ORIGINAL PAPER

SOME INTEGRAL CHARACTERIZATIONS OF TIMELIKE HELICES IN $\mathbb{R}_2^4$

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Manuscript received: 08.03.2022; Accepted paper: 15.09.2022;
Published online: 30.09.2022.

Abstract. *In this study, we examine timelike helices in $\mathbb{R}_2^4$ and some integral characterizations of these curves in terms of Frenet frame. In addition, we study timelike B_2 slant helices in $\mathbb{R}_2^4$ and present the differential equations for vector positions.*

Keywords: *timelike helix; slant helix; semi Euclid space.*

1. INTRODUCTION AND PRELIMINARIES

The curves are used in many different fields such as nature, art, technology and science. It is geometrically important to describe the behavior of the curve in a point on the curve. The curves are interpreted geometrically with the help of a frame in different spaces. There are many studies about curves and especially special curves in $\mathbb{R}_2^4$ [1-7].

The semi-Euclidean space $\mathbb{R}_2^4$ is the standard vector space given with the metric

$$\langle , \rangle = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2 ,$$

where $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4)$ is the coordinate system of $\mathbb{R}_2^4$. For the vector $\vec{v}$ in $\mathbb{R}_2^4$, if the condition $\langle \vec{v}, \vec{v} \rangle > 0$ is met, $\vec{v}$ is spacelike, if the condition $\langle \vec{v}, \vec{v} \rangle < 0$ is met, $\vec{v}$ is timelike, if the condition $\langle \vec{v}, \vec{v} \rangle = 0$, $\vec{v} \neq \vec{0}$ is met, $\vec{v}$ is null (lightlike) vector. The norm of the vector $\vec{v}$ is defined as $\|\vec{v}\| = \sqrt{|\langle \vec{v}, \vec{v} \rangle|}$. If $\langle \vec{v}, \vec{v} \rangle = \pm 1$, then $\vec{v}$ is the unit vector [8].

Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}_2^4$. Then there is only one angle between the vectors $\vec{v}$ and $\vec{w}$, such that;

i) if $\vec{v}$ and $\vec{w}$ are spacelike vectors, then $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \|\vec{w}\| \cos\theta$

ii) if $\vec{v}$ and $\vec{w}$ are timelike vectors, then $\langle \vec{v}, \vec{w} \rangle = -\|\vec{v}\| \|\vec{w}\| \cosh\theta$

iii) if $\vec{v}$ is spacelike vector and $\vec{w}$ is timelike vector, then $|\langle \vec{v}, \vec{w} \rangle| = \|\vec{v}\| \|\vec{w}\| \sinh\theta$ [9,10].

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This article was produced from Zeliha Erpehlivan's master's thesis.

Let the Frenet frame of the curve $\alpha(s)$ be $\{\mathrm{T(s), N(s), B_1(s), B_2(s)}\}$ in $\mathbb{R}_2^4$, where T is tangent vector field, N is principal normal vector field, B_1 is first binormal vector field, B_2 is the second binormal vector field.

Definition 1. If the tangent vector $\vec{T}$ of the curve $\alpha : I \to E^4$ makes constant angle with a unit vector $\vec{U}$ in E^4, the curve α is called general helix (inclined curve) [11].

Definition 2. If the principal normal vector $\vec{N}$ of the curve $\alpha : I \to E^4$ makes constant angle with a unit vector $\vec{U}$ in E^4, the curve α is called slant helix [12].

Let α be a timelike curve parametrized by arclength function s in $\mathbb{R}_2^4$. Let the vector N be timelike, B_1 and B_2 spacelike. In this case there exists only one Frenet frame $\{\mathrm{T, N, B_1, B_2}\}$ for which α is timelike curve with Frenet equations

$$\begin{aligned} T' &= -k_1 N \\ N' &= k_1 T + k_2 B_1 \\ B_1' &= k_2 N + k_3 B_2 \\ B_2' &= -k_3 B_1 \end{aligned} ,$$

where the vectors T, N, B_1, B_2 satisfy the equations:

$$\langle N, N\rangle = \langle T, T\rangle = -1, \langle B_1, B_1\rangle = \langle B_2, B_2\rangle = 1$$

and the functions $k_1 = k_1(s)$, $k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called the curvatures of the timelike curve $\alpha(s)$ [4,13].

