Topology Preserving Discretization Schemes for Digital Image Segmentation and Digital Models of the Plane

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Received 30 May 2014; revised 30 June 2014; accepted 9 July 2014

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Abstract

In this paper, discretization schemes are defined and studied that allow us to build digital models of 2-dimensional continuous objects with the same topological properties as their continuous counterparts. In particular, it is shown that the digital model of the plane is necessarily the digital 2-surface defined and studied in the context of digital topology. It is shown that in proposed discretization schemes, any required resolution can be used in any area of the object of interest. Finally, we describe a method, which transforms regions of interest produced by the image acquisition process into digital spaces with topological features of the regions.

Keywords

Digital Image, Discretization, Digitization, Cover, Plane, Digital Model

Subject Areas: Computer Vision, Computer Graphics and Visualization, Image Processing, Radiology & Medical Imaging

1. Introduction

Digital topology studies geometrical and topological properties of digital images arising in many areas of science including fluid dynamics, geoscience, neuroscience and medical imaging. Integrating topological features into discretization schemes in order to generate topologically correct digital models of anatomical structures is critical for many clinical and research applications, where particular regions of the object require a fine resolution while a relatively coarse resolution can be used over the rest of the object of interest (see [1] [2]).

In the present paper, discretization schemes preserving topology of the object of interest are introduced. These schemes let obtain more detailed geometrical and topological information about the specific regions of the object.

The shape and the size of individual elements can be arbitrary within the framework of the proposed scheme that allows representing objects with fine anatomical details. Another feature is that the digital model of a 2-dimensional continuous object is necessarily a digital 2-surface preserving the topology of the object. The approach studied in this paper is based on locally centered lump (LCL) discretization of n -dimensional objects.

In recent years, there has been a considerable amount of works devoted to building two-, three- and n -dimensional discretization schemes and digital images (see, e.g., [3]-[12]).

The material to be presented below begins with basic definitions and results related to discretization schemes for continuous objects in Section 2. We study properties of LCL covers of 1- and 2-dimensional continuous objects and formulate conditions for a cover of the plane to be the LCL cover. It is shown that one can choose an LCL cover with a required resolution in any region of interest, which is extremely important in medicine. In Section 3, we prove that for any LCL cover of the plane, the digital model is necessarily a digital 2-surface. A trivial result of this consideration is that there are infinitely many non-isomorphic digital models of the plane. A method is described that constructs digital models of areas of interest with any required resolution.

2. LCL Collections of 2-Cells and LCL Covers of Continuous Objects

In this section, we use intrinsic topology of an object, without reference to an embedding space.

We say that a set D is a 2-cell if it is homeomorphic to a closed unit square, a set D is a 1-cell or a segment if it is homeomorphic to a closed unit segment, a set C is a circle or a 1-sphere if it is homeomorphic to a unit circle. We denote the interior and the boundary of an n -cell D , $n = 1, 2$, by $\text{Int}D$ and ∂D respectively. Note that $D = D \cup \text{Int}D$. The boundary of a 1-cell is two endpoints, the boundary of a 2-cell is a circle. The 0-cell D is a single point for which $\partial D = \emptyset$. Facts about n -cells that we will need in this paper are stated below.

Facts 2.1

- If C is a circle and D is a 1-cell contained in C then $C - \text{Int}D$ is a 1-cell.
- Let C_1 and C_2 be 1-cells such that $C_1 \cap C_2 = \partial C_1 \cap \partial C_2 = v$ is an endpoint of C_1 and C_2 . Then $C_1 \cup C_2 = E$ is a 1-cell.
- Let D_1 and D_2 be 2-cells such that $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = D^1$ is a 1-cell. Then $D_1 \cup D_2 = B$ is a 2-cell.

Definition 2.1

Let $W = \{X_1, \dots, X_s\}$ be a collection of sets. W is called a *locally centered collection* (LC collection) if for all $m, k \in Q \subseteq \{1, \dots, s\}$, from condition $X_k \cap X_m \neq \emptyset, m \neq k$, it follows that $\bigcap \{X_k : k \in Q\} \neq \emptyset$.

Definition 2.2

- 1) Let $W = \{C_1, C_2, \dots\}$ be a collection of 1-cells. W is called a *locally lump collection* (LL collection) if:
 - a) From condition $C_k \cap C_m \neq \emptyset, m \neq k$, it follows that $C_k \cap C_m = \partial C_k \cap \partial C_m = v$ is a point.
 - b) The intersection of any three distinct 1-cells is empty.
- 2) W is called a *locally centered lump collection* (LCL collection) if W is a locally centered collection and a locally lump collection at the same time (Figure 1).

