EXACT ANALYTIC WAVE SOLUTIONS TO SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS FOR THE SHALLOW WATER WAVE ARISE IN PHYSICS AND ENGINEERING

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Abstract: In this article, we construct a family of closed form traveling wave solutions to the space-time fractional Equal Width equation (EW) and the space-time fractional modified Equal Width equation (mEW) by using newly proposed modified rational fractional ($D_\xi^\alpha G/G^2$)-expansion method and the exp-function method. The considered equations are turned into fractional order ordinary differential equations with the help of a complex fractional transformation along with conformable fractional derivative and then the methods are used to investigate their solutions. The achieved solutions are in terms of trigonometric function, hyperbolic function and rational function which might be used to analyze deeply the physical complex phenomena of real world as they are new and bear much more generality. Two more well-established methods, the (G'/G) -expansion method and the rational (G'/G) -expansion method, are also taken into account to unravel the suggested equations which do not provide any solution. The results reveal that the proposed method is efficient, straightforward and concise which might further be useful tool to examine any other nonlinear evolution equations of fractional order arising in various physical problems.

Keywords: The modified rational fractional ($D_\xi^\alpha G/G^2$)-expansion method; exp-function method; complex fractional transformation; conformable fractional derivative; exact solution; fractional order nonlinear evolution equation.

Mathematics Subject Classifications: 34A08, 35R11

1. Introduction

Fractional calculus originating from some speculations of Leibniz and L'Hospital in 1695 has gradually gained profound attention of many researchers for its extensive appearance in various fields of real world. The fractional order nonlinear evolution equations (FNLEEs) and their solutions in closed form play fundamental role in describing, modeling and predicting the underlying mechanisms related to the biology, bio-genetics, physics, solid state physics, condensed matter physics, plasma physics, optical fibers, meteorology, oceanic phenomena, chemistry, chemical kinematics, electromagnetic, electrical circuits, quantum mechanics, polymeric materials, neutron point kinetic model, control and vibration, image and signal processing, system identifications, the finance, acoustics and fluid dynamics [1-5]. Therefore, it has become the core aim in the research area of fractional related problems that how to develop a stable approach for investigating the solutions to FNLEEs in analytical or numerical form. Many researchers have

offered different approaches to construct analytic and numerical solutions to NLEEs of fractional order as well as integer order and put them forward for searching traveling wave solutions, such as the exponential decay law [6], the Ibragimov's nonlocal conservation method [7], the reproducing kernel method [8], the Jacobi elliptic function method [9], the (G'/G) -expansion method and its various modifications [10-14], the Exp-function method [15, 16], the sub-equation method [17, 18], the first integral method [19, 20], the functional variable method [21], the modified trial equation method [22, 23], the simplest equation method [24], the Lie group analysis method [25], the fractional characteristic method [26], the auxiliary equation method [27, 28], the finite element method [29], the differential transform method [30], the Adomian decomposition method [31, 32], the variational iteration method [33], the finite difference method [32], the various homotopy perturbation method [35-39] and the He's variational principle [40] etc. But no method is uniquely substantial to examine the closed form

solutions to all kind of FNLEEs. That is why; it is very much indispensable to establish new techniques. In this study, we propose a new technique, called the modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method, to construct closed form analytic wave solutions to some FNLEEs in the sense of conformable fractional derivative [41]. This effectual and reliable productive method shows its high performance through providing abundant fresh and general solutions to the suggested equations. The obtained solutions might bring up their importance through the contribution to analyze the inner mechanisms of physical complex phenomena of real world and make an acceptable record in the literature.

2. Preliminaries and Methodology

2.1. Conformable fractional derivative

A new and simple definition of derivative for fractional order introduced by Khalil et al. [41] is called conformable fractional derivative. This definition is analogous to the ordinary derivative

$$\frac{d\psi}{dx} = \lim_{\varepsilon \rightarrow 0} \frac{\psi(x + \varepsilon) - \psi(x)}{\varepsilon},$$

where $\psi(x) : [0, \infty] \rightarrow \mathbf{R}$ and $x > 0$. According to

this classical definition, $\frac{d(x^n)}{dx} = nx^{n-1}$. According to

this perception, Khalil has introduced α order fractional derivative of ψ as

$$T_\alpha \psi(x) = \lim_{\varepsilon \rightarrow 0} \frac{\psi(x + \varepsilon x^{1-\alpha}) - \psi(x)}{\varepsilon}, \quad 0 < \alpha \leq 1.$$

If the function ψ is α -differentiable in $(0, r)$ for $r > 0$ and $\lim_{x \rightarrow 0^+} T_\alpha \psi(x)$ exists, then the conformable

derivative at $x=0$ is defined as $T_\alpha \psi(0) = \lim_{x \rightarrow 0^+} T_\alpha \psi(x)$. The conformable integral of ψ is

$$I_\alpha^r \psi(x) = \int_r^x \frac{\psi(t)}{t^{1-\alpha}} dt, \quad r \geq 0, \quad 0 < \alpha \leq 1.$$

This integral represents usual Riemann improper integral.

