# COMPUTATIONAL AND MODEL SYSTEMS

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# CRITICAL PHENOMENA IN ANISOTROPIC MULTIDIMENSIONAL SYSTEMS

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A class of models, which describe the critical phenomena in anisotropic multidimensional systems with both higher-order spatial order parameter (OP) derivatives and higher OP nonlinearities is proposed. Such models may be useful in the study of phase transitions in early universe cosmology; inflation cosmology; superstring, p-branes and other non-point objects theories. Both the upper and the lower critical dimensions of the models were calculated. It allows one to define the ranges of the mean-field theory applicability for describing critical phenomena in the proposed models.

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## INTRODUCTION

The concept of the physical dimension of space has remained unchanged for a long time. The question of the number of dimensions was more a philosophical problem than a natural science problem. Of course, in physics and the corresponding areas of mathematics, various abstract spaces of different dimensions (configuration space, etc.) were used to describe various physical phenomena. However, there were no serious reasons to doubt the three-dimensionality of physical space. The progress of physics in the 20th century brought important changes to this picture. One of the most important steps along this path was the invention of the general theory of relativity. A clear "picture" of three-dimensional Euclidean space and one-dimensional time was replaced by a four-dimensional Riemannian space-time. Further progress in physics, especially in areas such as elementary particle physics and cosmology, raised many questions about the dimension of space. Is the dimension of space equal to three, is it constant. Spaces of different types of M-theory, spaces of different types of cosmology of the early universe, etc. are examples of spaces of different dimensions.

Other examples of changing ideas about the dimension of space can be seen in the theory of critical phenomena. The development of the fluctuation theory of phase transitions led to a significant revision of the function of the dimension of space in thermodynamics [1]. The dimension of space appears in the equations of thermodynamics along with other material parameters and behaves as a continuous real quantity.

Initially, the basic objects of application of the theory of phase transitions were various systems of the condensed state. However, such foundations of the phase transitions PT theory as spontaneous symmetry breaking and the group-theoretical approach are also the foundations of the theory of elementary particles [2]. Theories of the early universe, the concept of the multiverse also have a close relationship with the theory of phase transitions.

The object of the article is a model describing phase transitions in anisotropic systems of arbitrary dimensions.

## SPACE DIMENSION IN THE THEORY OF CRITICAL PHENOMENA

So, what role plays space dimension in the theory of critical phenomena. In the vicinity of points of "symmetry breaking" or, in other words, points of PT fluctuations of the order parameter and other physical values become very big. The influence of fluctuations strongly depends on space dimension. One of the consequences of these dependencies is the existence of two critical (borderline) dimensions (lower and upper).

The definition of the lower critical dimension ($d_l$) is as follows: if the dimension of space is less than $d_l$, then the phase transition at a nonzero temperature is not possible. I.e. the existence of any ordering states is possible only if $d > d_l$ [3].

The upper critical dimension $(d_u)$ determines the range of applicability of the mean-field approximation in the theory of critical phenomena. In addition, systems in the space of upper critical dimensions have some important properties. They are renormalizable and invariant under the scale transformations. These properties greatly facilitate the analysis and solution of the corresponding variational equations.

The critical dimensions divide axe $d$ into three regions [4]:

$d<dl$: the existence of any ordering states is impossible.

$dl<d<du$: phase transitions are possible, but the mean-field approximation is invalid.

$d>du$: fluctuations are suppressed, the mean-field approximation is valid.

Critical dimensions are important not only as boundaries separating spaces with different types of critical behavior. They are important when using methods based on the renormalization group theory.

## MULTICRITICAL AND LIFSHITZ POINTS

We want to introduce a model that describes system with critical point of new type. Let us consider the thermodynamic potential (TP) of the following form [5]:

$$\Phi = \Phi_0 + a_2 \cdot \eta^2 + a_4 \cdot \eta^4 + a_6 \cdot \eta^6 + \\ + c_1 \cdot (\nabla \eta)^2 + c_2 \cdot (\nabla^2 \eta)^2 + ..., \tag{1}$$

where $\eta$ is the order parameter depending on the space coordinate; $\nabla\eta$ is the gradient of order parameter; $a_i$, $c_j$ are the material parameters depending on the external fields.

To describe the phase transitions in the vicinity of the usual critical point it is sufficient to take into account the terms with coefficients $a_2$, $a_4$, $c_1$. In this case $a_4>0$, $c_1>0$ are the positive values and $a_2$ changes the sign in the point of the phase transition. It is an example of one of the simplest model with broken symmetry.