Collections of 1-cells shown in Figures 1(a)-(c) are LCL collections. (d) is not an LL collection because it violates condition (b) in Definition 2.2, i.e., $C_3 \cap C_4 \cap C_5 \neq \emptyset$. (e) is not an LL collection because it violates condition (a) in Definition 2.2, i.e., $C_4 \cap C_5 \neq \partial C_4 \cap \partial C_5$. (f) is not an LL collection because it violates condition (a) in Definition 2.2, i.e., $C_1 \cap C_2 = \partial C_1 \cap \partial C_2 = \{v_1, v_2\}$ is not a point. (g) is an LL but not LC collection.

From Definition 2.2, it follows that if $W = \{C_1, C_2, \dots\}$ is an LCL collection of 1-cells, and the union $M = C_1 \cup C_2 \cup \dots$ is a connected space, then M is either a circle or a segment or a line or a half line.

Definition 2.3

- 1) Let $W = \{D_1, D_2, \dots\}$ be a collection of 2-cells. W is called a *locally lump collection* (LL collection) if:
 - a) From condition $D_k \cap D_m \neq \emptyset, m \neq k$, it follows that $D_k \cap D_m = \partial D_k \cap \partial D_m = C_{km}$ is a 1-cell.
 - b) From condition $D_k \cap D_m \cap D_p \neq \emptyset, m \neq k, m \neq p, k \neq p$, it follows that $D_k \cap D_m \cap D_p = \partial D_k \cap \partial D_m \cap \partial D_p = C_{km} \cap C_{kp} = \partial C_{km} \cap \partial C_{kp} = v$ is a point.
 - c) The intersection of any four distinct 2-cells is empty.
- 2) W is called a *locally centered lump collection* (LCL collection) if W is a locally centered collection and a locally lump collection at the same time (Figure 2).

Collections of 2-cells are depicted in Figure 2. Collection (a) is an LC but not LL collection because non-empty intersections $D_1 \cap D_3$ and $D_2 \cap D_4$ are not 1-cells. Collection (b) is an LL but not LC collection because $D_1 \cap D_2 \neq \emptyset, D_1 \cap D_3 \neq \emptyset, D_2 \cap D_3 \neq \emptyset$ but $D_1 \cap D_2 \cap D_3 = \emptyset$. Collection (c) is an LCL collection of

Figure 1. Collections of 1-cells $C_1, C_2, \dots$ (a)-(c) are LCL collections. (d)-(f) are LC but not LL collections. (g) is an LL but not LC collection.

Figure 2. (a) is an LC but not an LL collection of 2-cells. (b) is an LL but not LC collection of 2-cells. (c) is an LCL collection of 2-cells.

2-cells. Collections W_1, W_2 and W_3 depicted in **Figure 8** are LCL collections of 2-cells. The following corollary is a direct consequence of Definitions 2.2 and 2.3.

Corollary 2.1

Any subcollection of an LCL collection of 1-cells or 2-cells is an LCL collection.

In papers [8], locally centered collections were studied it was shown that for a given object, the intersection graphs of all continuous, regular and contractible covers are homotopic to each other. There is a significant number of papers which investigate collections of sets in 2D and 3D (see, e.g., [3]-[5] [9]-[12]). In papers [10]-[12], properties of SN sets were studied. A normal collection W of convex non-generate polygons (intersection of any two of them is an edge, a vertex, or empty) was called strongly normal (SN) if for all $P, P_1, \dots, P_n$ ($n > 0$) $\in W$, if each P_i intersects P and $I = P_1 \cap \dots \cap P_n$ is non-empty, then I intersects P). In paper [4], it was shown that cells in strongly normal collections can be arbitrary topological disks.

The advantage of using LCL covers in discretization schemes is that the digital models of continuous objects have the same topology as their continuous counterparts. For example, if we use an LCL cover of the plane, then the digital model of the plane is necessarily a digital 2-surface, *i.e.*, an LCL discretization scheme preserves the topology of the plane. Besides, cells in an LCL collection can be of an arbitrary shape and size. This property allows us to use a fine resolution (with any necessary small size of cells) in areas of interest, where image details are required, and a coarse resolution in the other parts of the object.

Regard now the set of 2-cells intersecting a given 2-cell in an LCL collection. This set specifies local topological properties of the collection.

Proposition 2.1

Let $W = \{D_0, D_1, \dots\}$ be an LCL collection of 2-cells, $U = \{D_1, D_2, \dots, D_s\}$ be a collection of all 2-cells intersecting D_0 , and $V = \{C_1, C_2, \dots, C_s\}$ be the collection of 1-cells $C_i = D_0 \cap D_i \neq \emptyset, i = 1, \dots, s$. Then V is an LCL collection.