The conformable fractional derivative satisfies the following useful properties [41]:

If the functions $u(x)$ and $v(x)$ are α -differentiable at any point $x > 0$, for $\alpha \in (0, 1]$, then

$$(a) \quad T_\alpha (au + bv) = aT_\alpha(u) + bT_\alpha(v) \\ \forall a, b \in \mathbf{R}.$$

$$(b) \quad T_\alpha(x^n) = nx^{n-\alpha} \quad \forall n \in \mathbf{R}.$$

$$(c) \quad T_\alpha(c) = 0, \text{ where } c \text{ is any constant.}$$

$$(d) \quad T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u).$$

$$(e) \quad T_\alpha(u/v) = \frac{vT_\alpha(u) - uT_\alpha(v)}{v^2}.$$

$$(f) \quad \text{if } u \text{ is differentiable, then}$$

$$T_\alpha(u)(x) = x^{1-\alpha} \frac{du}{dx}(x).$$

Many researchers used this new derivative of fractional order in physical applications due to its convenience, simplicity and usefulness [42-44].

2.2. Methodology

In this section, we discuss the main steps of the above, mentioned methods to investigate exact analytic solutions of FNLEEs. Consider the FNLEE in the independent variables $t, x_1, x_2, \dots, x_n$ as

$$Q(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, D_{x_1}^\beta u_1, \dots, D_{x_1}^\beta u_k, \dots, D_{x_n}^\beta u_1, \dots, D_{x_n}^\beta u_k, \dots) = 0, \quad (1)$$

where $u_i = u_i(t, x_1, x_2, \dots, x_n)$, $i = 1, \dots, k$ are unknown functions, F is a polynomial in u_i and it's various partial derivatives of fractional order.

Making use of the fractional composite transformation

$$u_i = u_i(t, x_1, x_2, \dots, x_n) = U_i(\xi), \\ \xi = \xi(t, x_1, x_2, \dots, x_n) \quad (2)$$

Eq. (1) is turned into the following ordinary differential equation of fractional order with respect to the variable ξ :

$$Q(u_1, \dots, u_k, D_\xi^\alpha u_1, \dots, D_\xi^\alpha u_k, D_\xi^\beta u_1, \dots, D_\xi^\beta u_k, \dots) = 0 \quad (3)$$

We may, if possible, take the anti-derivative of Eq. (3) term by term one or more times and integral constant can be set to zero as soliton solutions are sought. Then the following two methods are employed to construct closed form analytic solutions of Eq. (3).

2.2.1. The modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method

In this subsection, the main steps of the modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method is discussed for finding exact analytic solutions to FNLEEs.

Step 1: Suppose that the traveling wave solution can be expressed as follows:

$$U(\xi) = \frac{\sum_{i=0}^n a_i (D_\xi^\alpha G/G^2)^i}{\sum_{i=0}^n b_i (D_\xi^\alpha G/G^2)^i}, \quad (4)$$

where at least one of a_n and b_n is nonzero; $a_i (i = 0, 1, 2, \dots, n)$, $b_i (i = 0, 1, 2, \dots, n)$ and d are unknown parameters, and $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation of fractional order:

$$G^2 D_\xi^{2\alpha} G - (2G + \lambda)(D_\xi^\alpha G)^2 - \mu G^4 = 0 \quad (5)$$

It can be written as

$$D_{\xi}^{\alpha}(D_{\xi}^{\alpha}G/G^2) = \mu + \lambda(D_{\xi}^{\alpha}G/G^2)^2 \quad (6)$$

The nonlinear fractional complex transformation $G(\xi) = H(\eta)$, $\eta = \xi^{\alpha}/\alpha$ reduces Eq. (5) into the following ordinary differential equation:

$$H^2 H'' - (2G + \lambda)H'^2 - \mu H^4 = 0 \quad (7)$$

whose solutions are well-known. Since $D_{\xi}^{\alpha}G(\xi) = D_{\xi}^{\alpha}H(\eta) = H'(\eta)D_{\xi}^{\alpha}\eta = H'(\eta)$, with the aid of the solutions of Eq. (7), we can obtain the solutions of Eq. (5) as follows:

$$(D_{\xi}^{\alpha}G/G^2) = \sqrt{\mu/\lambda} \times \frac{A \cos(\sqrt{\lambda\mu\xi^{\alpha}/\alpha} + B \sin(\sqrt{\lambda\mu\xi^{\alpha}/\alpha})}{B \cos(\sqrt{\lambda\mu\xi^{\alpha}/\alpha} - A \sin(\sqrt{\lambda\mu\xi^{\alpha}/\alpha})}, \lambda\mu > 0 \quad (8)$$

$$(D_{\xi}^{\alpha}G/G^2) = -\sqrt{|\lambda\mu|/\lambda} \times \frac{A \sinh(2\sqrt{\lambda\mu\xi^{\alpha}/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^{\alpha}/\alpha}) + B}{A \sinh(2\sqrt{\lambda\mu\xi^{\alpha}/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^{\alpha}/\alpha}) - B}, \lambda\mu < 0 \quad (9)$$

$$(D_{\xi}^{\alpha}G/G^2) = -\frac{A\alpha}{\lambda(A\xi^{\alpha} + B\alpha)}, \mu = 0, \lambda \neq 0 \quad (10)$$

