are red. If either of X_3 or X_4 join to X_1 or X_2 via red edges, say X_1 and X_3 are joined by red edges, then $x^2x_1y_2y_1x_3$ is a red P_5, again giving a contradiction (see the first image in Figure 2). So, assume that X_3 and X_4 join to X_1 and X_2 via blue edges (see the second image in Figure 2. Then $x_1x_3y_1x_4x_2$ is a blue P_5, giving a contradiction. Thus, it follows that no such critical coloring exists with $q = 4$.

<u>Case 3.</u> Assume that $q = 5$. Since $t + 3 \geq 6$, the pigeonhole principle implies that some X_i contains more than one vertex. Assume that $|X_1| \geq 2$ and let $x_1, y_1 \in X_1$. For each i such that $2 \leq i \leq t$, select a single vertex in X_i and denote it by x_i. Suppose that at least three of X_2, X_3, X_4, X_5 join to X_1 via the same color edge. Without loss of generality, assume that X_2, X_3, X_4 all join to X_1 via red edges (see the first image in Figure 3). Then $x_2x_1x_3y_1x_4$ is a red P_5, which is a contradiction.

If at most two of X_2, X_3, X_4, X_5 join to X_1 via the same color edges, then exactly two of them will join in each color. Without loss of generality, assume that X_2 and X_3 join to X_1 via red edges and X_4 and X_5 join to X_1 via blue edges (see the second image in Figure 3). If x_2x_5 is red then, $x_5x_2x_1x_3y_1$ forms a red P_5. If x_2x_5 is blue, then $x_2x_5x_1x_4y_1$ is a blue P_5. In all cases, there is a monochromatic P_5, so no such critical coloring exists with $q = 5$.

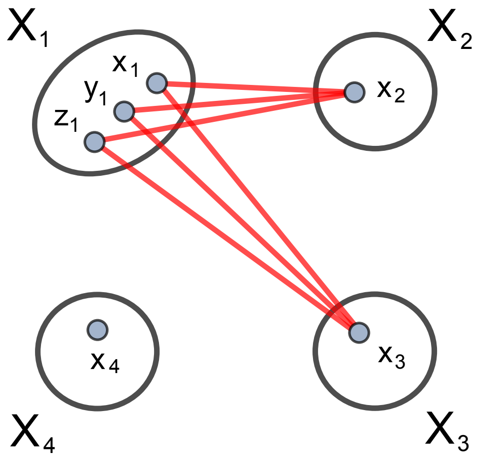

Figure 1: The case where $q = 4$ and some block has order at least 3.

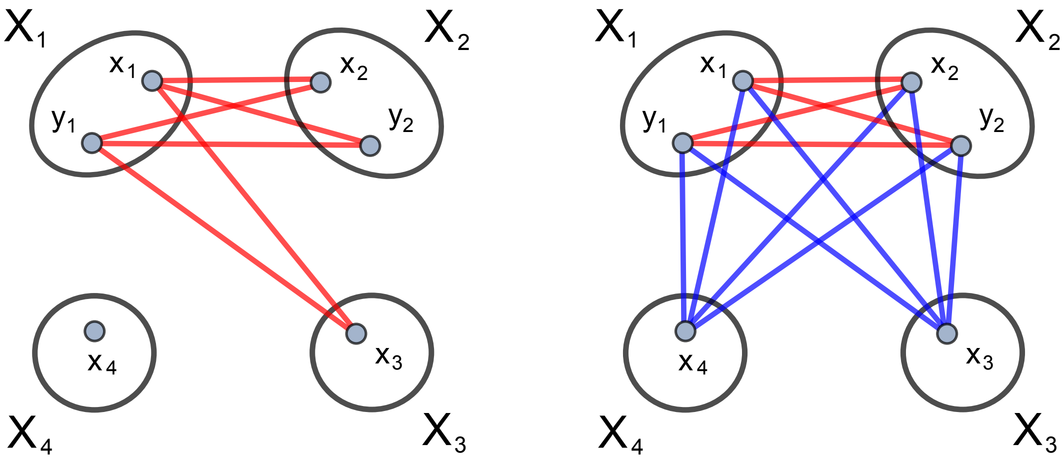

Figure 2: Two cases where $q = 4$ and all blocks have cardinality at most 2.