A well-known example of an anisotropic system is a system with the Lifshitz point [6]. Originally, the conception of the Lifshitz point was introduced to describe the phase transition in systems with anisotropic magnetic properties. Recently it has been often used in the theory of black halls, quantum gravity, and cosmology. The critical phenomena in the systems with the critical Lifshitz point of the first order might be described by taking into account the fact that the coefficient $c_1$ vanishes at the point of phase transition. Thus, it is necessary to consider the term with a positive coefficient $c_2$. If the system has the Lifshitz point of the p-1 order, then in TP the terms with the coefficients $c_1$, $c_2$, $c_{p-1}$ can change their signs, so the terms up to $c_p(\nabla^p\eta)^2$ must be taken into account. The TP for a system with Lifshitz point of (d, m) type in the vicinity of the critical point:

$$\Phi = \Phi_0 \int d^m x_i d^{d-m} x_c \left\{ \frac{r}{2}\eta^2 + \frac{\gamma}{2}\left(\Delta^{\frac{1}{2}}{}_i \eta\right)^2 + \right. \\ \left. \frac{\delta}{2}\left(\Delta^{\frac{1}{2}}{}_c \eta\right)^2 + \frac{\beta}{2}\left(\Delta^{\frac{p}{2}}{}_i \eta\right)^2 + u\eta^4 \right\}. \tag{2}$$

Here $\eta(x)$ is the one-component order parameter; $d$ is the physical space dimension and $r, \gamma, \delta, \beta, u$ are the material parameters. We assume that the physical space consists two subspaces of dimensions $m$ and $d$-$m$ denoted by $i$ and $c$ respectively. There are the wave modulation vectors in the first subspace and are not in the second one. We consider $d$ and $m$ as continuous variables and. $\Delta_i$ and $\Delta c$ are the Laplacian operators in the corresponding subspaces. Such model even in simplest one-dimensional case allows one to describe a parametric evolution of ordep parameter field with different tipes of soliton-like states [7].

The highest OP nonlinearity in model (2) equals 4. Systems with OP nonliniarities higher than 4 are known as systems with multicritical points [8]. TP for such systems looks as follows:

$$\Phi = \frac{r}{2}\eta^2 + ... + u\eta^{N+1}.$$

In [9] critical behavior in a system with joint tricritical ($N = 4$) and Lifshitz point was described.

In [10] was proposed a model which describes the system with joint multicritical (arbitrary $N$) and Lifshitz behavior. Corresponding TP has the following form:

$$\Phi = \Phi_0 \int d^m x_i d^{d-m} x_c \left\{ \frac{r}{2}\eta^2 + \frac{\gamma}{2}\left(\Delta^{\frac{1}{2}}{}_i \eta\right)^2 + \right. \\ \left. \frac{\delta}{2}\left(\Delta^{\frac{1}{2}}{}_c \eta\right)^2 + \frac{\beta}{2}\left(\Delta^{\frac{p}{2}}{}_i \eta\right)^2 + u\eta^{N+1} \right\}. \tag{3}$$

The corresponding dispersion law looks as:

$$\nu(q) \sim \left(r + \gamma q_i^2 + \beta q_i^{2p} + \delta q_c^2\right). \tag{4}$$

The main property of systems with Lifshitz point is the anisotropic scaling. The considered system is invariant under variational scale transformation if the dimension of space coincides with upper critical one.

Generator of corresponding scale transformation:

$$\widehat{X} = -\frac{N-1}{2}\frac{\partial}{\partial x_c} - \frac{N-1}{2p}\frac{\partial}{\partial x_i} + \frac{\partial}{\partial \varphi}. \tag{5}$$

As was mentioned above the existence of symmetry (5) is very helpful while analyzing the variational equations which describe the OP spatial distributions.

In models (2) and (3) the whole physical space is considered to be separated into two subspaces with two different types of critical behavior. In this paper, we generalize these models considering the physical space separating into an arbitrary number of subspaces with arbitrary dimensions.

## MODEL WITH AN ARBITRARY NUMBER OF SUBSPACES WITH ARBITRARY DIMENSIONS

Lets consider a system with space consisting of $\boldsymbol{k}$ subspaces (indexed by α) with dimensions $\boldsymbol{m_\alpha}$. The order of gradient in subspace denoted by $\alpha$ is equal to $p_\alpha$.