Proof

Consider the collection $V = \{C_1, C_2, \dots, C_s\}$ of 1-cells. Suppose that $C_k \cap C_i \neq \emptyset$. Then $C_k \cap C_i = \partial C_k \cap \partial C_i = (D_0 \cap D_k) \cap (D_0 \cap D_i) = D_0 \cap D_i \cap D_k \neq \emptyset$. Since W is an LCL collection, then $D_0 \cap D_i \cap D_k = C_k \cap C_i = \partial C_k \cap \partial C_i = v$ is a point according to Definition 2.3. Consider the intersection $C_k \cap C_m \cap C_p, m \neq k, m \neq p, k \neq p$. Then $C_k \cap C_m \cap C_p = D_k \cap D_m \cap D_p \cap D_0$. Since W is an LCL collection, then $D_k \cap D_m \cap D_p \cap D_0 = \emptyset$ according to Definition 2.3. Therefore, $C_k \cap C_m \cap C_p = \emptyset$. Thus, $V = \{C_1, C_2, \dots, C_s\}$ is an LCL collection of 1-cells according to Definition 2.2. $\square$

In order to prove the following proposition, let us remind the definition of isomorphic sets. A collection $W = \{A_0, A_1, \dots\}$ of sets is isomorphic to a collection $V = \{B_0, B_1, \dots\}$ of sets, if there exists one-one onto correspondence $f: W \rightarrow V$ such that

$$A_i \cap \dots \cap A_k \cap \dots \cap A_p \neq \emptyset \text{ if and only if } f(A_i) \cap \dots \cap f(A_k) \cap \dots \cap f(A_p) \neq \emptyset.$$

Proposition 2.2

Let $W = \{D_0, D_1, \dots\}$ be an LCL collection of 2-cells, $U = \{D_1, D_2, \dots, D_s\}$ be a collection of all 2-cells intersecting D_0 and $V = \{C_1, C_2, \dots, C_s\}$ be an LCL collection of 1-cells $C_i = D_0 \cap D_i \neq \emptyset, i = 1, \dots, s$. Then collections $U = \{D_1, D_2, \dots, D_s\}$ and $V = \{C_1, C_2, \dots, C_s\}$ are isomorphic.

Proof

Let $U = \{D_1, D_2, \dots, D_s\}$, $V = \{C_1, C_2, \dots, C_s\}$. $D_i \in U, D_k \in U, i \neq k$, and $D_i \cap D_k \neq \emptyset$.

Suppose that $D_i \in U, D_k \in U, i \neq k$, and $D_i \cap D_k \neq \emptyset$. Since $D_0 \cap D_i \neq \emptyset, D_0 \cap D_k \neq \emptyset$, and W is an LCL collection, then $D_0 \cap D_i \cap D_k \neq \emptyset$. Since $C_i \cap C_k = D_0 \cap D_i \cap D_k$, then $C_i \cap C_k \neq \emptyset$. Suppose that $D_i \in U, D_k \in U, D_p \in U, i \neq k, i \neq p, p \neq k$, and consider the intersection $D_p \cap D_i \cap D_k$. Since W is an LCL collection, then $D_p \cap D_i \cap D_k = D_p \cap D_i \cap D_k \cap D_0 = \emptyset$.

Since $C_p \cap C_i \cap C_k = D_p \cap D_i \cap D_k \cap D_0$, then $D_p \cap D_i \cap D_k = \emptyset$. Similarly, if $C_i \cap C_k \neq \emptyset$, then $D_i \cap D_k \neq \emptyset$, and since $C_p \cap C_i \cap C_k = \emptyset$, then $D_p \cap D_i \cap D_k = \emptyset$. Therefore, $f: U \rightarrow V$ such that $f(D_i) = C_i, i = 1, \dots, s$, is an isomorphism. $\square$

The number of elements in an LCL collection can be reduced by merging couples of adjacent 1-cells or 2-cells.

Proposition 2.3

Let $W = \{C_0, C_1, \dots\}$ be an LCL collection of 1-cells, $C_0 \cap C_1 \neq \emptyset$ and $B = C_0 \cup C_1$. Let $V = \{B, C_2, \dots\}$ be a collection of 1-cells obtained by merging C_0 and C_1 , C_i is adjacent to C_0 , C_k is adjacent to C_1 .

- If $C_i \cap C_k = \emptyset$, then $V = \{B, C_2, \dots\}$ is an LCL collection of 1-cells (**Figure 3(a), Figure 3(b)**).
- If $C_i \cap C_k \neq \emptyset$, then $V = \{B, C_2, \dots\}$ is not an LCL collection of 1-cells (**Figure 3(c), Figure 3(d)**).

Proof

Suppose $C_i \cap C_k = \emptyset$. Then $B \cap C_i = C_0 \cap C_i = v_1$, and $B \cap C_k = C_0 \cap C_k = v_2$, are points, and V is an LCL collection of 1-cells by construction of V (see **Figure 3(a), Figure 3(b)**).