2. TIMELIKE HELICES IN THE SEMI-EUCLIDEAN SPACE $\mathbb{R}_2^4$

2.1. TIMELIKE HELICES

Theorem 1. Let $\alpha : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}_2^4$ be timelike curve parametrized by arclength function s. The necessary and sufficient condition to be a timelike helix of the curve α is

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2 + \frac{1}{k_3^2(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2 = constant. \tag{1}$$

Proof: Let $\alpha : \mathrm{I} \subset \mathbb{R} \to \mathbb{R}_2^4$ be a timelike helix. In this case, the tangent vector $\vec{T}$ of the curve α makes constant angle with a spacelike unit vector $\vec{U}$. Thus, the equality $\langle \vec{\mathrm{T}}, \vec{U}\rangle = \|\vec{\mathrm{T}}\|\|\vec{U}\| \sinh\theta = constant$ (or $\langle \vec{T}, \vec{U}\rangle = -\|\vec{T}\|\|\vec{U}\| \cosh\theta = const.$, for $\vec{U}$ is timelike) is written. If this equality is differentiated, the following equation is obtained using the Frenet frame:

$$\left\langle \vec{N}, \vec{U} \right\rangle = \vec{0}.$$

So, it is clear that $\vec{N} \perp \vec{U}$. The vector $\vec{U}$ is written as

$$\vec{U} = u_1 \vec{\mathrm{T}} + u_2 \vec{\mathrm{B}}_1 + u_3 \vec{\mathrm{B}}_2.$$

Differentiating this equality, we get

$$\vec{0} = u_1' \vec{\mathrm{T}} + \left(-u_1 k_1(s) + u_2 k_2(s)\right) \vec{N} + \left(u_2' - u_3 k_3(s)\right) \vec{\mathrm{B}}_1 + \left(u_2 k_3(s) + u_3'\right) \vec{\mathrm{B}}_2,$$

from here the following system is easily visible

$$\left.\begin{array}{l} u_1' = 0 \\ -u_1 k_1(s) + u_2 k_2(s) = 0 \\ u_2' - u_3 k_3(s) = 0 \\ u_2 k_3(s) + u_3' = 0. \end{array}\right\} \tag{2}$$

It is clear that $u_1' = 0 \Rightarrow u_1 = c = constant$, and from the second equality of the system (2)

$$u_2 = \frac{k_1(s)}{k_2(s)} u_1 = c \frac{k_1(s)}{k_2(s)} \tag{3}$$

If the value u_2 found from the last equality of the system (2) is equalized with the Eq.(3), then

$$u_2 = \frac{k_1(s)}{k_2(s)} c = -\frac{1}{k_3(s)} \frac{du_3}{ds} \tag{4}$$

Using this value of u_2 in the third equation of the system (2), we get

$$u_3 = \frac{1}{k_3(s)} u_2' = \frac{c}{k_3(s)} \frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right) \tag{5}$$

Also, since $\frac{du_2}{ds} = u_3 k_3(s)$, $\frac{d}{ds}\left(\frac{1}{k_3(s)} \frac{du_3}{ds}\right) + u_3 k_3(s) = 0$. If the transformation $t = \int_0^s k_3(s) ds$ is made, $dt = k_3(s) ds \Rightarrow k_3(s) = \frac{dt}{ds}$ is obtained. Substituting this equality in the equality (5), we get

$$\frac{d}{ds}\left(\frac{du_3}{dt}\right) + \frac{dt}{ds} u_3 = 0.$$

Multiplying both sides of this equation by $\frac{ds}{dt}$, we get

$$\frac{d^2u_3}{dt^2}+u_3=0.$$

This equation is the second order, constant coefficient, linear, homogeneous differential equation. The solution of this equation is calculated as

$$u_3=\mu_1\cos t+\mu_2\sin t,\quad \mu_1,\mu_2\in\mathbb{R}, \tag{6}$$

and it is clear that

$$u_3=\frac{c}{k_3(s)}\cdot\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)=\mu_1\cos\left(\int_0^s k_3(s)ds\right)+\mu_2\sin\left(\int_0^s k_3(s)ds\right).$$

