Weak Insertion of a Continuous Function between Two Comparable α -Continuous (C-Continuous) Functions*

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Abstract

A sufficient condition in terms of lower cut sets is given for the insertion of a continuous function between two comparable real-valued functions.

Keywords

Weak Insertion, Strong Binary Relation, C-Open Set, Semi-Preopen Set, α -Open Set, Lower Cut Set

Subject Areas: Topology

1. Introduction

The concept of a C-open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in 1996 [1]. The authors define a set S to be a C-open set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a C-closed set if its complement is C-open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an α -open set and a C-open set. This enables them to provide the following decomposition of continuity: a function is continuous if and only if it is α -continuous and C-continuous.

Recall that a subset A of a topological space (X, τ) is called α -open if A is the difference of an open and a nowhere dense subset of X . A set A is called α -closed if its complement is α -open or equivalently if A is union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called semi-preopen or β -open. A set is semi-preclosed or β -closed if its complement is semi-preopen or β -open.

The concept of a set A was β -open if and only if $A \subseteq Cl(Int(Cl(A)))$ was introduced by J. Dontchev in 1998 [2].

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Recall that a real-valued function f defined on a topological space X was called A -continuous if the preimage of every open subset of $\mathbb{R}$ belongs to A , where A was a collection of subset of X and this the concept was introduced by M. Przemski in 1993 [3]. Most of the definitions of function used throughout this paper are consequences of the definition of A -continuity. However, for unknown concepts, the reader might refer to papers introduced by J. Dontchev in 1995 [4], M. Ganster and I. Reilly in 1990 [5].

Hence, a real-valued function f defined on a topological space X is called C -continuous (resp. α -continuous) if the preimage of every open subset of $\mathbb{R}$ is C -open (resp. α -open) subset of X .

Results of Katětov in 1951 [6] and in 1953 [7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which was due to Brooks in 1971 [8], were used in order to give necessary and sufficient conditions for the strong insertion of a continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X , we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X .

The following definitions were modifications of conditions considered in paper introduced by E. Lane in 1976 [9].

A property P defined relative to a real-valued function on a topological space is a c -property provided that any constant function has property P and provided that the sum of a function with property P and any continuous function also has property P . If P_1 and P_2 are c -property, the following terminology is used: A space X has the *weak c -insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2 , then there exists a continuous function h such that $g \leq h \leq f$.

In this paper, it is given a sufficient condition for the weak c -insertion property. Also several insertion theorems are obtained as corollaries of this result.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all α -open, α -closed, C -open and C -closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ) . Respectively, we define the α -closure, α -interior, C -closure and C -interior of a set A , denoted by $\alpha Cl(A)$, $\alpha Int(A)$, $C Cl(A)$ and $C Int(A)$ as follows:

$$\begin{aligned}\alpha Cl(A) &= \bigcap \{F : F \supseteq A, F \in \alpha C(X, \tau)\} \\ \alpha Int(A) &= \bigcup \{O : O \subseteq A, O \in \alpha O(X, \tau)\} \\ C Cl(A) &= \bigcap \{F : F \supseteq A, F \in CC(X, \tau)\} \\ \text{and} \\ C Int(A) &= \bigcup \{O : O \subseteq A, O \in CO(X, \tau)\}.\end{aligned}$$

Respectively, we have $\alpha Cl(A), C Cl(A)$ are α -closed, semi-preclosed and $\alpha Int(A), C Int(A)$ are α -open, semi-preopen.

The following first two definitions are modifications of conditions considered in [6] [7].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S .

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

- 1) If $A_i \rho B_j$ for any $i \in \{1, \dots, m\}$ and for any $j \in \{1, \dots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \dots, m\}$ and any $j \in \{1, \dots, n\}$.
- 2) If $A \subseteq B$, then $A \bar{\rho} B$.
- 3) If $A \rho B$, then $Cl(A) \subseteq B$ and $A \subseteq Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined [8] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if

$\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ , then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ .

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on a topological space X with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$, then there exists a continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \rho A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \bar{\rho} F(t_2)$, $G(t_1) \bar{\rho} G(t_2)$, and $F(t_1) \rho G(t_2)$. By Lemmas 1 and 2 of [7] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \rho H(t_2)$, $H(t_1) \rho H(t_2)$ and $H(t_1) \rho G(t_2)$.

For any x in X , let $h(x) = \inf \{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \text{Int}(H(t_2)) \setminus \text{Cl}(H(t_1))$. Hence $h^{-1}(t_1, t_2)$ is an open subset of X , i.e., h is a continuous function on X . ■

The above proof used the technique of proof of Theorem 1 of [6].

3. Applications

The abbreviations αc and Cc are used for α -continuous and C -continuous, respectively.

Corollary 3.1. If for each pair of disjoint α -closed (resp. C -closed) sets F_1, F_2 of X , there exist open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X has the weak c -insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).

Proof. Let g and f be real-valued functions defined on the X , such that f and g are αc (resp. Cc), and $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $\alpha \text{Cl}(A) \subseteq \alpha \text{Int}(B)$ (resp. $C \text{Cl}(A) \subseteq C \text{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is an α -closed (resp. C -closed) set and since $\{x \in X : g(x) < t_2\}$ is an α -open (resp. C -open) set, it follows that $\alpha \text{Cl}(A(f, t_1)) \subseteq \alpha \text{Int}(A(g, t_2))$ (resp. $C \text{Cl}(A(f, t_1)) \subseteq C \text{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint α -closed (resp. C -closed) sets F_1, F_2 , there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then every α -continuous (resp. C -continuous) function is continuous.

Proof. Let f be a real-valued α -continuous (resp. C -continuous) function defined on the X . Set $g = f$, then by Corollary 3.1, there exists a continuous function h such that $g = h = f$. ■

Corollary 3.3. If for each pair of disjoint subsets F_1, F_2 of X , such that F_1 is α -closed and F_2 is C -closed, there exist open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ then X have the weak c -insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$.

Proof. Let g and f be real-valued functions defined on the X , such that g is αc (resp. Cc) and f is Cc (resp. αc), with $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $C \text{Cl}(A) \subseteq \alpha \text{Int}(B)$ (resp. $\alpha \text{Cl}(A) \subseteq C \text{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X . If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a C -closed (resp. α -closed) set and since $\{x \in X : g(x) < t_2\}$ is an α -open (resp.

C -open) set, it follows that $CCl(A(f, t_1)) \subseteq \alpha Int(A(g, t_2))$ (resp. $\alpha Cl(A(f, t_1)) \subseteq CInt(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1) \rho A(g, t_2)$. The proof follows from Theorem 2.1. ■

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