Step 2: Determine the positive constant n appeared in Eq. (4) by using homogenous balance between the highest order derivative term and nonlinear term in Eq. (3).

Step 3: Use Eqs. (4) and (5) in Eq. (3) with the value of n found in step 2 to obtain a polynomial in $(D_{\xi}^{\alpha}G/G^2)$. Set each coefficient of this polynomial to zero and solve them by computer software Maple to gain the values of the unknown parameters available in Eq. (4).

Step 4: Utilizing the values calculated at step 3 in Eq. (4) along with Eqs. (8)-(10) makes available exact analytic solutions to Eq. (1).

2.2.2. The Exp-function method

In this subsection, the main steps of the Exp-function method are discussed for finding exact analytic solutions of nonlinear partial differential equations of fractional order.

Step 1: Consider the wave solution to Eq. (1) as

$$U(\xi) = \frac{\sum_{n=-c}^d a_n \exp(n\xi)}{\sum_{m=-p}^q b_m \exp(m\xi)}, \quad (11)$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants.

Step 2: Balance the linear term of lowest order of Eq. (3) with the lowest order nonlinear term to determine the values of c and p . Similarly, to determine the values of d and q , balance the linear term of highest order of Eq. (3) with highest order nonlinear term.

Step 3: Substitute Eq. (11) into Eq. (3) with the values of c, d, p and q obtained in step 2, we obtain polynomials in $e^{r\xi}$, for any integer r . Equating like terms to zero gives a system of algebraic equations for a_i 's and b_i 's. Solve this system for a_i 's and b_i 's by means of the symbolic computation software, such as Maple.

Step 4: Substitute the values appeared in step 3 into Eq. (11), we obtain traveling wave solutions of Eq. (1) in closed form.

3. Formulation of the solutions

In this section, the suggested methods are employed to ravel the space-time fractional EW equation and the space-time fractional mEW equation for their solutions in closed form.

3.1. The space-time fractional EW equation

Consider the space-time fractional EW equation

$$D_t^{\alpha}u + \delta D_x^{\alpha}u^2 - \tau D_{xxt}^{\alpha}u = 0 \quad (12)$$

which is used as a model in partial differential equation and handles the simulation of one-dimensional wave transmission in nonlinear media with dispersion processes.

The wave variable

$$u(x, t) = U(\xi), \xi = k^{1/\alpha}x + c^{1/\alpha}t \quad (13)$$

reduces Eq. (12) to the following fractional order ordinary differential equation with respect to the variable ξ :

$$cD_{\xi}^{\alpha}u + k\delta D_{\xi}^{\alpha}u^2 - ck^2\tau D_{\xi}^{\alpha}u = 0 \quad (14)$$

Taking anti-derivative of Eq. (14) yields

$$cu + k\delta u^2 - ck^2\tau D_{\xi}^{\alpha}u = 0 \quad (15)$$

3.1.1. Solutions via modified rational fractional $(D_{\xi}^{\alpha}G/G^2)$ -expansion method

Applying the homogeneous balance method to Eq. (15) along with Eq. (4) we obtain $n = 2$. Then the solution Eq. (4) takes the form

$$U(\xi) = \frac{a_0 + a_1(D_{\xi}^{\alpha}G/G^2) + a_2(D_{\xi}^{\alpha}G/G^2)^2}{b_0 + b_1(D_{\xi}^{\alpha}G/G^2) + b_2(D_{\xi}^{\alpha}G/G^2)^2} \quad (16)$$

Eq. (15) under the use of Eq. (16) and Eq. (6) creates a polynomial in $(D_{\xi}^{\alpha}G/G^2)$ whose coefficients assigning to zero gives a set of algebraic equations. Solving these equations by Maple provides the following results:

$$\text{Set 1: } a_0 = 0, a_1 = \pm \frac{4b_0c\lambda\sqrt{-3b}}{3a}, a_2 = \frac{8bb_0c\lambda^2}{3a\sqrt{b\lambda\mu}}, b_1 = \pm \frac{2b_0\lambda}{\sqrt{-3\lambda\mu}}, b_2 = -\frac{b_0\lambda}{3\mu}, k = \frac{1}{2\sqrt{b\lambda\mu}} \quad (17)$$

where b_0 and c are arbitrary constants.

$$\text{Set 2: } a_0 = 0, a_1 = \pm \frac{4b_0c\lambda\sqrt{-3b}}{3a}, a_2 = -\frac{8bb_0c\lambda^2}{3a\sqrt{b\lambda\mu}}, b_1 = \mp \frac{2b_0\lambda}{\sqrt{-3\lambda\mu}}, b_2 = -\frac{b_0\lambda}{3\mu}, k = \frac{1}{2\sqrt{b\lambda\mu}} \quad (18)$$

where b_0 and c are arbitrary constants.