Differentiating this equality, we get

$$\frac{du_3}{ds}=-\mu_1\sin\left(\int_0^s k_3(s)ds\right)k_3(s)+\mu_2\cos\left(\int_0^s k_3(s)ds\right)k_3(s) \tag{7}$$

Substituting this equality in the equality (4), we get

$$u_2=-c\cdot\frac{k_1(s)}{k_2(s)}=\mu_1\sin\left(\int_0^s k_3(s)ds\right)-\mu_2\cos\left(\int_0^s k_3(s)ds\right). \tag{8}$$

The solving equations (7) and (8) together, we get

$$\mu_1=\frac{c}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\cos\left(\int_0^s k_3(s)ds\right)+c\frac{k_1(s)}{k_2(s)}\cdot\sin\left(\int_0^s k_3(s)ds\right),$$

$$\mu_2=-c\frac{k_1(s)}{k_2(s)}\cos\left(\int_0^s k_3(s)ds\right)+\frac{c}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\sin\left(\int_0^s k_3(s)ds\right).$$

Since $\mu_1,\mu_2\in\mathbb{R}$ and $\mu_1^2+\mu_2^2\in\mathbb{R}$, $\mu_1^2+\mu_2^2=constant$. And so we get

$$\mu_1^2+\mu_2^2=c^2\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{c^2}{k_3^2(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2=constant \text{ and}$$

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{1}{k_3^2(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2=constant.$$

Let

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{1}{k_3^{\,2}(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2=constant\,. \tag{9}$$

Let's take a vector in the form

$$\vec{U}=\left[-\vec{\mathrm{T}}-\frac{k_1(s)}{k_2(s)}\vec{\mathrm{B}}_1-\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\vec{\mathrm{B}}_2\right]s\operatorname{inh}\theta\,. \tag{10}$$

It is clear that $\left\langle\vec{U},\vec{\mathrm{T}}\right\rangle=\sinh\theta=-u_1$. Differentiating the equality (10), we get

$$\frac{d\vec{U}}{ds}=\left[-\left(\frac{k_1(s)}{k_2(s)}\right)k_3(s)-\left[\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]'\right]\overrightarrow{\mathrm{B}_2}\,. \tag{11}$$

Differentiating the equality (9), we get

$$\left[\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]'=-\frac{k_1(s)k_3(s)}{k_2(s)}\,. \tag{12}$$

So, $\dfrac{d\vec{U}}{ds}=\vec{0}$ and $\vec{U}$ =constant. Therefore, the curve α is a timelike helix curve. Thus, the proof is completed.

Corollary 1. Let $\alpha:\mathrm{I}\subset\mathbb{R}\to\mathbb{R}_2^4$ be a timelike curve with parameters s. In order for the curve $\alpha(s)$ to be a timelike helix, the equation

$$k_3(s)\frac{k_1(s)}{k_2(s)}+\frac{d}{ds}\left[\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]=0\,. \tag{13}$$

must be satisfied.

Proof: Let $\alpha:\mathrm{I}\subset\mathbb{R}\to\mathbb{R}_2^4$ be a timelike curve. In that case, we get the equality

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{1}{k_3^{\,2}(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2=constant.$$

Differentiating this equality, we get

$$\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\left\{\left(\frac{k_1(s)}{k_2(s)}\right)+\frac{1}{k_3(s)}\frac{d}{ds}\left[\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]\right\}=0\,. \tag{14}$$

where

i) for $\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)=0$, α is a timelike helix,

ii) for $\left(\frac{k_1(s)}{k_2(s)}\right)+\frac{1}{k_3(s)}\frac{d}{ds}\left[\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]=0$, this equality is multiplied with $k_3(s)$ and the proof is completed.