Suppose $C_i \cap C_k \neq \emptyset$. By construction of $V = \{C_i, B, C_k\}$, $C_i \cap C_k \neq \emptyset$, $C_i \cap B \neq \emptyset$, $B \cap C_k \neq \emptyset$. Since $B \cap C_i \cap C_k = \emptyset$, then V is not a locally centered collection (see **Figure 3(c), Figure 3(d)**). Thus, V is not an LCL collection. $\square$

Proposition 2.4

Let $W = \{D_0, D_1, \dots\}$ be an LCL collection of 2-cells, $D_0 \cap D_1 \neq \emptyset$ and $B = D_0 \cup D_1$. Let $V = \{B, D_2, \dots\}$ be a collection of 2-cells obtained by merging D_0 and D_1 .

- If for any D_i such that $D_0 \cap D_i \neq \emptyset, D_1 \cap D_i = \emptyset$, and for any D_k such that $D_1 \cap D_k \neq \emptyset, D_0 \cap D_k = \emptyset$, it follows that $D_k \cap D_i = \emptyset$, then $V = \{B, D_2, \dots\}$ is an LCL collection of 2-cells (**Figure 4(a), Figure 3(b)**).
- If there are 2-cells D_i and D_k such that $D_0 \cap D_i \neq \emptyset, D_1 \cap D_i = \emptyset, D_1 \cap D_k \neq \emptyset, D_0 \cap D_k = \emptyset$, and $D_k \cap D_i \neq \emptyset$, then $V = \{B, D_2, \dots\}$ is not an LCL collection of 2-cells (**Figure 4(c)**).

Proof

Suppose that $B \cap D_i \neq \emptyset, B \cap D_k \neq \emptyset$ and $D_k \cap D_i \neq \emptyset$. Since W is an LCL collection, there can be only constructions depicted in **Figures 4(a)-(c)**. For simplicity, we omit cases obtained by replacing indexes “0” with “1”, “1” with “0”, “1” with “k” and “k” with “1”.

Consider an LCL collection $W = \{D_0, D_1, \dots\}$ and the collection $V = \{B, D_2, \dots\}$ depicted in **Figure 4(a)**.

Since $B \cap D_i \cap D_k = v$, $B \cap D_k = C_{k0}$, then V is an LCL collection. Consider an LCL collection $W = \{D_0, D_1, \dots\}$ and the collection $V = \{B, D_2, \dots\}$ depicted in **Figure 4(b)**. It follows from construction of V that $B \cap D_i \cap D_k = v$. $B \cap D_k = C_{k0} \cup C_{k1}$, i.e., V is an LCL collection. Consider an LCL collection $W = \{D_0, D_1, \dots\}$ and the collection $V = \{B, D_2, \dots\}$ depicted in **Figure 4(c)**. By construction of V , $B \cap D_i \neq \emptyset, B \cap D_k \neq \emptyset, D_k \cap D_i \neq \emptyset$, but $B \cap D_i \cap D_k = \emptyset$. Therefore, V is not an LCL collection. $\square$

3. Digital 1- and 2-Surfaces and Digital Models of the Plane

Traditionally, a digital image has the structure of a simple graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n, \dots\}$ is a set of vertices, and $E = \{(v_p, v_q), \dots\} \subseteq V \times V$ is a set of edges. Edges between vertices define the nearness in the neighborhood of any vertex and such notions as the connectedness, the dimensionality and the distance on G are

Figure 3. (a) An LCL collection W of 1-cells. (b) An LCL collection V of 1-cells. (c) An LCL collection W of 1-cells. (d) Collection V is not an LCL collection.

Figure 4. (a) W is an LCL collection. V is an LCL collection. (b) W is an LCL collection. V is an LCL collection. (c) W is an LCL collection. V is not an LCL collection.

completely defined by sets V and E (see, e.g., [6]-[9] [13] [14]).

We use the notations $v_p \in G$ and $(v_p v_q) \in G$ if $v_p \in V$ and $(v_p v_q) \in E$ respectively if no confusion can result.

For two graphs $G = (V, E)$ and $H = (Y, W)$ with disjoint vertex sets V and Y , their join $G \oplus H$ is the graph that contains G , H , and edges joining every vertex in G with every vertex in H .

An induced subgraph $O(v) \subseteq G$ containing all vertices of G adjacent to v is called the rim or the neighborhood of v in G , the subgraph $U(v) = \{v\} \oplus O(v)$ is called the ball of v [14]. If $G = \{v\}$ is a singleton, we write v rather than $\{v\}$ if no confusion can result.

Two graphs G and H are called homotopy equivalent if one of them can be converted to the other one by a sequence of contractible transformations. Properties of contractible transformations were studied in [9] [14]. In particular, it was shown that contractible transformations do not change the Euler characteristic and the homology groups of graphs.