Inserting the values appeared in Eqs. (17) and (18) in Eq. (16) grants the following expressions for analytic solutions:

$$U_1(\xi) = \frac{\mu}{a}\sqrt{-3/b} \times \frac{4bc\lambda\{2\lambda(D_{\xi}^{\alpha}G/G^2) \pm \sqrt{-3\lambda\mu}\}(D_{\xi}^{\alpha}G/G^2)}{\pm 6\lambda\mu(D_{\xi}^{\alpha}G/G^2) - \sqrt{-3\lambda\mu}\{3\mu - \lambda(D_{\xi}^{\alpha}G/G^2)^2\}}, \quad (19)$$

$$U_2(\xi) = \frac{\mu}{a}\sqrt{-3/b} \times \frac{4bc\lambda\{-2\lambda(D_{\xi}^{\alpha}G/G^2) \pm \sqrt{-3\lambda\mu}\}(D_{\xi}^{\alpha}G/G^2)}{\mp 6\lambda\mu(D_{\xi}^{\alpha}G/G^2) - \sqrt{-3\lambda\mu}\{3\mu - \lambda(D_{\xi}^{\alpha}G/G^2)^2\}}, \quad (20)$$

where $\xi = \{1/(2\sqrt{b\lambda\mu})\}^{1/\alpha}x + c^{1/\alpha}t$.

Eq. (19) together with Eqs. (8)-(10) makes available the following traveling wave solutions:

When $\lambda\mu > 0$,

$$U_1^1(\xi) = \frac{4bc\lambda\mu}{a} \sqrt{-3\mu/b\lambda} \left\{ 2\lambda \sqrt{\frac{\mu}{\lambda}} \times \frac{\frac{A\cos\sqrt{\lambda\mu\xi^\alpha} + B\sin\sqrt{\lambda\mu\xi^\alpha}}{B\cos\sqrt{\lambda\mu\xi^\alpha} - A\sin\sqrt{\lambda\mu\xi^\alpha}} \pm \sqrt{-3\lambda\mu}} \right\} \times \frac{\frac{A\cos\sqrt{\lambda\mu\xi^\alpha} + B\sin\sqrt{\lambda\mu\xi^\alpha}}{B\cos\sqrt{\lambda\mu\xi^\alpha} - A\sin\sqrt{\lambda\mu\xi^\alpha}}}{\pm 6\lambda\mu \sqrt{\frac{\mu}{\lambda}} \times \frac{\frac{A\cos\sqrt{\lambda\mu\xi^\alpha} + B\sin\sqrt{\lambda\mu\xi^\alpha}}{B\cos\sqrt{\lambda\mu\xi^\alpha} - A\sin\sqrt{\lambda\mu\xi^\alpha}}}} \times \frac{1}{- \mu \sqrt{-3\lambda\mu} \left\{ 3 - \left(\frac{A\cos\sqrt{\lambda\mu\xi^\alpha} + B\sin\sqrt{\lambda\mu\xi^\alpha}}{B\cos\sqrt{\lambda\mu\xi^\alpha} - A\sin\sqrt{\lambda\mu\xi^\alpha}} \right)^2 \right\}}, \quad (21)$$

where $\xi = \{1/(2\sqrt{b\lambda\mu})\}^{1/\alpha} x + c^{1/\alpha} t$.

When $\lambda\mu < 0$,

$$U_1^2(\xi) = -\frac{4bc\lambda\mu}{a} \sqrt{-3\mu/|\lambda\mu|} \times \left\{ 2\lambda \sqrt{|\lambda\mu|/\lambda} \times \frac{\frac{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + B}{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) - B} \pm \sqrt{-3\lambda\mu}} \right\} \times \frac{\frac{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + B}{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) - B}}{\mp 6\lambda\mu \sqrt{|\lambda\mu|/\lambda} \times \frac{\frac{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + B}{A\sinh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) + A\cosh(2\sqrt{\lambda\mu\xi^\alpha}/\alpha) - B}}}, \quad (22)$$

where $\xi = \{1/(2\sqrt{b\lambda\mu})\}^{1/\alpha} x + c^{1/\alpha} t$.