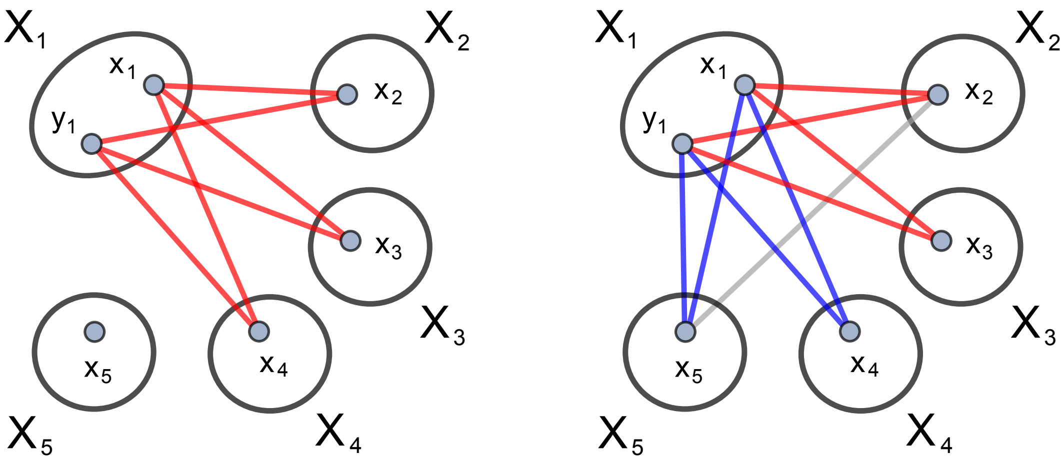

Figure 3: Two cases where $q = 5$.

Overall, we have shown that every critical coloring for $gr^t(P_5)$ has a Gallai partition containing two blocks, one of which consists of a single vertex. This vertex is incident with edges in only one color class. $\square$

Theorem 2.1. *For all* $t \geq 2$, $gr_*^t(P_5) = t + 1$.

Proof. To show that $t + 1$ is a lower bound for $gr_*^t(P_5)$, start with the construction that led to the lower bound given by Inequality (1). Specifically, begin with a copy of K_4 in color 1, which we denote by G_1. Then for each i such that $1 \leq i \leq t-1$, recursively form G_{i+1} by introducing vertex x_i and joining x_i to all of the vertices in G_i using edges in color $i+1$. The result is that G_t is a Gallai t-colored complete graph of order $t+3$ that avoids a monochromatic P_5, producing a critical coloring for $gr^t(P_5)$. Next, introduce a vertex v and join v to x_i using color $i+1$. Also, join v to a single vertex in the original K_4 using color 2 (see Figure 4). The result is a Gallai t-coloring of $K_{t+3} \sqcup K_{1,t}$ that avoids a monochromatic P_5. It follows that $gr_*^t(P_5) \geq t+1$.

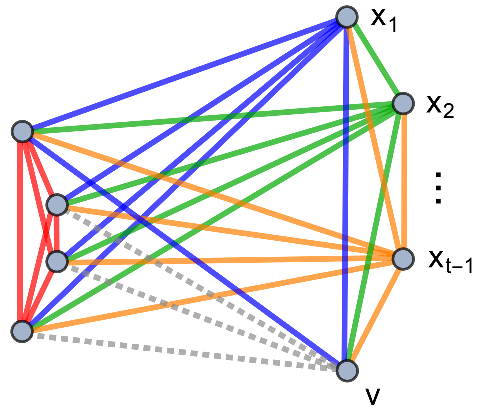

Figure 4: A Gallai t-coloring of $K_{t+3} \sqcup K_{1,t}$ that avoids a monochromatic P_5, from which it follows that $gr_*^t(P_5) \geq t+1$.