If we consider a graph as a digital space, we use the word space for a graph and the word point for a vertex. Notice that in this paper, a subspace is an induced subspace. In papers [9] [13], digital n -surfaces and n -manifolds were defined and investigated.

Definition 3.1

- A digital 0-dimensional surface (sphere) is a graph $S^0(x, y)$ with just two non-adjacent points x and y .
- A connected space M is called a digital 1-surface, if for any point $v \in M$, the rim $O(v)$ is a digital 0-dimensional surface.

A digital 1-surface has four or more points. Digital 1-spheres G_0 - G_3 are depicted in Figure 5. A digital line G_4 has an infinite number of points (see Figure 5).

Consider a sequence of transformations, which reduce the number of points in a given digital 1-surface G . In graph theory, the contraction of points x and y in a graph G is the replacement of x and y with a point z such that z is adjacent to the points to which points x and y were adjacent. The resulting graph has one point less than G . Therefore, G must contain two adjacent points x and y such that after their contraction, the obtained space

$N = (G - \{x, y\}) \cup z$ is a digital 1-surface, which is homotopy equivalent to G . For example, a minimal digital 1-sphere is the join $S_{\min}^1 = S^0(u_1, v_1) \oplus S^0(u_2, v_2)$ of two copies of a zero-dimensional sphere (G_5 in **Figure 5**) [9] [13]. A digital 1-surface M is called a digital 1-sphere (circle) if it can be converted to $S_{\min}^1$ by a sequence of contracting adjacent points.

Definition 3.2

Let M be a digital 1-surface and x and y be adjacent points of M . We say that $\{x, y\}$ is a simple pair if the space $U(x) \cup U(y) - \{x, y\}$ is a digital 0-sphere.

In **Figure 6(a)**, $\{x, y\}$ is a simple pair.

Proposition 3.1

Let $\{x, y\}$ be a simple pair lying in a digital 1-surface M . Then the space $N = (M - \{x, y\}) \cup z$, obtained by contraction of $\{x, y\}$ is a digital 1-surface.

Proof

The proof is depicted in **Figure 6**. Since $\{x, y\}$ is a simple pair of points, then $U(x) \cup U(y) - \{x, y\} = \{v, u\}$ is a digital 0-sphere (**Figure 6(a)**). Glue a point z to M in such a way that $O(z) = U(x) \cup U(y)$, and delete points x and y (**Figure 6(b)**, **Figure 6(c)**). By construction, the obtained space $N = (M - \{x, y\}) \cup z$ is a digital 1-surface. The proof is complete. $\square$

Remark 3.1

It is easy to see from Proposition 3.3, that a digital 1-surface with a finite number of points is a digital 1-sphere, which can be transformed to a minimal digital 1-sphere by sequential contraction of simple pairs. Digital 1-surfaces G_1 - G_3 depicted in **Figure 5** can be transformed to a minimal 1-sphere G_0 by sequential contracting simple pairs. Thus, a digital 1-surface is either a digital 1-sphere or a digital line.

Figure 5. $G_0 - G_3$ are digital closed 1-surfaces (digital circles). G_4 is the digital line. $G_5 - G_8$ are digital closed 2-surfaces. G_9 is a portion of the digital plane.

Figure 6. (a) M is a digital 1-sphere with six points. (b) $\{x, y\}$ is a topological pair of points. (c) $N = (M \cup z) - \{x, y\}$ is a digital 1-sphere with five points. (d)-(f) $U(x) \cup U(y)$ is a contractible space. $U(x) \cup U(y) - \{x, y\}$ is a digital 1-sphere. $\{x, y\}$ is a topological pair of points. (g) $\{a, b\}$ is not a topological pair of points. $U(a) \cup U(b) - \{a, b\}$ is not a digital 1-sphere.

Definition 3.3

- A connected digital space M is called a digital 2-surface if for any point v of M , the rim $O(v)$ is a digital 1-sphere [13].

A digital 2-surface with a finite number of points is called a digital closed 2-surface. In **Figure 5**, $G_5 - G_8$ are digital closed 2-surfaces. G_9 is a portion of a digital plane.

Consider a sequence of transformations, which reduce the number of points in a given digital 2-surface M . As above, this is a contraction of two adjacent points x and y such that after their contraction, the obtained space $N = (M - \{x, y\}) \cup z$ is a digital 2-surface.

Definition 3.4

Let M be a digital 2-surface and x and y be adjacent points of M . We say that $\{x, y\}$ is a simple pair if the space

$U(x) \cup U(y) - \{x, y\}$ is a digital 1-sphere (see **Figure 6(d)**, **Figure 6(e)**).

In **Figure 6(d)**, $\{x, y\}$ is a simple pair. A pair $\{a, b\}$ depicted in **Figure 6(g)** is not a simple pair.