When $\mu = 0$, $\lambda \neq 0$,

$$U_1^3(\xi) = -\frac{4bc\lambda\mu}{a} \sqrt{-3/b}, \quad (23)$$

where $\xi = \{1/(2\sqrt{b\lambda\mu})\}^{1/\alpha} x + c^{1/\alpha} t$.

The above solutions are in terms of trigonometric function, hyperbolic function and rational function involving many free parameters. In the same way, Eq. (20) with the aid of Eqs. (8)-(10) provides more exact analytic solutions.

3.1.2. Solutions through the exp-function method

Considering the homogenous balance between the highest order derivative and the nonlinear term appearing in Eq. (15), the solution Eq. (11) takes the form

$$U(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1e^\xi}{b_{-1}e^{-\xi} + b_0 + b_1e^\xi} \quad (24)$$

Substituting Eq. (24) into Eq. (15) the left hand side becomes a polynomial in e^ξ . Setting each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, c$ and k . Solving this set of equations with the aid of computer algebra, like Maple, provides the following results:

Set 1: $a_{-1} = \pm \frac{c\sqrt{-b}}{a}$, $a_1 = \pm \frac{aa_0^2}{16c\sqrt{-b}}$, $b_{-1} = 1$,

$$b_0 = \mp \frac{aa_0}{2c\sqrt{-b}}, b_1 = -\frac{a^2a_0^2}{16bc^2}, k = \mp \frac{1}{\sqrt{-b}}, \quad (25)$$

where a_0 and c are arbitrary constants.

Set 2: $a_{-1} = 0$, $a_1 = 0$, $b_{-1} = 1$,

$$b_0 = \pm \frac{aa_0}{3c\sqrt{-b}}, b_1 = \frac{a^2a_0^2}{36bc^2}, k = \mp \frac{1}{\sqrt{-b}}, \quad (26)$$

where a_0 and c are arbitrary constants.

Using the values from Eq. (25) into Eq. (24), the expression for exact solutions to Eq. (12) is obtained as

$$U_1(\xi) = \frac{bc}{a} \times \frac{\mp 16bc^2e^{-\xi} + 16aa_0c\sqrt{-b} \pm a^2a_0^2e^\xi}{16bc^2\sqrt{-b}e^{-\xi} \pm 8abca_0 - a^2a_0^2\sqrt{-b}e^\xi}, \quad (27)$$

where $\xi = (\pm 1/\sqrt{-b})^{1/\alpha} x + c^{1/\alpha} t$.

If we choose $a = 4$, $b = -1$ and $c = a_0$, then we obtain the following hyperbolic function solution:

$$U_1^1(\xi) = \frac{a_0}{8} (2 + \operatorname{sech}^2(\xi/2)), \quad (28)$$

and

$$U_1^2(\xi) = -\frac{a_0}{8} (2 - \operatorname{cosech}^2(\xi/2)), \quad (29)$$

where $\xi = (\pm 1)^{1/\alpha} x + a_0^{1/\alpha} t$.

Putting the values appeared in Eq. (26) into Eq. (24), the expression for closed form solutions of Eq. (12) attain

$$U_2(\xi) = \frac{36a_0bc^2}{36bc^2e^{-\xi} \pm a^2a_0^2e^\xi + 12aa_0\sqrt{b}c}, \quad (30)$$

where $\xi = (\mp 1/\sqrt{-b})^{1/\alpha} x + c^{1/\alpha} t$.

Assign the values for the parameters as $a = 6$, $b = -1$ and $c = a_0$, then Eq. (30) provides the hyperbolic function solutions

$$U_2^1(\xi) = \frac{a_0}{4} \operatorname{sech}^2(\xi/2), \quad (31)$$

and

$$U_2^2(\xi) = -\frac{a_0}{4} \operatorname{cosech}^2(\xi/2), \quad (32)$$

where $\xi = (\pm 1)^{1/\alpha} x + a_0^{1/\alpha} t$.

3.2. The space-time fractional mEW equation

The following nonlinear space-time fractional mEW equation is used to handle the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes:

$$D_t^\alpha u + aD_x^\alpha u^3 - bD_{xxt}^3 u = 0 \quad (33)$$

The compound fractional transformation

$$u(x, t) = U(\xi), \xi = k^{1/\alpha} x + c^{1/\alpha} t \quad (34)$$

reduces Eq. (33) to the following ordinary differential equation fractional order with respect to the variable ξ :

$$cu + akD_\xi^\alpha u^3 - bck^2D_\xi^{3\alpha} u = 0 \quad (35)$$

Taking anti-derivative of Eq. (35) yields

$$cu + aku^3 - bck^2D_\xi^{2\alpha} u = 0 \quad (36)$$

3.2.1. Solutions obtained by $(D_\xi^\alpha G/G^2)$ -expansion method

Apply the homogeneous balance to Eq. (36) after using Eq. (4) and obtain $n = 1$. Then the solution Eq. (4) becomes

$$U(\xi) = \frac{a_0 + a_1(D_\xi^\alpha G/G^2)}{b_0 + b_1(D_\xi^\alpha G/G^2)} \quad (37)$$