Theorem 2. Let $\alpha=\alpha(s):\mathrm{I}\subset\mathbb{R}\rightarrow\mathbb{R}_2^4$ be a unit speed timelike curve. The curve $\alpha(s)$ is a timelike helix if and only if there is a differentiable

$$f(s)=-\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)=m_1\sin\left(\int_0^s k_3(s)ds\right)-m_2\cos\left(\int_0^s k_3(s)ds\right),$$

$$f'(s)=\frac{k_1(s)k_3(s)}{k_2(s)}.$$

Proof: Let $\alpha=\alpha(s):\mathrm{I}\subset\mathbb{R}\rightarrow\mathbb{R}_2^4$ be a unit speed timelike helix. We get the equation

$$\frac{dt}{ds}\frac{k_1(s)}{k_2(s)}+\frac{d}{ds}\left[\frac{d}{dt}\left(\frac{k_1(s)}{k_2(s)}\right)\right]=0$$

by applying the variable change $t=\int_0^s k_3(s)ds$ in the equation (13). And thus, the 2nd order, constant coefficient, linear, homogeneous differential equation depending on $\frac{k_1(s)}{k_2(s)}$ is obtained as

$$\frac{d^2}{dt^2}\left(\frac{k_1(s)}{k_2(s)}\right)+\frac{k_1(s)}{k_2(s)}=0.$$

The solving this equation, we get

$$\frac{k_1(s)}{k_2(s)}=m_1\cos t+m_2\sin t=m_1\cos\left(\int_0^s k_3(s)ds\right)+m_2\sin\left(\int_0^s k_3(s)ds\right). \tag{15}$$

$m_1,m_2\in\mathbb{R}$. Let's define the equality

$$f(s)=-\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)=\frac{\left(\frac{k_1(s)}{k_2(s)}\right)\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)}{\frac{d}{ds}\left(\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right)}, \tag{16}$$

over the equation (14). The solution (15) is used in the equality (16) to get

$$f(s)=m_1\sin\left(\int_0^s k_3(s)ds\right)-m_2\cos\left(\int_0^s k_3(s)ds\right).$$

Also, it is obvious from the equations (14) and (16) that

$$\left(\frac{k_1(s)}{k_2(s)}\right)\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)+\left(\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right)\frac{d}{ds}\underbrace{\left(\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right)}_{-f(s)}=0,$$

$$f'(s)=\frac{k_1(s)k_3(s)}{k_2(s)}.$$

Let the equations

$$f(s)=-\frac{1}{k_3(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)=m_1\sin\left(\int_0^s k_3(s)ds\right)-m_2\cos\left(\int_0^s k_3(s)ds\right)$$

and $f'(s)=\dfrac{k_1(s)k_3(s)}{k_2(s)}$ be given. Let's define the function

$$\varphi(s)=\frac{d}{ds}\left[\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{1}{k_3^{\,2}(s)}\left\{\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right\}^2\right]=\frac{d}{ds}\left[\frac{1}{k_3^{\,2}(s)}\left(f'(s)\right)^2+\left(f(s)\right)^2\right],$$

$$\varphi(s)=2f'(s)f''(s)\frac{1}{k_3^{\,2}(s)}-\frac{2k_3'(s)}{k_3^{\,3}(s)}\left(f'(s)\right)^2+2f(s)f'(s).$$

Differentiating $f'(s)$ again, we get

$$f''(s)=\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)k_3(s)+\frac{k_1(s)}{k_2(s)}\frac{d}{ds}\left(k_3(s)\right),$$

and thus

$$f'(s)f''(s)=k_3^{\,2}(s)\frac{k_1(s)}{k_2(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)+\left(\frac{k_1(s)}{k_2(s)}\right)^2k_3(s)\frac{d}{ds}\left(k_3(s)\right).$$

Also, it is clear that

$$f(s)f'(s)=-\frac{k_1(s)}{k_2(s)}\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right).$$

Since $\varphi(s)=0$ for these equalities, $\left(\frac{k_1(s)}{k_2(s)}\right)^2+\frac{1}{k_3{}^2(s)}\left[\frac{d}{ds}\left(\frac{k_1(s)}{k_2(s)}\right)\right]^2$ = constant. Thus, the curve $\alpha(s)$ is a timelike helix.

