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Generalized 2N Color Theorem

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ABSTRACT

2N-Color Theorem

This article gives a standard proof of the famous Four-Color theorem and generalizes it be the 2N-Color problem. The article gives a number of possible applications of the 2N-Color problem that is the essence of orientation. Orientation is fundamental to many fields of scientific knowledge. The Four-color theorem applies to map making by the knowledge that only four colors are necessary to color a planar map. The Six-color theorem applies to three-dimensional space implying that a space station could be ideally designed to have six compartments adjacent to one another allowing a door from any one of the compartments to the other five. The 2N-color generalization applies to the physical reality of quantum physics. Bubble chamber investigations suggest that the universe is four or more dimensions. Thus the 2N-color theorem applies to the N dimensional universe. At this time string theorists have suggested that the universe could be greater than four dimensions. Physics has not as of yet proven the exact dimension of the universe that could even be infinite as a possibility.

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Introduction

In a prior article the author shows how one can construct all possible regular graphs. The author's investigation into the structure of regular graphs motivated applying them to constructing a standard (non-computer proof) of the famous Four-Color Theorem. After having discovered the basic method for solving the Four-Color Theorem in 1989, the author subjected the first unrefined proof to the review of a number of mathematicians throughout the world. Based on these formal and informal reviews the author made several minor fixes to the proof without disturbing its basic flow. In 1993, the author generalized the proof to n-dimensions opening the door to a new way of approaching science and engineering. The 2N Color Theorem should be regarded as a basic building block for a wide range of technical fields of practical interest such as, logistics engineering, human factors engineering, particle physics, molecular engineering, hardware engineering, network design, and much more.

Basic Concepts and Notation

Next we summarize theory, definitions and terminology employed in the proof of the 2N Color Theorem. Most of the concepts employed in this article are standard with the exception of the definition of semi-regularity defined later.

A graph is planar, if it lies in a plane and vertices are the only places where edges intersect. Graphs that are not planar are said to be non-planar. We can alternatively regard a non-planar graph as one that requires exiting two dimensions to avoid intersections that are not vertices. This way of viewing non-planarity is more visual than the first and is equivalent. A proper coloring of a graph is one in which no two adjacent vertices are of the same color. The chromaticity of a graph is the minimum number of colors required to color the graph's vertices so that vertices are not of the same color. Clearly, regions of a map can be represented by a graph where the vertices are equivalent to regions in a map and the boundaries of the regions are equivalent to edges in a graph.

The concept of a planar graph leads one to consider the topology of the plane. To that end, the well-known Jordan Simple Curve Theorem plays a dominant role. Basically, the Jordan Curve Theorem says that every continuous non intersecting closed curve J divides the plane into two open sets, one called the interior and the other the exterior, in such a way that any line joining a point from the interior to a point in the exterior must intersect J. The theorem seems obviously true, but is very difficult to prove. The Jordan Curve Theorem is a basic building block for deducing results related to the design of a network. The next stated result relies on the Jordan Curve Theorem.

We define K_n to be the graph with n vertices such that each of the vertices has exactly one edge to each of the other n 1 vertices. Two adjacent vertices having two or more edges between them are said to be linked. A graph has a link if at least one pair of adjacent vertices are linked. The next result stated is a special case of the well-known Kuratowski theorem and so we omit its proof.

Theorem 1

K_n for $n>4$ is non-planar.

The degree of a vertex is defined to be the number of edges emanating from the vertex. A p semi-regular graph is a graph having degree p for all of its vertices. We permit links in this definition, but not loops. A p-semi-regular graph with no links is p-regular.

Theorem 2
Every 5-chromatic 4-semiregular graph is non-planar.

pf. The previous theorem provides an ideal basis for finite induction, because K5 is non planar, requires 5 colors, and is the only 4 semiregular graph having five vertices where each vertex must be adjacent to the remaining colors. Assume that every 4 semiregular 5-chromatic graph G(k) with k vertices or less is non-planar. Let G(k+1) be a 4 semiregular 5-chromatic graph with k+1 Vertices. Temporarily assume a special case of the theorem where each vertex of G(k+1) is colored in such a way that every vertex is adjacent to each of the other colors. I.e. G(k+1) is like K5 in the sense that G(k+1) can only be colored with each vertex being adjacent to each of the 4 other colors. I.e. G(k+1) is 5-chromatic and 4-semiregular.

The next critical Lemma (which essentially is the Four-Color Theorem) deals with this special class of 4 semiregular graphs which for all we care could consist only of disconnected unions of K_5 without affecting the logic of the proof and that is what happens!! We assume k>5. For the special class of G(k+1) (temporarily assumed) at least two vertices must be identically colored for each color when k>5.