Proposition 3.2

Let $\{x, y\}$ be a simple pair lying in a digital 2-surface M . Then the space $N = (M - \{x, y\}) \cup z$ obtained by contraction of $\{x, y\}$ is a digital 2-surface.

Proof

The proof is similar to the proof of Proposition 3.3. Delete from M points x and y and glue a point z in such a way that $O(z) = U(x) \cup U(y) - \{x, y\}$. Therefore, the rim of point z is a digital 1-sphere according to Definition 3.6 (see **Figure 6(f)**). The rims of all other points belonging to N are digital 1-surfaces by construction. Therefore, N is a digital 2-surface. The proof is complete. $\square$

The join $S_{\min}^2 = S^0(u_1, v_1) \oplus S^0(u_2, v_2) \oplus S^0(u_3, v_3)$ of three copies of a zero-dimensional sphere is a minimal digital 2-sphere (G_5 in **Figure 5**). A digital 2-surface M is called a digital 2-sphere if it is homotopy equivalent to a minimal digital 2-sphere [8] [9]. It can be checked directly that a digital 2-sphere G_6 depicted in **Figure 5** can be converted to G_5 by a sequential contraction of simple pairs. G_7 is a digital torus, and G_8 is a digital projective plane. G_7 and G_8 have no simple pairs, and cannot be converted to G_5 by sequential contractions of simple pairs. Properties of a digital sphere, a digital torus and a digital projective plane were studied in [8] [9].

Digital 2-surfaces can be used to construct topologically correct digital models of segmented medical images.

4. Digital Models of LCL Collections and Digital Models of the Plane

Let us investigate a connection between a segmented image of an object of interest and the digital model of this image. We are going to show that an LCL segmentation of an object incorporates topological features directly into the segmentation process.

Remind the definition of the intersection graph of a family of sets. Let $W = \{D_1, D_2, \dots, D_n, \dots\}$ be a finite or countable family of sets. Then the graph $G(W)$ with points $\{x_1, x_2, \dots, x_n, \dots\}$ is called the intersection graph of W , if points x_k and x_i are adjacent whenever $D_k \cap D_i \neq \emptyset$. In other words, $f: G(W) \rightarrow W$ such that $f(x_i) = D_i$ is an isomorphism.

Definition 4.1

- Let $W = \{D_1, D_2, \dots\}$ be an LCL collection of 1-cells or 2-cells, and M be the union of all cells, $M = D_1 \cup D_2 \cup \dots$. Then the collection W is called an *LCL cover* of M .
- Let $W = \{D_1, D_2, \dots\}$ be an LCL collection of 1-cells or 2-cells, and $G(W)$ be the intersection graph of W . Then $G(W)$ is called the digital model of $M = D_1 \cup D_2 \cup \dots$ in regard to W (**Figure 7**, **Figure 8**).

Theorem 4.1

Let M be a circle and $W = \{C_0, C_1, \dots, C_n\}$ be an LCL cover of $M = C_0 \cup \dots \cup C_n$ by 1-cells. Then the digital model $G(W)$ of M in regard to W is a digital 1-surface.

Proof

Let $W = \{C_0, C_1, \dots, C_n\}$ be an LCL cover of a circle M by 1-cells such that $C_k \cap C_i \neq \emptyset$ if $k = (i+1) \bmod (n+1)$. Let $G(W) = \{x_0, x_1, \dots, x_n\}$ be the intersection graph of W , where $x_i = f(C_i)$ (**Figures 7(a)-(c)**). This means that points x_k and x_i are adjacent whenever $C_k \cap C_i \neq \emptyset$. Consider 1-cell C_1 . Then $C_1 \cap C_0 = v_0$ and $C_1 \cap C_2 = v_2$ are endpoints of C_1 , $C_1 \cap C_i = \emptyset, i \neq 0, 2$, and $C_0 \cap C_2 = \emptyset$. Therefore, point x_1 is adjacent only to points x_0 and x_2 and points x_0 and x_2 are non-adjacent. Therefore, the rim $O(x_1)$ in $G(W)$ is a digital 0-surface consisting of two non-adjacent points x_0 and x_2 . For the same reason, the rim of any point in

Figure 7. (a), (b) and (c) W_1, W_2, W_3 are LCL covers of a circle C , G_1, G_2, G_3 are the corresponding digital models of C . (d) W_4 is an LCL cover of line L , G_4 is the corresponding digital model of L .

Figure 8. W_1, W_2 and W_3 are LCL covers of the plane E^2 . $G(W_1), G(W_2)$ are digital models of E^2 .