To prove the reverse inequality, we will proceed by using induction on $t \geq 2$, with $r_*^2(P_5) \leq 3$ (see [7] and [8]) serving as the base case. Assume that $gr_*^{t-1}(P_5) \leq t$ and consider a Gallai t-coloring of $K_{t+3} \sqcup K_{1,t+1}$, where v denotes the center vertex of the missing star. Deleting vertex v results in a Gallai t-coloring of K_{t+3}. If a monochromatic copy of P_5 is to be avoided, then by Lemma 2.1, some vertex must be incident with edges in only one color. Label this vertex x, the other vertices $y_1, y_2, \ldots, y_{t+2}$, and without loss of generality, assume that xy_i receives color 1, for all $1 \leq i \leq t+2$. We identify color 1 with the color red and consider the following cases.

Case 1. Suppose that at least two edges in the subgraph induced by $\{y_1, y_2, \ldots, y_{t+2}\}$ are red. Regardless of whether or not two red edges are adjacent, there exists a red P_5. For example, if y_1y_2 and y_2y_3 are red, then $y_1y_2y_3xy_4$ is a red P_5 and if y_1y_2 and y_3y_4 are red, then $y_1y_2xy_3y_4$ is a red P_5.

Case 2. Suppose that exactly one edge in the subgraph induced by $\{y_1, y_2, \ldots, y_{t+2}\}$ is red and suppose it is edge y_1y_2. For each i such that $3 \leq i \leq t+2$, the edges y_1y_i and y_2y_i must receive the same color since rainbow triangles are avoided. At most, $t-1$ colors are used on the edges joining $\{y_1, y_2\}$ to $\{y_3, y_4, \ldots, y_{t+2}\}$. By the pigeonhole principle, there exists distinct numbers $i, j \in \{3, 4, \ldots, t+2\}$ such that all edges joining $\{y_i, y_j\}$ to $\{y_1, y_2\}$ are the same color. We now complete this case by considering serval subcases.

Subcase 2.1. If for any three distinct numbers $i, j, k \in \{3, 4, \ldots, t+2\}$, the edges joining $\{y_1, y_2\}$ to $\{y_i, y_j, y_k\}$ are the same color, then $y_iy_1y_jy_2y_k$ is a monochromatic P_5.

Subcase 2.2. If distinct y_i, y_j, y_k, y_ℓ are such that $\{y_i, y_j\}$ joins to $\{y_1, y_2\}$ via one color (say, blue) and $\{y_k, y_\ell\}$ joins to $\{y_1, y_2\}$ via another color (say, green), then consider edge y_iy_k. In order for a rainbow triangle to be avoided, y_iy_k is either blue or green. If it is blue, then $y_ky_iy_1y_jy_2$ is a blue P_5. If it is green, then $y_iy_ky_1y_\ell y_2$ is a green P_5.