The sequel shows how to prove a special case of Theorem 2 by induction. The special case of Theorem 2 is all that is needed to prove the Four-Color theorem and its generalization. Theorem 2 may be near impossible to prove directly but is unnecessary for proving the Four-Color Theorem. We prove the necessary special case of Theorem 2 needed to prove the Four-Color Theorem next.

Let P(k) notate the k^{th} case of proposition k. P(1) is K5 the graph that consists of 5 vertices each connected to the others. K5 is 5-chromatic, four semi-regular and nonplanar. P(2)is the extension of P(1) to add a vertex that satisfies four semi-regularity, and is 5-chromatic. Adding a new vertex to P(1) forces adding four more vertices to maintain four semi-regularity that is 5 chromatic including the added new vertex because it cannot connect to the vertices from P(1). I.e. P(2) results in K5∪K5. The same logic that applies to forcing P(2)to be the union of two K5s applies to P(k+1) the addition of K5 retains the nonplanarity found in the case of P(k+1) By induction we have established the special case of Theorem 2 required to prove the Four Color Theorem.

At this point Theorem 2 is established but only for the special class of 4 semiregular 5 chromatic graphs temporarily assumed. I.e. Theorem 2 is only partially proved for a special case of it.

For the Four-Color proof we require that only this special class of 4 semiregular graphs (ones like K_5) are proven non-planar. After employing this partial result to prove the Four-Color Theorem the rest of the cases of the preceding theorem, being a special case of the Four-Color Theorem itself, follows immediately!

We emphasize that the preceding Theorem 2 is not proven completely, yet. Only the special case of it needed to prove the Four-Color Theorem is proven at this point! (Where is it written in stone that a proof must follow a linear sequence?) We should think of the Theorem 2 as being on hold while we use a proven special case of it to prove the Four-Color Theorem, which can then in turn imply the rest of Theorem 2. The argument almost seems circular!

A vertex is saturated if in a proper coloring the vertex is adjacent to vertices which exhaust the remaining colors. A path in a graph is a sequence of adjacent vertices with their connecting edges. A graph is connected if there exists a path between every pair of vertices in it.

Lemma
If a connected graph G with more than 5 vertices is 5-chromatic and a proper coloring requires at least one non-removably saturated vertex then G is non-planar.

pf. By assumption there exists a non-removably saturated vertex v1. Removably saturated means that the finite process described in this proof can be employed to recolor G so that all of V1's adjacent vertices exhaust the colors other than the one of v1. If all vertices of G are either removably saturated or unsaturated then four adjacent vertices each are missing a color and it cannot be the color of V1 since all of the vertices saturating v1 are adjacent to v1, G is assumed 5-chromatic. Therefore one of the 5 possible colors is missing in all four of the adjacent vertices of v1. We can replace the one missing color in all four of the adjacent vertices of v1 with the color of v1 contradicting that v1 is non-removably saturated as assumed.

Without loss of generality we assume that V1 is any non-removable saturated vertex of G. In the diagram let V1 be adjacent to V2,V3,V4 and v5 exhausting the 5 colors of the assumed proper coloring as depicted in the diagram.

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         V2     X3__________Y3___
         |      |            |
         |      |            |
V5_______V1_____V3__________Y2________
         |      |            |
         |      |            |
         V4     X1           Y1
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V1 may have more than one adjacent vertex of the same color. To simplify the above we only depict one vertex representing each color. We can repeat the foregoing argument for every other non-removable saturated node in a graph G proving that all such non-removably saturated vertices must be adjacent to another such non-removable saturated vertex.

The same sequence of reasoning applies to each of the colors saturating V1. One of the non-removable saturated vertices of G, say V3, must connect to a nonadjacent vertex of V1 or the graph G would not be connected. We can assume that there exists at least one vertex in G nonadjacent to V1. The preceding sequence of reasoning applied to V1 may be repeated on V3 to create a sequence of non-removable saturated vertices V1,V3,... As the graph is finite, the procession must terminate in the evolution of a 4-regular sub-graph requiring 5 colors where each vertex is non-removably saturated. i.e. Eventually, the vertices whose existence are forced by the process are already included in the 4-regular sub-graph being generated. The conclusion follows from the proven special case of the previous partially proven Theorem 2.