2.2. TIMELIKE B_2-*SLANT HELIX*

Theorem 3. Let $\alpha=\alpha(s):\mathrm{I}\subset\mathbb{R}\to\mathbb{R}_2^4$ be a unit speed timelike curve. The curve $\alpha(s)$ is a timelike B_2-slant helix if and only if

$$\left(\frac{k_3(s)}{k_2(s)}\right)^2+\frac{1}{k_1^2(s)}\left\{\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right\}^2=constant,\ for$$

Proof: Let $\alpha=\alpha(s):\mathrm{I}\subset\mathbb{R}\to\mathbb{R}_2^4$ be a timelike B_2-slant helix. In this case, the second binormal vector B_2 of the curve α makes a constant angle with a spacelike unit vector U. So $\left\langle\overrightarrow{\mathrm{B}_2},\vec{U}\right\rangle=\left\|\overrightarrow{\mathrm{B}_2}\right\|\left\|\vec{U}\right\|\cos\theta=constant.$(or $\left\langle\overrightarrow{\mathrm{B}_2},\vec{U}\right\rangle=\left\|\overrightarrow{\mathrm{B}_2}\right\|\left\|\vec{U}\right\|\sinh\theta=const.$, for $\vec{U}$ is timelike)

Differentiating this equality, we get

$$\left\langle\overrightarrow{\mathrm{B}_1},\vec{U}\right\rangle=0.$$

Thus, $\overrightarrow{\mathrm{B}_1}\perp\vec{U}$ and

$$\vec{U}=u_1\vec{\mathrm{T}}+u_2\vec{\mathrm{N}}+u_3\overrightarrow{\mathrm{B}_2}$$

are written. Differentiating this equality, we get

$$\vec{\mathrm{T}}\left(u_1'+u_2k_1(s)\right)+\vec{\mathrm{N}}\left(-u_1k_1(s)+u_2'\right)+\overrightarrow{\mathrm{B}_1}\left(u_2k_2(s)-u_3k_3(s)\right)+\overrightarrow{\mathrm{B}_2}u_3'=\vec{0}.$$

So we get the system

$$\left.\begin{array}{l} u_1'+u_2k_1(s)=0, \\ -u_1k_1(s)+u_2'=0, \\ u_2k_2(s)-u_3k_3(s)=0, \\ u_3'=0. \end{array}\right\}. \tag{17}$$

It is clear that $u_3'=0 \Rightarrow u_3=c=constant$, and from the third equality of the system (17)

$$u_2=\frac{k_3(s)}{k_2(s)}c.. \tag{18}$$

If the value u_2 found from the first equality of the system (17) is equalized with the Eq.(18), then

$$u_2 = \frac{k_3(s)}{k_2(s)}c = -\frac{1}{k_1(s)}\frac{du_1}{ds}. \tag{19}$$

Using this value of u_2 in the second equation of the system (17), we get

$$u_1 = \frac{1}{k_1(s)}\frac{du_2}{ds} = \frac{c}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right). \tag{20}$$

Using this value of u_1 in the first equation of the system (17), we get

$$\frac{d}{ds}\left(\frac{1}{k_1(s)}\frac{du_2}{ds}\right) + u_2 k_1(s) = 0. \tag{21}$$

If the transformation $t = \int_0^s k_1(s)ds$ is made, $dt = k_1(s)ds \Rightarrow k_1(s) = \frac{dt}{ds}$ is obtained. Substituting this equality in the equality (21), we get

$$\frac{d}{ds}\left(\frac{du_2}{dt}\right) + u_2\frac{dt}{ds} = 0.$$

Multiplying both sides of this equation by $\frac{ds}{dt}$, we get

$$\frac{d^2u_2}{dt^2} + u_2 = 0 .$$

This equation is the second order, constant coefficient, linear, homogeneous differential equation. The solution of this equation is calculated as

$$u_2 = \omega_1 \cos t + \omega_2 \sin t, \ \ \omega_1, \omega_2 \in \mathbb{R},$$

and it is clear that

$$u_2 = c\frac{k_3(s)}{k_2(s)} = \omega_1 \cos\left(\int_0^s k_1(s)ds\right) + \omega_2 \sin\left(\int_0^s k_1(s)ds\right). \tag{22}$$

Differentiating this equality, we get

$$u_1 = -\omega_1 \sin\left(\int_0^s k_1(s)ds\right) + \omega_2 \cos\left(\int_0^s k_1(s)ds\right).$$