$G(W)$ is a digital 0-surface. Hence, $G(W)$ is a digital 1-surface according to Definition 3.1. $\square$

In **Figure 7**, W_1, W_2 and W_3 are LCL covers of a circle C , and G_1, G_2 and G_3 are their intersection graphs. Evidently, G_1, G_2 and G_3 are digital 1-surfaces according to Definition 3.1. W_4 is an LCL cover of a line L . The intersection graph G_4 of W_4 is a digital 1-surface. A cover $V = \{C_1, C_2\}$ of a circle shown in **Figure 3(c)** is not an LCL cover, and the intersection graph $G(V)$ containing two adjacent points is not a digital 1-surface. Similarly, a cover $U = \{C_1, C_2, C_3\}$ of a circle shown in **Figure 3(d)** is not an LCL cover, and the intersection graph $G(U)$ containing three adjacent points is not a digital 1-surface.

Theorem 4.2

Let M be a circle, $W = \{C_0, C_1, C_2, \dots, C_n\}$ be an LCL cover of $M = C_0 \cup \dots \cup C_n$ by 1-cells, and $G(W) = \{x_0, x_1, x_2, \dots, x_n\}$ be a digital model of M in regard to W . Let $V = \{C_1, B, C_3, \dots, C_n\}$ be the cover of M by 1-cells obtained by merging C_1 and C_2 , $B = C_1 \cup C_2$, and $G(V) = \{x_0, z, x_3, \dots, x_n\}$ be the digital model of M in regard to V .

If $V = \{C_1, B, C_3, \dots, C_n\}$ is an LCL cover of M , then $\{x_1, x_2\}$ is a simple pair in the digital 1-surface $G(W)$ and $G(V) = (G(W) - \{x_1, x_2\}) \cup z$ is a digital 1-surface obtained from $G(W)$ by contracting $\{x_1, x_2\}$.

Proof

According to Proposition 2.3, $C_0 \cap C_3 = \emptyset$. Therefore, points x_0 and x_3 are not adjacent and $\{x_1, x_2\}$ is a simple pair in $G(W)$. This means that the rim $O(z)$ is a digital 0-sphere $S_{\min}^0(x_1, x_3)$. The rims of all other points of $G(V)$ are also digital 0-spheres by construction of $G(V)$. $\square$

A collection W_2 depicted in **Figure 7(b)** is an LCL cover of a circle C . Any pair of adjacent 1-cells of W_2 is a simple pair. By merging any two adjacent 1-cells, W_2 can be transformed to an LCL cover W_1 containing four elements (**Figure 7(a)**). Similarly, the intersection graph G_2 containing a simple pair of points can be transformed to a 1-surface G_1 by contracting these points (see **Figure 7(a)** and **Figure 7(b)**). An LCL cover W_2 is a minimal LCL cover of a circle, and G_2 does not contain simple pairs (see **Figure 7(a)**).

In paper [8], the intersection graphs of covers of convex 2-cells were studied. Let $F = \{U_1, U_2, \dots, U_n, \dots\}$ be a set of all unit squares on E^2 whose vertices are located in points with integer coordinates, and whose edges are parallel to the coordinate axes. F is a cover of E^2 . Suppose that M is a convex 2-cell on E^2 , and

$W = \{U_1, U_2, \dots, U_n\}$ is a collection of all squares intersecting M . It was shown that the intersection graph $G(W)$ of W is contractible, i.e., can be converted to a one-point graph by a sequence of contractible transformations.

Using this result and the notion of isomorphic covers it was shown that the intersection graphs of covers of a sphere, a torus and a projective plane are homotopy equivalent to digital 2-surfaces G_5 , G_7 and G_8 respectively depicted in [Figure 5](#).

Theorem 4.3

Let $M = E^2$ be the plane, and $W = \{D_0, D_1, \dots\}$ be an LCL cover of $M = D_0 \cup D_1 \cup \dots$ by 2-cells such that for any $D_i \in W$, the collection of all 2-cells of W intersecting D_i is finite. Then the digital model $G(W)$ of M is a digital 2-surface.

Proof

Suppose that $W = \{D_0, D_1, \dots\}$ is an LCL cover of M , and $G(W)$ is the intersection graph of W such that $x_i = f(D_i)$, $i = 0, 1, \dots$. Consider the collection $U = \{D_1, D_2, \dots, D_s\}$ containing all 2-cells which intersect D_0 . Then the collection $V = \{C_1, C_2, \dots, C_s\}$, where $C_i = D_0 \cap D_i$, $i = 1, \dots, s$, is an LCL collection of 1-cells, and collections U and V are isomorphic to each other according to Proposition 2.2. Therefore, the intersection graph $G(U)$, $G(U) \subseteq G(W)$, of collection U is the intersection graph $G(V)$ of collection V , where $x_i = f(C_i)$, $i = 1, \dots, s$. Since $G(V)$ is a digital 1-sphere according to Theorem 3.1, then $G(U)$ is a digital 1-sphere. Therefore, the rim $O(x_0) = G(U)$ of x_0 in $G(W)$ is a digital 1-sphere. For the same reason, the rim of any point in $G(W)$ is a digital 1-sphere. Hence, $G(W)$ is a digital 2-surface according to Definition 3.3. $\square$

[Figure 8](#) contains LCL covers W_1 , W_2 and W_3 of the plane E^2 and digital models $G(W_1)$ and $G(W_2)$ which are digital 2-surfaces. One can check directly that $G(W_3)$ is also a digital 2-surface.