The thermodynamic potential:

$$\Phi = \Phi_0 \int d^{m_1} x_1 ... d^{m_k} x_k \left\{ b\varphi^2 + \sum_{\alpha=1}^{k} b_\alpha \left(\nabla_\alpha^{p_\alpha}\varphi\right)^2 + u\varphi^{N+1} \right\}. \tag{6}$$

The corresponding dispersion law looks as:

$$\nu(q) \sim \sum_{\alpha=1}^{k} b_\alpha \left(q_\alpha^{p_\alpha}\right)^2. \tag{7}$$

This model allows one to study the critical phenomena in the multidimensional anisotropic systems with higher nonlinearities. We want to calculate lower and upper critical dimensions for such systems.

To calculate the lower critical dimension we have to consider a fluctuation contribution to entropy in vicinity of critical point.

$$S_{fl}(\tau) = s \cdot \tau^{\sigma(d)}. \tag{8}$$

Here $\tau=(T-T_c)/T_c$ is reduced temperature and $\sigma$ is a function of space dimension. We want to study the critical behavior of $S_{fl}$ in the vecinity of the critical point:

$$\lim_{\tau \to 0} S_{fl}(\tau) = \begin{cases} 0, \sigma(d) > 0, \\ \infty, \sigma(d) < 0. \end{cases} \tag{9}$$

One can see that under condition $T_c \neq 0$ the fluctuation contribution to entropy is a divergent function if $\sigma(d)<0$, so appearance of ordering states is impossible. Otherwise, the fluctuation contribution to entropy goes to zero. Thus, one can find lower critical dimension from condition:

$$\sigma\left(d_l\right)=0. \qquad (10)$$

To find temperature dependencies of entropy we use the following expression:

$$S_{fl}(\tau)=-\frac{\partial G_{fl}}{\partial T}=-\frac{1}{T_c}\frac{\partial G_{fl}}{\partial \tau}. \qquad (11)$$

Here $G_{fl}$ is a fluctuation contribution to thermodynamic potential [11]:

$$G_{fl}=A\int \ln\frac{\left(\beta\tau+\sum_{\alpha=1}^{k} b_\alpha\left(q_\alpha^{p_\alpha}\right)\right)^2}{\pi T_c}\prod_{\alpha=1}^{k} d^{m_\alpha}\vec{q}_\alpha . \qquad (12)$$

From eq.(11) and eq.(12) we have:

$$S_{fl}=\beta A\int\left(\beta\tau+\sum_{\alpha=1}^{k} b_\alpha\left(q_\alpha^{p_\alpha}\right)^2\right)^{-1}\prod_{\alpha=1}^{k} d^{m_\alpha}\vec{q}_\alpha . \qquad (13)$$

After some manipulations:

$$S_{fl}(t)=t^{\sigma(d)}\cdot I\left(q\right)$$

Here $I(q)$ does not depend on temperature. And:

$$\sigma\left(d\right)=\sum_{\alpha=1}^{k}\frac{m_\alpha}{2p_\alpha}-1.$$

In order to find $d_l$ let $p_k=1$, we can do it by renumbering the subspaces.

$$\begin{aligned}&\sum_{j=1}^{k}\frac{m_j}{p_j}=m_k+\sum_{j=1}^{k-1}\frac{m_j}{p_j}=\\&d-\sum_{j=1}^{k-1}m_j+\sum_{j=1}^{k-1}\frac{m_j}{p_j}=d+\sum_{j=1}^{k-1}m_j\left(\frac{1}{p_j}-1\right).\end{aligned} \qquad (14)$$

And finally:

$$d_l=2+\sum_{j=1}^{k-1}m_j\left(1-\frac{1}{p_j}\right). \qquad (15)$$

We see that lower critical dimension does not depend on power of nonlinearities of system. Some properties of $d_l$ we will discuss later.

There are a few ways to calculate upper critical dimension.

First of them is similar to way we have calculated $d_l$. One should compare fluctuation contribution to entropy with its equilibrium value.

The second one is to find $d_u$ from stability condition of fixed point of corresponding renomgroup transformation.