Subcase 2.3. Without loss of generality, assume that $\{y_3, y_4\}$ joins to $\{y_1, y_2\}$ via edges in color 2 (which we identify with blue), and y_k joins to $\{y_1, y_2\}$ via color $k-2$, for each $5 \leq k \leq t+2$. If any y_k joins to $\{y_3, y_4\}$ via a blue edge, then a blue P_5 is formed. For example, if y_3y_k is blue, then $y_ky_3y_1y_4y_2$ is a blue P_5. In order for a rainbow triangle to be avoided, y_k must the join to $\{y_3, y_4\}$ via edges in color $k-2$. Since vertex v joins to at least t of the vertices in $\{y_1, y_2, \ldots, y_{t+2}\}$, it must join to at least two of the vertices in $\{y_1, y_2, y_3, , y_4\}$. If any such edge is red, then a red P_5 is formed. For example, if vy_1 is red, then $vy_1y_2xy_3$ is a red P_5, and if vy_3 is red, then $y_1y_2xy_3v$ is a red P_5. Likewise, if any edge joining v to $\{y_1, y_2, y_3, y_4\}$ is blue, then a blue P_5 is formed. If v joins to any of the pairs $\{y_1, y_2\}$, $\{y_1, y_3\}$, $\{y_1, y_4\}$, $\{y_2, y_3\}$, or $\{y_2, y_4\}$, then the two edge must be the same color (and not red or blue) and a monochromatic P_5 is formed. For example, if vy_1 and vy_3 are both given color k, where $3 \leq k \leq t$, then $y_3vy_1y_{k+2}y_2$ is a P_5 in color k. The only case that remains is if v only joins to $\{y_3, y_4\}$. If the two edges receive the same color (say, color k, with $3 \leq k \leq t$, then $y_3vy_4y_{k+2}y_1$ is a P_5 in color k. If edges vy_3 and vy_4 receive different colors, say colors i and j, respectively, then edge y_3y_4 receives one of colors i or j. If y_3y_4 is given color i, then $y_4y_3vy_{i+2}y_1$ is a monochromatic P_5 and if y_3y_4 is given color j, then $y_3y_4vy_{j+2}y_1$ is a monochromatic P_5.

Case 3. Suppose that the subgraph induced by $\{y_1, y_2, \ldots, y_{t+2}\}$ does not contain any red edges and note that v joins to at least t vertices in this set. If v joins to $\{y_1, y_2, \ldots, y_{t+2}\}$ via t edges in colors $2, 3, \ldots, t$, then they form a $(t-1)$-colored $K_{t+2} \sqcup K_{1,t}$, which contains a monochromatic P_5 by the inductive hypothesis. So, assume that v joins to $\{y_1, y_2, \ldots, y_{t+2}\}$ using at least one red edge. If two such edges are red, say vy_1 and vy_2, then $y_1vy_2xy_3$ is a red P_5. So, only one such red edge exists. Without loss of generality, assume that vy_1 is red. Without loss of generality, assume that v also joins to vertices $y_2, y_3, \ldots, y_t$, using only colors $2, 3, \ldots, t$. In order for a rainbow triangle to be avoided, for each k such that $2 \leq k \leq t$, the edges vy_k and y_1y_k receive the same color.

Subcase 3.1. If for any three distinct numbers $i, j, k \in \{2, 3, \ldots, t\}$, the edges joining $\{v, y_1\}$ to $\{y_2, y_3, \ldots, y_t\}$ receive the same color, then a monochromatic P_5 is formed. For example, if the edges joining $\{v, y_1\}$ to $\{y_2, y_3, y_4\}$ are all blue, then $y_2y_1y_3vy_4$ is a blue P_5.

Subcase 3.2. If distinct y_i, y_j, y_k, y_ℓ, where $i, j, k, \ell \in \{2, 3, \ldots, t\}$, are such that $\{y_i, y_j\}$ joins to $\{v, y_1\}$ via one color (say, blue) and $\{y_k, y_\ell\}$ joins to $\{v, y_1\}$ via another color (say, green), then consider edge y_iy_k, which must be either blue or green. If y_iy_k is blue, then $y_ky_iy_1y_jv$ is a blue P_5. If y_iy_k is green, then $y_iy_ky_1y_\ell v$ is a green P_5.