Definition of Derivative Graph
Assume a graph G is 5-chromatic. The derivative graph D(G) is formed by partitioning the vertices of G into subsets A, B, C, D, and E according to a proper coloring. The partitioned subsets are treated as though they are the vertices of a new graph. Defining the edges of the derivative graph is slightly more complicated. First, we consider all paths starting with each vertex in each of the partitioned sets. Each path emanating from one of the partitioned sets traverses a vertex e (not necessarily in the same partition set

as the initial vertex of the path containing e) where a maximum number of colors are adjacent. Select any one such vertex. An edge is defined from the partition set A to B, if the selected vertex e is adjacent to the color represented by B. Similarly, all edges are defined. Obviously, a graph can have a multiplicity of derivative graphs.

We might note that the derivative graph of K_5 is itself and in general derivative graphs of complete graphs are themselves. From a purely mathematical point of view it may be of interest to attempt to characterize all graphs that are their own derivative graph. I.e. are complete graphs the only ones that are their own derivative? The derivative graph is a way to systematically address all possible coloring relationships in a comprehensive way that allows proving the Four-Color Theorem 3. The derivative graph concept gives a convenient way of systematically treating all color relationships in a graph format. The author originally invented the concept, because of a desire to be certain that all interactions between coloring and network logic are considered.

Theorem 3

A planar graph requires no more than four colors for a proper coloring.

pf. Assume that there exists a planar graph G requiring a minimum of 5 colors so that no two colors are adjacent. There must be at least one vertex that is non-removably saturated or the graph could be colored with less than 5 colors. Let G' be a derivative graph of G that allows at least one vertex in G to be saturated. Note that the derivative graph gives a snap shot comprehensive representation of the overall coloring of G. G has many possible derivative graphs G' and at least one that allows a vertex connected to all of the four colors or G would not be 5-chromatic. We can assume without loss of generality that there are vertices in G that have 1,2, or 3 of the other colors adjacent because we could modify G to have additional vertices, if necessary, that satisfy this assumption without loss of generality. G' has vertices that are adjacent to either 2 or 3 of its nodes at most, because it cannot connect to four of G's vertices by the preceding lemma that would force nonplanarity contrary to the assumption that G is planar. I.e. G or a modified version of G has a derivative graph that admits a G' with a vertex that is saturated and vertices that are unsaturated with 1,2, or 3 adjacent different colors. Therefore, G must be 4-chromatic contrary to assumption because none of its vertices require more than 2 or 3 colors to properly color all of its adjacencies according to the Lemma contradicting the assumption that G is planar for Theorem 3. This proves that only four colors are necessary to color G contrary to the assumption that G is planar and is 5-chromatic.

The last Theorem 3 is referred to as the Four-Color Theorem and has been claimed to have been verified by a case study method employing computer algorithms. The Four-Color Theorem is listed as one of the three most famous unsolved mathematical problems with Fermat's conjecture in number theory and Riemann's hypothesis in function theory [1]. The origin of the problem is traceable to a Physics student by the name of Frances Guthrie, who attended the University College of London in 1852. Mathematicians became interested in the problem in about 1878, when the well known English mathematician Cayley posed the problem in a meeting of the London Mathematical Society. The quest to prove the theorem has motivated mathematicians to develop much new interesting theory, but failed to produce a standard proof until now. The proof given here relies only on traditional methods of proving theorems.

Recent articles on graph coloring, indicate that much interest still exists in this area of research [2,3]. One recent article defines a graph to be (m,k) colorable if its vertices can be colored with m colors so that each vertex is forced to be contiguous to at most k vertices of the same color [2]. The Four-Color Theorem says that all planar graphs are (4,0) colorable. Another recent article generalizes the notion of graph coloring as a function f from V(G) into a finite set of integers i=1,2...k, such that the inverse image of each i has property (color) P [3]. These recent articles employ their way of conceptualizing chromaticity to develop interesting new theory.

Extension to Higher Dimensions and Potential Applications

The Four-Color Theorem generalized to space becomes the Six Color Theorem. In fact, we can generalize the Four-Color Theorem to N dimensions to be the 2N Color Theorem. The proofs of the previous section all generalize with minor modifications.

The 2N Color Theorem is essentially about orientation. In one dimensional space, only two directions are possible and correspondingly relative to one dimension the 2N Color Theorem is the Two-Color theorem. Confining to the two-dimensional plane results in the Four-Color Theorem and three-dimensional space in the Six Color Theorem. We can also correspond orientations to faces of a hypercube. For example, the square seems to be the ideal shape representing orientation in the plane, while the cube with its six faces is ideal for representing orientation in space. Undoubtedly, a 2N faced hypercube with N-1 dimensioned faces is ideal for representing orientation in N dimensions.