Eq. (36) under the use of Eq. (37) and Eq. (6) put together a polynomial in $(D_\xi^\alpha G/G^2)$ whose coefficients

assigning to zero gives a system of equations. Solving these equations by Maple yields the following results:

$$\text{Set 1: } a_1 = -\frac{a_0 b_0 \lambda}{b_1 \mu}, c = \pm \frac{aa_0^2 \lambda}{b_1^2 \mu \sqrt{2b\lambda\mu}}, k = \pm \frac{1}{\sqrt{2b\lambda\mu}} \quad (38)$$

where a_0, b_0 and b_1 are arbitrary constants.

$$\text{Set 2: } a_0 = 0, b_1 = 0, c = \pm \frac{aa_1^2 \sqrt{2b\lambda\mu}}{2bb_0^2 \lambda}, k = \pm \frac{1}{\sqrt{2b\lambda\mu}} \quad (39)$$

where a_1 and b_0 are arbitrary constants.

$$\text{Set 3: } a_1 = 0, b_0 = 0, c = \pm \frac{aa_0^2 \sqrt{2b\lambda\mu}}{2bb_1^2 \mu^2}, k = \pm \frac{1}{\sqrt{2b\lambda\mu}} \quad (40)$$

where a_0 and b_1 are arbitrary constants.

Utilizing the values available in Eqs. (38)-(40) into Eq. (37) yields the following expressions for desired solutions:

$$U_1(\xi) = \frac{a_0}{b_1 \mu} \times \frac{b_1 \mu - b_0 \lambda (D_\xi^\alpha G/G^2)}{b_0 + b_1 (D_\xi^\alpha G/G^2)}, \quad (41)$$

$$\text{where } \xi = \left(\pm \frac{1}{\sqrt{2b\lambda\mu}} \right)^{1/\alpha} x + \left(\pm \frac{aa_0^2 \lambda}{b_1^2 \mu \sqrt{2b\lambda\mu}} \right)^{1/\alpha} t. \\ U_2(\xi) = \frac{a_1}{b_0} \times (D_\xi^\alpha G/G^2), \quad (42)$$

$$\text{where } \xi = \left(\pm \frac{1}{\sqrt{2b\lambda\mu}} \right)^{1/\alpha} x + \left(\pm \frac{aa_1^2 \sqrt{2b\lambda\mu}}{2bb_0^2 \lambda} \right)^{1/\alpha} t. \\ U_3(\xi) = \frac{a_0}{b_1} \times \frac{1}{(D_\xi^\alpha G/G^2)}, \quad (43)$$

$$\text{where } \xi = \left(\pm \frac{1}{\sqrt{2b\lambda\mu}} \right)^{1/\alpha} x + \left(\pm \frac{aa_0^2 \sqrt{2b\lambda\mu}}{2bb_1^2 \mu^2} \right)^{1/\alpha} t.$$

Making use of Eqs. (8)-(10) into solution Eq. (41) give the following three type solutions:

Under the condition $\lambda\mu > 0$, the trigonometric function solution is

$$U_1^1(\xi) = \frac{a_0}{b_1 \mu} \times \frac{b_1 \mu - b_0 \lambda \sqrt{\mu/\lambda} \times \frac{A \cos \sqrt{\lambda\mu\xi^\alpha/\alpha} + B \sin \sqrt{\lambda\mu\xi^\alpha/\alpha}}{B \cos \sqrt{\lambda\mu\xi^\alpha/\alpha} - A \sin \sqrt{\lambda\mu\xi^\alpha/\alpha}}}{b_0 + b_1 \sqrt{\mu/\lambda} \times \frac{A \cos \sqrt{\lambda\mu\xi^\alpha/\alpha} + B \sin \sqrt{\lambda\mu\xi^\alpha/\alpha}}{B \cos \sqrt{\lambda\mu\xi^\alpha/\alpha} - A \sin \sqrt{\lambda\mu\xi^\alpha/\alpha}}}, \quad (44)$$

When $\lambda\mu < 0$, the hyperbolic function solution is

$$U_1^2(\xi) = \frac{a_0}{b_1 \mu} \times \frac{b_1 \mu + b_0 \lambda \sqrt{|\lambda\mu|/\lambda} \times \frac{A \sinh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + B}{A \sinh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) - B}}{b_0 - b_1 \sqrt{|\lambda\mu|/\lambda} \times \frac{A \sinh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + B}{A \sinh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) + A \cosh(2\sqrt{\lambda\mu\xi^\alpha/\alpha}) - B}}, \quad (45)$$