Substituting this equality in the equality (20), we get

$$u_1=\frac{c}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)=-\omega_1\sin\left(\int_0^s k_1(s)ds\right)+\omega_2\cos\left(\int_0^s k_1(s)ds\right). \tag{23}$$

The solving equations (22) and (23) together, we get

$$\omega_1=c\frac{k_3(s)}{k_2(s)}\cos\left(\int_0^s k_1(s)ds\right)-\frac{c}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\sin\left(\int_0^s k_1(s)ds\right),$$
$$\omega_2=\frac{c}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\cos\left(\int_0^s k_1(s)ds\right)+c\frac{k_3(s)}{k_2(s)}\sin\left(\int_0^s k_1(s)ds\right).$$

Since $\omega_1,\omega_2\in\mathbb{R}$, $A^2+B^2\in\mathbb{R}$, for $A=\omega_1+\omega_2,\ \ B=\omega_1-\omega_2$. And so we get

$$A^2+B^2=\left(\frac{k_3(s)}{k_2(s)}\right)^2+\frac{1}{k_1^2(s)}\left(\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)^2=constant\,.$$

Let

$$\left(\frac{k_3(s)}{k_2(s)}\right)^2+\frac{1}{k_1^2(s)}\left(\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)^2=constant. \tag{24}$$

Let's take a vector in the form

$$\vec{U}=\left\{\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\vec{\mathrm{T}}+\frac{k_3(s)}{k_2(s)}\vec{\mathrm{N}}+\vec{\mathrm{B}_2}\right\}\cos\theta. \tag{25}$$

It is clear that $\left\langle\vec{U},\vec{\mathrm{B}_2}\right\rangle=\cos\theta=constant$. Differentiating the equality (25), we get

$$\frac{d\vec{U}}{ds}=\vec{\mathrm{T}}\left\{\left(\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)'+\frac{k_3(s)k_1(s)}{k_2(s)}\right\}\cos\theta\,.$$

Differentiating the equality (11), we get,

$$\left[\frac{1}{k_1(s)}\left(\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)\right]'=-\frac{k_1(s)k_3(s)}{k_2(s)},$$

So, $\dfrac{d\vec{U}}{ds}=\vec{0}$ and $\vec{U}$ =constant. Therefore, the timelike curve α is B_2-slant helix. Thus, the proof is completed.

Corollary 2. Let $\alpha : \mathrm{I} \subset \mathbb{R} \rightarrow \mathbb{R}_2^4$ be a timelike curve with parameters s. In order for the curve $\alpha(s)$ to be a timelike B_2 −slant helix, the equation

$$k_1(s)\frac{k_3(s)}{k_2(s)}+\frac{d}{ds}\left[\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right]=0 \tag{26}$$

must be satisfied.

Proof: Let $\alpha : \mathrm{I} \subset \mathbb{R} \rightarrow \mathbb{R}_2^4$ be a timelike B_2 −slant helix. In that case,

$$\left(\frac{k_3(s)}{k_2(s)}\right)^2+\frac{1}{k_1^{\,2}(s)}\left[\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right]^2=constant.$$

Differentiating this equality, we get

$$\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\left\{\left(\frac{k_3(s)}{k_2(s)}\right)+\frac{1}{k_1(s)}\frac{d}{ds}\left[\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right]\right\}=0, \tag{27}$$

where

i) for $\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)=0$, α is a timelike B_2 −slant helix,

ii) for $\left(\frac{k_3(s)}{k_2(s)}\right)+\frac{1}{k_1(s)}\frac{d}{ds}\left[\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right]=0$, this equality is multiplied with $k_1(s)$ and the proof is completed.