From the logistics network design point of view the generalization to three-dimensional space has limited value for the following reasons [4]. (1) Space communities barely exist. (2) The fourth dimension cannot be employed to bridge space the way the third-dimension permits bridging out of the plane. However, since for the most part we live in a three-dimensional world and hardware is designed three dimensionally, the Six Color Theorem has many potential applications. For example, reducing the probability of mid-air collision depends on spatiality considerations. Optimizing design of processes taking place in a production line constrained to three dimensions is another potential application involving spatiality considerations. The author has macro-designed a space station that offers a best-design for optimizing logistics flow based on the Six Color Theorem.

In general, the Six Color Theorem applies to design of hardware systems, production lines, and any three-dimensional activity where spatiality plays a role. Assuring optimal proximity of required switches on a control panel to one or more simultaneous users of a control panel requires considering relative position carefully to minimize motion as well as maximize convenience of operation and minimizing collisions of users. Both quality of motion and convenience play a dominant role in setting up a user-friendly production line, control panel, and user interface for most equipment. The Army's MANPRINT (Manpower, Personnel and Integration) program especially highlights design as it relates to the man machine interface (MMI). For electronic systems, electro-magnetic interference and radiation are design concerns. For such systems, the objective of minimizing interference relates directly to spatiality in the circuitry. In chemistry, three dimensional networks representing complex molecules have been employed extensively by Pauling and others to further the boundaries of biology and chemistry. The Six Color Theorem places constraints on possible bonding of atoms which could lead to new discovery in biology and chemistry.

The extension to the Eight Color Theorem seems likely to apply to the physics of the space time continuum whereas higher dimension color theorems will no doubt eventually find application in particle physics. The definition of non-spatiality to follow will clarify this point. The author believes that the extension of the Four-Color Theorem to N dimensions goes to the heart of what reality must be. It is essence even more primitive than subatomic particles. The Color Theorems represent non physical reality preceding the smallest particle. As a point of entry to physics, color theorems are more primitive and therefore more likely to unify physical phenomena, such as electro-magnetism, nuclear forces and gravity. In this regard, at risk of being an anti-biblical teaching we might suggest editing Genesis to say "First there was orientation and then there was light."

In space all bodies have a gravitational field influencing the gravitational fields of other bodies sharing their environment. Distance between the bodies is the main factor in measuring the degree of influence. The interactions of gravitational fields of bodies in space determines the curvature of the space surrounding them as well as their motion relative to each other. The bodies do not have to be planets, stars or other large massed bodies. Basic physical principles apply to the behavior of subatomic particles with additional subatomic forces to consider. When applying the basic truth of the 2N Color Theorem at the subatomic level we are in the domain of quantum or particle physics. When applying it to bodies of large masses like stars and planets, we are in the domain of astronomy. The same basic (non-physical) logic characterized by the 2N Color Theorem in the space time continuum applies to either the micro or macro level physics. Therefore, if a unified field theory exists then the 2N Color Theorem seems to be the most logical point of entry to such a theory.

Summing up application, we should reflect on the all-pervasive nature of the 2N Color Theorem, which is equivalent to orientation and characterizes maximal adjacency. Engineers and scientists have a need to assimilate basic truths, such as exemplified by the 2N Color Theorem, as a starting point for detailed investigation and exploitation of nature. Survival and quality of life depends on man's understanding of his universe. Application possibilities for the Six Color Theorem to the three-dimensional world in which we live is limited only by our imagination. Next we prove the Six Color Theorem realizing that the steps of the proof depend in no way on three dimensions and extend easily to N dimensions. Take note of the space station depicted at the end of this article shows how one can design a space capsule with six separate rooms such that each one can access the other five rooms through a doorway.

Theorem 4

Every 6-semiregular 7-chromatic graph is non-spatial.