For $\mu = 0, \lambda \neq 0$, the solution is

$$U_1^3(\xi) = \frac{a_0 \lambda}{b_1 \mu} \times \frac{b_1 \mu (A \xi^\alpha + B \alpha) + A b_0 \alpha}{b_0 \lambda (A \xi^\alpha + B \alpha) - A b_1 \alpha}, \quad (46)$$

$$\text{where } \xi = \left(\pm \frac{1}{\sqrt{2b\lambda\mu}} \right)^{1/\alpha} x + \left(\pm \frac{aa_0^2 \lambda}{b_1^2 \mu \sqrt{2b\lambda\mu}} \right)^{1/\alpha} t.$$

Eqs. (42), (43) along with Eqs. (8)-(10) provide more three type solutions such as trigonometric solution, hyperbolic solution and rational solution. For the convenience of readers we avoid the solutions to be recorded here.

3.2.2. Solutions via exp-function method

Applying the homogenous balance method to Eq. (27) reduces solution Eq. (11) to Eq. (24). Eq. (24) along with Eq. (36) yields a polynomial in e^ξ . Equating like terms of the polynomial to zero gives a set of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, c$ and k . Solving this set of equations with the aid of computer algebra, like Maple, provides the following results:

$$\text{Case 1: } a_1 = -\frac{a_{-1}^2 b_0^2 - a_0^2}{4a_{-1}}, b_{-1} = 1, b_1 = \frac{a_{-1}^2 b_0^2 - a_0^2}{4a_{-1}^2}, \\ c = \pm \frac{2aa_{-1}^2}{\sqrt{-2b}}, k = \mp \frac{2}{\sqrt{-2b}}, \quad (47)$$

where a_{-1}, a_0 and b_0 are unknown parameters.

$$\text{Case 2: } a_{-1} = -\frac{a_1}{b_1}, a_0 = 0, b_0 = 0, b_{-1} = 1, \\ c = \pm \frac{aa_1^2}{b_1^2 \sqrt{-2b}}, k = \mp \frac{1}{\sqrt{-2b}}, \quad (48)$$

where a_1 and b_1 are unknown constants.

$$\text{Case 3: } a_0 = -a_{-1} b_0, a_1 = 0, b_{-1} = 1, b_1 = 0, \\ c = \pm \frac{2aa_{-1}^2}{\sqrt{-2b}}, k = \mp \frac{2}{\sqrt{-2b}}, \quad (49)$$

where a_{-1} and b_0 are unknown constants.

$$\text{Case 4: } a_{-1} = 0, a_1 = 0, b_0 = 0, b_{-1} = 1, \\ b_1 = \pm \frac{aa_0^2}{8\sqrt{b}c}, k = \mp \frac{1}{\sqrt{-2b}}, \quad (50)$$

where a_0 and c are arbitrary constants.

Inserting the values available in Eqs. (47)-(50) into Eq. (24) makes available the following four exact traveling wave solutions:

$$U(\xi) = a_{-1} \cdot \frac{4a_{-1}^2 e^{-\xi} + 4a_0 a_{-1} - (a_{-1}^2 b_0^2 - a_0^2) e^\xi}{4a_{-1}^2 e^{-\xi} + 4b_0 a_{-1}^2 + (a_{-1}^2 b_0^2 - a_0^2) e^\xi}. \quad (51)$$

In particular, if $a_0 = a_{-1}, b_0 = \sqrt{5}$, then Eq. (51) gives the hyperbolic function solution

$$U(\xi) = a_{-1} \cdot \frac{1 - 2 \sinh \xi}{\sqrt{5} + 2 \cosh \xi}. \quad (52)$$

Putting the value of ξ in Eq. (52) becomes

$$U_{1,2}(x, t) = a_{-1} \cdot \frac{1 - 2 \sinh(k^{1/\alpha} x + l^{1/\alpha} t)}{\sqrt{5} + 2 \cosh(k^{1/\alpha} x + l^{1/\alpha} t)}, \quad (53)$$

where $k = \mp \frac{2}{\sqrt{-2b}}$ and $l = \pm \frac{2aa_{-1}^2}{\sqrt{-2b}}$.

$$U(\xi) = \frac{a_1(b_1 e^\xi - e^{-\xi})}{b_1(b_1 e^\xi + e^{-\xi})}. \quad (54)$$

For $b_1 = 1$, Eq. (54) becomes

$$U(\xi) = a_1 \tanh \xi. \quad (55)$$

Substitute the value of ξ possesses

$$U_{3,4}(x, t) = a_1 \tanh(k^{1/\alpha} x + l^{1/\alpha} t), \quad (56)$$

where $k = \mp \frac{1}{\sqrt{-2b}}$ and $l = \pm \frac{aa_1^2}{b_1^2 \sqrt{-2b}}$.