Subcase 3.3. Without loss of generality, assume that the edges joining $\{v, y_1\}$ to $\{y_2, y_3\}$ are given color 2 (blue) and for $4 \leq k \leq t$, the edges joining y_k to $\{v, y_1\}$ receive color $k - 1$. If any edge joining y_k to $\{v, y_1, y_2, y_3\}$ is blue, then a blue P_5 is formed. In order to avoid rainbow triangles, the edges joining y_k to $\{y_2, y_3\}$ must all receive color $k - 1$. Note that if any edge joining $\{y_{t+1}, y_{t+2}\}$ to $\{y_1, y_2, y_3\}$ is blue, then a blue P_5 is formed. It follows that all edges joining y_{t+1} to $\{y_1, y_2, y_3\}$ must be a color other than blue and they must all be the same color. If they receive color $k - 1$, then $y_3y_{t+1}y_1y_ky_2$ is a P_5 in color $k - 1$. Thus, all edges joining y_{t+1} to $\{y_1, y_2, y_3\}$ must receive color t. The same argument can be made for the edges joining y_{t+2} to $\{y_1, y_2, y_3\}$, and it follows that $y_1y_{t+1}y_2y_{t+2}y_3$ is a P_5 in color t.

Subcase 3.4. Without loss of generality, assume that for each k such that $2 \leq k \leq t$, the edges joining $\{v, y_1\}$ to y_k receive color k. Then the edge y_1y_{t+1} must have a color the same as some other edge y_1y_k. The same argument applies to the edge y_1y_{t+2}. If for $i \neq j$, edge y_1y_{t+1} receives color i and y_1y_{t+2} receives color j, then consider edge $y_{t+1}y_{t+2}$, which must also be one of the colors i or j. If it has color i, then $y_{t+2}y_{t+1}y_1y_iv$ is a P_5 in color i. If it has color j, then $y_{t+1}y_{t+2}y_1y_jv$ is a P_5 in color j. So, assume that y_1y_{t+1} and y_1y_{t+2} both receive color k. If for any $\ell \in \{2, 3, \ldots, k-1, k+1 \ldots, t\}$, the edge $y_\ell y_{t+1}$ has color k, then $y_\ell y_{t+1}y_1y_kv$ is a P_5 in color k. The same is true for the edges $y_\ell y_{t+2}$. In order for rainbow triangles to be avoided, the edges $y_\ell y_{t+1}$ and $y_\ell y_{t+2}$ must receive color ℓ, for all $\ell \in \{2, 3, \ldots, k-1, k+1 \ldots, t\}$. Now, if edge $y_{t+1}y_{t+2}$ is given color k, then $y_{t+1}y_{t+2}y_1y_kv$ is a P_5 in color k. So, assume that $y_{t+1}y_{t+2}$ receives color m, where $m \neq 1, k$. Finally, consider the edges joining y_k to $\{y_{t+1}, y_{t+2}\}$. If either such edge has color k, say y_kt_{t+1}, then $vy_ky_{t+1}y_1y_{t+2}$ is a P_5 in color k. If either such edge has color m, say y_ky_{t+1}, then $vy_my_{t+2}y_{t+1}y_k$ is a P_5 in color m. So, both edges y_ky_{t+1} and y_ky_{t+2} must both receive the same color n, for some $n \neq 1, k, m$, and $vy_ny_{t+1}y_ky_{t+2}$ is a P_5 in color n.

In all cases, it has been shown that a Gallai t-coloring of $K_{t+3} \sqcup K_{1,t+1}$ contains a monochromatic P_5. It follows that $gr_*^t(P_5) \leq t + 1$. □

While it has not been explicitly discussed here, the cases in the proof of the upper bound for $gr_*^t(P_5)$ in Theorem 2.1 may assist in the complete classification of the critical colorings for $gr^t(P_5)$. It is also worth considering that there may be other ways in which one can prove Theorem 2.1 than the proof given here. In particular, in the proof that $gr^t(P_5) = t + 3$ given in [2], the authors made use of the fact that every Gallai coloring of a complete graph contains a monochromatic spanning *broom* (consisting of a path $x_1x_2 \cdots x_n$ along with a star with center vertex x_n). This result appeared in [6] and may serve as an alternate tool in a revised proof of the upper bound in Theorem 2.1.

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