pf. As before, we have an ideal basis for finite induction, because K7 is non-spatial, requires 7 colors and is a 6 semiregular graph having seven vertices. Assume that every 6 semiregular 7 chromatic graph G(k) with k vertices is non-spatial. Let G(k+1) be any 6 semiregular 7-chromatic graph with k+1 vertices. Temporarily assume that each vertex of G(k+1) is colored in such a way that every vertex must be adjacent to each of the other colors required. I.e. G(k+1) is like K7 in every respect including that G(k+1) can only be colored with each vertex being adjacent to each of the 6 other colors. The following Lemma is all about this special class of 6 semiregular graphs which consists of disconnected unions of K7!! We assume k>7. For the special class of G(k+1) (temporarily assumed) at least two vertices must be identically colored for each color when k>7. Remove any vertex from G(k+1) colored the same as another vertex in G(k+1). The loose ends of the six edges that were connected to the removed vertex may be tied together to form a new 6 semiregular graph G'(k) having k vertices. The resulting graph G' must require 7 colors or the excised vertex could be placed back to rebuild G(k+1) colored with only one vertex of the original color removed, contradicting that two such vertices had to exist for the special class of 6 semiregular graphs considered temporarily. By the induction assumption G'(k) is non-spatial. Therefore, G(k+1) must also be non-spatial because G'(k) is non-spatial. By finite induction the Theorem is established for the special class of 6 semiregular 7 chromatic graphs temporarily assumed. For the rest of the Six Color proof to be valid only this special class of 6 semiregular graphs (ones like K7) need be non-spatial. Once the Six Color Theorem is proven the preceding theorem, being a special case of the Six Color Theorem, must necessarily be true just as it was in the Four-Color case. The reader should note as before that the preceding theorem is not proven yet, but as in the Four-Color Theorem the special case needed to prove the Six Color Theorem is proven at this point. As before, we should view the preceding theorem as being on hold while we use its induction hypothesis to prove the Six Color Theorem, itself. The proof is analogous to the preceding one used to prove the Four-Color Theorem 3. Obviously, it can be generalized to any dimension as can the Four Color-Theorem itself.

Optimal adjacency requirements in space dictate that the basic design of the ideal space station should conform to the Six Color Theorem since one cannot have a seven-compartment space station that does not have at least two compartments nonadjacent! Six compartments are the best that one can do according to the Six Color Theorem. Of course, one can sub-compartmentalize each of the six compartments to maximize their internal adjacencies. A space station designed with each of its six main compartments compartmentalized into six sub-compartments will have thirty-six compartments to maximize adjacencies. This fractal-like sub-optimization may be carried out as many times as necessary and practical.

Lemma

If a connected graph G with more than 7 vertices is 7-chromatic and a proper coloring requires at least one saturated vertex then the graph is non-spatial.

pf. The proof is identical to the analogous one for the Four-Color Theorem substituting 6 for 4 and 7 for 5.

Generalization of Derivative Graph

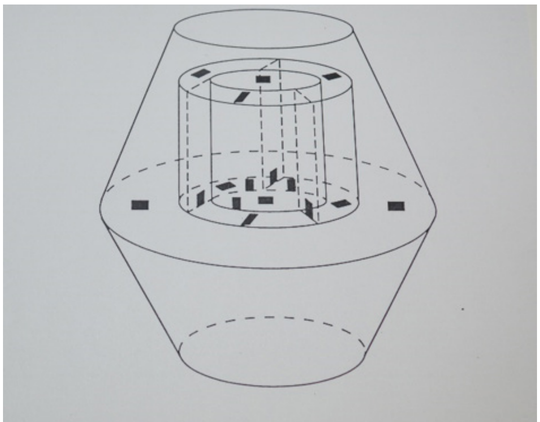
Assume a graph G is n-chromatic. To define the general derivative graph we partition the vertices of G into subsets A_1, A_2,...A_n according to color. The partitioned subsets are treated as though they are the vertices of a new graph. The definition is a straightforward analog to the Four-Color definition.

Theorem

An hyper-spatial graph represented in N dimensions requires at most 2N colors to ensure that no two adjacent vertices have the same color.

pf. Suppose a hyper-spatial graph G requires 2N+1 colors. Symmetry forces G' to have a (2N-1)-regular derivative graph. G' cannot have an arc determined by a maximally saturated vertex by the Lemma. The Structure Theorem for regular graphs forces G' to be 2N-1 or 2N-2 regular. This means G is 2N-chromatic contrary to assumption.

The author proved in a previous unpublished article that in general a graph with an odd number of vertices cannot be odd regular [4]. The earlier article by the author gives some basic theory related to extending the Color Theorems to N dimensions. The previous article also gives several interesting structural theorems about regular graphs and a method for constructing all regular graphs. It was no accident that the prior article set the groundwork for generalizing and solving the Four-Color Theorem. Attempting to find a standard proof of the Four-Color Theorem was one of the principle motivations for the earlier article, too.



The 6-Color Theorem Implies Space Station is Best Designed with Six main Compartments. Each Compartment is Directly Accessible by Every Other Compartment Facilitating Optimal Logistics! (i.e. No need to go through any other compartment to get to another compartment.)

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