$$U(\xi) = \frac{a_{-1}(e^{-\xi} - b_0)}{e^{-\xi} + b_0}. \quad (57)$$

After assigning $b_0 = 1$ and simplifying, we find

$$U(\xi) = a_{-1} \cdot \tanh(\xi/2). \quad (58)$$

With the value of ξ Eq. (58) reduces to

$$U_{5,6}(x, t) = a_{-1} \cdot \tanh\{(k^{1/\alpha} x + l^{1/\alpha} t)/2\}, \quad (59)$$

where $k = \mp \frac{2}{\sqrt{-2b}}$ and $l = \pm \frac{2aa_{-1}^2}{\sqrt{-2b}}$.

$$U(\xi) = \frac{8a_0 \sqrt{b} l}{8\sqrt{b} l e^{-\xi} \pm aa_0^2 e^\xi}. \quad (60)$$

Using $a = 8c$ and $b = a_0^4$, Eq. (60) provides

$$U(\xi) = \frac{a_0}{e^{-\xi} + e^{\xi}}. \quad (61)$$

The simplification of Eq. (61) gives the following two results:

$$U_7(\xi) = \frac{a_0}{2} \operatorname{sech} \xi. \quad (62)$$

and

$$U_8(\xi) = \frac{a_0}{2} \operatorname{cosech} \xi. \quad (63)$$

The substitution for the value of ξ yields

$$U_7(x, t) = \frac{a_0}{2} \operatorname{sech}(k^{1/\alpha} x + l^{1/\alpha} t), \quad (64)$$

and

$$U_8(x, t) = \frac{a_0}{2} \operatorname{cosech}(k^{1/\alpha} x + l^{1/\alpha} t), \quad (65)$$

where $k = \mp \frac{1}{\sqrt{-2b}}$ and l is an arbitrary constant.

4. Results and Discussions:

The modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method has been proposed and used for finding exact analytic solutions to the space-time fractional EW equation which provided twelve solutions in terms of trigonometric, hyperbolic and rational functions while the space-time fractional mEW equation through the same method made available eighteen solutions in the same form. The exp-function method has also been applied to the suggested equations which gave four solutions for the first equation and eight for the second. Hosseini, and Ayati applied Kudryashov method to the space-time fractional EW equation and the space-time fractional mEW equation and obtained only four solutions (**Appendix A, B**) in terms of hyperbolic function for each equation [45]. We have applied two well-established methods, the (G'/G) -expansion method [46] and the rational (G'/G) -expansion method [47], to investigate the exact solutions of the considered equations but we haven't got any solutions.

5. Conclusion

The core aim of this study was to make available further general and fresh closed form analytic solutions to the space-time fractional EW equation and the space-time fractional mEW equation through the proposed modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method and the exp-function method. Both methods have successfully presented attractive solutions to the suggested equations though the performance of first one is perceptible. So far we know the achieved solutions are not available in the literature and might create a milestone in research area. We have also utilized the (G'/G) -expansion method and the rational (G'/G) -expansion method to unravel the suggested equations but not found any solution. Therefore, it may be claimed that the modified rational fractional $(D_\xi^\alpha G/G^2)$ -expansion method in deriving the closed form analytical solutions is simple, straightforward and productive. This method may be taken into account for

further implementation to investigate any fractional order nonlinear evolution equations arising in various fields of science and engineering.

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Appendix

A. Hosseini, and Ayati applied Kudryashov method to the space-time fractional EW equation and obtained the following solutions [45]:

$$u_{1,2}(x, t) = \frac{c\sqrt{-\delta}}{\varepsilon} \left(\pm 1 \mp \frac{6}{1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)} \pm \frac{6}{\left(1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)\right)^2} \right).$$

$$u_{3,4}(x, t) = \frac{6c\sqrt{-\delta}}{\varepsilon} \left(\pm \frac{1}{1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)} \mp \frac{1}{\left(1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)\right)^2} \right).$$

B. Hosseini, and Ayati studied the space-time fractional modified EW equation by using Kudryashov method and obtained the solutions [45]

$$u_{1,2}(x, t) = -\frac{\sqrt{2}\sqrt{\pm\sqrt{2}}\varepsilon c\sqrt{-\delta}}{2\varepsilon} + \frac{\sqrt{2}\sqrt{\pm\sqrt{2}}\varepsilon c\sqrt{-\delta}}{\varepsilon\left(1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)\right)}.$$

$$u_{3,4}(x, t) = \frac{\sqrt{2}\sqrt{\pm\sqrt{2}}\varepsilon c\sqrt{-\delta}}{2\varepsilon} - \frac{\sqrt{2}\sqrt{\pm\sqrt{2}}\varepsilon c\sqrt{-\delta}}{\varepsilon\left(1+d \exp\left(\frac{k}{\Gamma(1+\alpha)}x^\alpha - \frac{c}{\Gamma(1+\alpha)}t^\alpha\right)\right)}.$$