Theorem 4. Let $\alpha=\alpha(s): \mathrm{I} \subset \mathbb{R} \rightarrow \mathbb{R}_2^4$ be a unit speed timelike curve. The curve $\alpha(s)$ is a timelike B_2 −slant helix if and only if there is a differentiable

$$f(s)=-\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)=\ell_1\sin\left(\int_0^s k_1(s)ds\right)-\ell_2\cos\left(\int_0^s k_1(s)ds\right),$$

$$f'(s)=\frac{k_1(s)k_3(s)}{k_2(s)}.$$

Proof: Let $\alpha=\alpha(s): \mathrm{I} \subset \mathbb{R} \rightarrow \mathbb{R}_2^4$ be a unit speed timelike B_2 −slant helix. We get the equation

$$\frac{dt}{ds}\frac{k_3(s)}{k_2(s)}+\frac{d}{ds}\left[\frac{d}{dt}\left(\frac{k_3(s)}{k_2(s)}\right)\right]=0,$$

by applying the variable change $t=\int_0^s k_1(s)ds$ in the equation (26). And thus, the second order, constant coefficient, linear, homogeneous differential equation depending on $\frac{k_3(s)}{k_2(s)}$ is obtained as

$$\frac{k_3(s)}{k_2(s)}+\frac{d^2}{dt^2}\left(\frac{k_3(s)}{k_2(s)}\right)=0.$$

The solving this equation, we get

$$\frac{k_3(s)}{k_2(s)}=\ell_1\cos t+\ell_2\sin t=\ell_1\cos\left(\int_0^s k_1(s)ds\right)+\ell_2\sin\left(\int_0^s k_1(s)ds\right), \tag{28}$$

$\ell_1,\ell_2\in\mathbb{R}$. Let's define the equality

$$f(s)=-\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)=\frac{\left(\frac{k_3(s)}{k_2(s)}\right)\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)}{\frac{d}{ds}\left(\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)} \tag{29}$$

over the equation (27). The solution (28) is used in the equality (29) to get

$$f(s)=\ell_1\sin\left(\int_0^s k_1(s)ds\right)-\ell_2\cos\left(\int_0^s k_1(s)ds\right).$$

Also, it is obvious from the equations (27) and (29) that

$$\left(\frac{k_3(s)}{k_2(s)}\right)\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)+\left(\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)\frac{d}{ds}\underbrace{\left(\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right)}_{-f(s)}=0,$$

$$f'(s)=\frac{k_1(s)k_3(s)}{k_2(s)}.$$

Let the equations

$$f(s)=-\frac{1}{k_1(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)=-\frac{1}{k_1(s)}\frac{d}{ds}\left[\ell_1\cos\left(\int_0^s k_1(s)ds\right)+\ell_2\sin\left(\int_0^s k_1(s)ds\right)\right]$$

and $f'(s)=\frac{k_1(s)k_3(s)}{k_2(s)}$ be given. Let's define the function

$$\varphi(s)=\frac{d}{ds}\left[\frac{1}{k_1^{2}(s)}\left(f'(s)\right)^2+\left(f(s)\right)^2\right],$$

$$\varphi(s)=2f'(s)\,f''(s)\frac{1}{k_1^{2}(s)}-\frac{2k_1'(s)}{k_1^{3}(s)}\left(f'(s)\right)^2+2f(s)\,f'(s).$$

Differentiating $f'(s)$ again, we get

$$f''(s)=\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)k_1(s)+\frac{k_3(s)}{k_2(s)}\frac{d}{ds}\left(k_1(s)\right),$$

and thus

$$f'(s)\,f''(s)=k_1^{2}(s)\frac{k_3(s)}{k_2(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)+\left(\frac{k_3(s)}{k_2(s)}\right)^2k_1(s)\frac{d}{ds}\left(k_1(s)\right).$$

Also, it is clear that

$$f(s)\,f'(s)=-\frac{k_3(s)}{k_2(s)}\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right).$$

Since $\varphi(s)=0$ for these equalities, $\left(\frac{k_3(s)}{k_2(s)}\right)^2+\frac{1}{k_1^{2}(s)}\left[\frac{d}{ds}\left(\frac{k_3(s)}{k_2(s)}\right)\right]^2=$ *constant*. Thus, the curve $\alpha(s)$ is a timelike B_2-slant helix.

3. CONCLUSION

In this study, the integral characterizations are given for the timelike helices according to the Frenet frame in $\mathbb{R}_2^4$. In addition, the timelike helix, the timelike B_2-slant helix concepts are examined in $\mathbb{R}_2^4$ and the differential equations for vector positions are presented.

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