We will find it from the condition of scale variational invariance. Scale variational invariance is very important property of models of type (6). It allows one to simplify the analysis of the corresponding differential equations [12] Lets consider the invariance of model (6) under the following transformations:

$$x_\alpha=\mu_\alpha x_\alpha^{*},\varphi=\tau\varphi^{*}.$$

This condition looks as a set of $k+1$ equations under $k+1$ variables $(\mu_\alpha, \tau)$:

$$\begin{cases}\sum_{\alpha=1}^{k}\mu_\alpha m_\alpha+2\tau-2\mu_1 p_1=0,\\ \dots\dots\dots\\ \sum_{\alpha=1}^{k}\mu_\alpha m_\alpha+2\tau-2\mu_k p_k=0,\\ \sum_{\alpha=1}^{k}\mu_\alpha m_\alpha+\left(N+1\right)\tau=0.\end{cases} \qquad (16)$$

Compatibility condition of the equations (16) is vanishing of determinant:

$$D=\begin{vmatrix} m_k-2p_k & m_{k-1} & \dots & m_1 & 2\\ m_k & m_{k-1}-2p_{k-1} & \dots & m_1 & 2\\ . & . & \dots & . & .\\ m_k & m_{k-1} & \dots & m_1-2p_1 & 2\\ m_k & m_{k-1} & \dots. & m_1 & N+1\end{vmatrix}$$

After some manipulations:

$$D=\sum_{j=1}^{k}m_j\left(-1\right)^k\left(1-N\right)2^{k-1}\prod_{i\neq j}^{k}p_j+\left(-2\right)^k\left(1+N\right)\prod_{i=1}^{k}p_j.$$

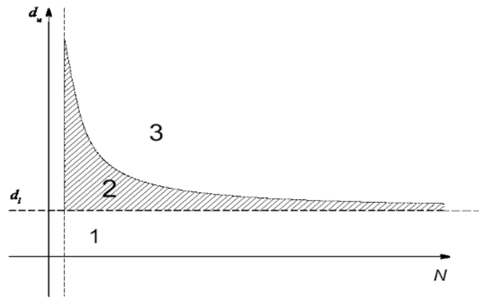
From $D=0$, taking into account (14) one can find expression for $d_u$:

$$d_u=2\frac{N+1}{N-1}+\sum_{j=1}^{k-1}m_j\left(1-\frac{1}{p_j}\right). \qquad (17)$$

## CONCLUSIONS

From (15) and (17) one can find a range of the fluctuation region. It depends on power of the nonlinearity and does not depend on anisotropic features of our system (Figure).

$$d_u-d_l=\frac{4}{N-1}. \qquad (18)$$




*Dependency of the range of the fluctuation region on the power of nonlinearity of the model*

From 18 follows:

$$\lim_{N\to\infty}\left(d_u-d_l\right)=0. \qquad (19)$$

So, the “fluctuation region” decreases as function of power of nonlinearity. This fact is physically reasonable. Strong coupling damps the fluctuations.

Lets consider what types of anisotropic system in classic 3 and relativistic 4 dimensional spaces allow the existing of ordering states.

There are 2 types of anisotropic systems in three-dimensional space:

1) usual critical point (without anisotropy);
2) 1-axe Lifshitz point.

In case of 4-dimensional space:

1) usual critical point (without anisotropy);

2) 1; 2; 3 – axe Lifshitz point;
3) 1; 2 – axe Lifshits point of order p>2.

In general case: for some initial order of Lifshitz point – *p*, the ordering in space with dimensionality $d>2p$ with any kind of anisotropy is possible. In case of $d \le 2p$ there are various situations and this case demands extra investigation.

As expected, the lower critical dimension of any systems is not less than 2. The obtained results are correct for classical PTs. As it is known, in the theory of quantum PTs the effective dimension of a system in the vicinity of the quantum critical point is higher than a dimension of space [13]. Therefore, it is apparent that there are more possible types of PTs in a quantum case. In particular, our results do not contradict a possibility of quantum PTs in two-dimensional systems.

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### КРИТИЧНІ ЯВИЩА В АНІЗОТРОПНИХ БАГАТОМІРНИХ СИСТЕМАХ

***А.В. Бабіч, В.Ф. Клепіков***

Запропоновано клас моделей, що описують критичні явища в анізотропних багатовимірних системах як з вищими просторовими похідними параметрів порядку (ПП), так і з вищими нелінійностями ПП. Такі моделі можуть бути корисні при вивченні фазових переходів у космології раннього Всесвіту; інфляційної космології; теорії суперструн, p-бран та теоріях інших багатовимірних протяжних об'єктів. Були розраховані як верхня, так і нижня критичні розмірності запропонованих моделей. Це дозволяє визначити області застосування теорії середнього поля для опису критичних явищ у пропонованих моделях.