Consider now the replacement of two adjacent 2-cells with their union. Find conditions which guarantee that a new cover of E^2 is an LCL cover.

Theorem 4.4

Let $M = E^2$ be the plane, $W = \{D_0, D_1, \dots\}$ be an LCL cover of $M = D_0 \cup D_1 \cup \dots$ by 2-cells and $G(W) = \{x_0, x_1, \dots\}$ be a digital model of M in regard to W . Let $V = \{D_0, B, D_3, \dots\}$ be the cover of $M = D_0 \cup B \cup D_3 \cup \dots$ by 2-cells obtained by merging D_1 and D_2 , $B = D_1 \cup D_2$, and $G(V) = \{x_0, z, x_3, \dots\}$ be the digital model of M in regard to V . If for any D_i such that $D_0 \cap D_i \neq \emptyset$, $D_1 \cap D_i = \emptyset$, and for any D_k such that $D_1 \cap D_k \neq \emptyset$, $D_0 \cap D_k = \emptyset$, it follows that $D_i \cap D_k = \emptyset$, then $\{x_1, x_2\}$ is a simple pair in the digital 2-surface $G(W)$ and $G(V) = (G(W) - \{x_1, x_2\}) \cup z$ is a digital 2-surface obtained from $G(W)$ by contracting $\{x_1, x_2\}$.

Proof

According to Proposition 2.4, $V = \{D_1, B, D_3, \dots\}$ is an LCL collection of 2-cells. Therefore, the rim $O(z)$ of point z in $G(V)$ is a digital 1-sphere. Since $U(x_2) \cup U(x_1) - \{x_1, x_2\} = O(z)$ is a digital 1-sphere, then $\{x_0, x_1\}$ is a simple pair according to Definition 3.4 and $G(V) = (G(W) - \{x_1, x_2\}) \cup z$, is a digital 2-surface obtained from $G(W)$ by contracting $\{x_1, x_2\}$. The proof is complete. $\square$

It is easy to check directly (see [Figure 8](#)) that merging any pair of adjacent 2-cells converts W_2 to non-LCL covers of the plane. Similarly, the intersection graph $G(W_2)$ do not contain simple pairs. In W_3 , all white 2-cells can be merged together and the obtained cover is an LCL collection.

In paper [7], Eckhardt and Latecki studied digital topologies in Z^2 (Z^2 is the set of points with integer coordinates in E^2). They showed that “there are only two topologies in Z^2 whose connected sets are connected in the intuitive sense.” In the present paper, we do not use Z^2 as a basic set of points for building a topology. In this paper, a digital point corresponds to a 2-cell in an LCL cover of the continuous plane. Since elements of an LCL cover can be chosen arbitrarily (within LCL cover), then there are infinitely many non-isomorphic digital models of the plane.

As one can see from [Figure 8](#), 2-cells have different shapes and sizes within the same LCL cover. There is a direct similarity between the plane and its digital models which are digital 2-surfaces. In E^2 , an open neighborhood of a point x contains an open 2-disk D whose boundary is a circle C . In $G(W_1)$ and $G(W_2)$, a point v belongs to the ball $U(v) = \{v\} \oplus O(v)$ whose boundary is the rim $O(v)$, i.e., a digital circle according to Definition 3.3. Thus, a digital model preserves the topology of a continuous space.

Now we are able to describe a method that lets us transform regions of interest produced by the image acquisition process into digital spaces with topological features of the regions.

In the first step, construct an LCL cover W of the region M in accordance with conditions and restrictions imposed by requirements defined by the accuracy and correctness of the representation.

In the second step, build the digital model $G(W)$ of the region. Using an LCL cover guarantees that the digital model is a digital space preserving the topology of the object.

5. Conclusions

This paper proposes a new discretization scheme for constructing digital models of continuous two-dimensional objects. LCL covers of a circle C and Euclidean spaces E^1 and E^2 are defined, and it is shown that digital models of C and E^1 are necessarily digital 1-surfaces; the digital model of E^2 is necessarily the digital 2-surface. It is shown that digital models preserve topology of their continuous counterparts, which is particularly important for many clinical and research applications.

It is shown that one can choose an LCL cover with a required resolution in any region of interest. A simple result of this consideration is that there are infinitely many non-isomorphic digital models of the plane. We describe a method that constructs digital models of areas of interest with any required resolution.

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