New Approach to FM/FM/1 Queue and its Performance Measures

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ABSTRACT: In this paper we discussed uncertainty in the classical queueing model M/M/1 assuming that arrival rate is a fuzzy number $\tilde{\lambda} = (\lambda_{\min} / \bar{\lambda} / \lambda_{\max})$ and service rate is a fuzzy number $\tilde{\mu} = (\mu_{\min} / \bar{\mu} / \mu_{\max})$. We have also discussed its performance measures which were also fuzzy numbers using the concept of α -cuts. Finally we presented a numerical example showing the efficiency of this new approach.

Keywords: Queueing Theory, Fuzzy Numbers, Interval Arithmetic, α -cuts, Performance Measures.

1. Introduction

Traditional crisp queueing models are not suitable in real world because in crisp queues, we calculate performance measures depending on arrival rate λ and departure rate μ without giving attention to uncertainty in these two values.^[1]

In fuzzy queues we study the uncertainty in λ , μ and its effect on uncertainty of performance measures L_s , L_q , W_s , W_q , i.e. we could know if λ varies from λ_1 , λ_2 how the performance measures would vary?

2. The Simple Crisp Queue M/M/1

Suppose a simple queue where arrival rates are following Poisson distribution with parameter λ and departure rates are also following Poisson distribution with parameter μ , We have only one server. The calling source is set to be infinite and service discipline is FCFS (first comes first served).

This queue is stable if and only if $\lambda < \mu$, i.e. $\rho < 1$ where $\rho = \frac{\lambda}{\mu}$. At the steady state we can see that:^{[2],[3]}

$$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda} \quad (2.1)$$

$$L_q = L_s - \rho = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu^2 - \lambda\mu} \quad (2.2)$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu(1-\rho)} = \frac{1}{\mu-\lambda} \quad (2.3)$$

$$W_q = \frac{L_q}{\lambda} = \frac{\rho}{\mu(1-\rho)} = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu^2 - \lambda\mu} \quad (2.4)$$

3. Fuzzy Set Theory

3.1. Fuzzy set:

A fuzzy set A in Ω is a set characterized by its membership function as follows: $A : \Omega \rightarrow [0,1]$.^[4]

3.2. Triangular fuzzy number:

A triangular fuzzy number $\tilde{A}$ is defined by: $\tilde{A} = (a_1 / a_2 / a_3)$; $a_1 < a_2 < a_3 \in \mathbf{R}$ and its membership function is:^[5]

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & ; x < a_1 \\ \frac{a_3-x}{a_3-a_2} & ; a_1 \leq x \leq a_2 \\ 0 & ; \text{otherwise} \end{cases} \quad (3.1)$$

3.3. α -cuts:

An α -cut of $\tilde{A}$ (written $\tilde{A}[\alpha]$) is defined as:

$$\tilde{A}[\alpha] = \{x \in \Omega ; \mu_{\tilde{A}}(x) \geq \alpha\}$$

In special case, if $\tilde{A}$ is a triangular fuzzy number then: (3.2) ^{[5], [6]}

3.4. Fuzzy arithmetic based on interval arithmetic:

Let $[a_1, b_1], [a_2, b_2]$ be two closed, bounded, intervals of real numbers. If $*$ denotes addition, subtraction, multiplication, or division, then: ^{[5], [6]}

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$$

$$[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$$

$$[a_1, b_1] / [a_2, b_2] = [a_1, b_1] \cdot \left[\frac{1}{b_2}, \frac{1}{a_2} \right]$$

$$[a_1, b_1] \cdot [a_2, b_2] = [\alpha, \beta]$$

Where:

$$\alpha = \min \{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}$$

$$\beta = \max \{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}$$

Assuming that zero doesn't belong to $[a_2, b_2]$ in division.

Multiplication and division may be further simplified in some special cases as follows:

If $a_1 \geq 0$ and $a_2 \geq 0$ then:

$$[a_1, b_1] \cdot [a_2, b_2] = [a_1 a_2, b_1 b_2]$$

If $b_1 < 0$ and $a_2 \geq 0$ then:

$$[a_1, b_1] \cdot [a_2, b_2] = [a_1 b_2, a_2 b_1]$$

If $b_1 < 0$ and $b_2 < 0$ then:

$$[a_1, b_1] \cdot [a_2, b_2] = [b_1 b_2, a_1 a_2]$$

If $a_1 \geq 0$ and $b_2 < 0$ then:

$$[a_1, b_1] \cdot [a_2, b_2] = [a_2 b_1, b_2 a_1]$$

Now we can define operations on two fuzzy numbers $\tilde{A}, \tilde{B}$ using interval arithmetic rules on $\tilde{A}[\alpha], \tilde{B}[\alpha]$ as follows:

$$\tilde{A}[\alpha] * \tilde{B}[\alpha] = \tilde{C}[\alpha]$$

Where $*$ denotes addition, subtraction, multiplication, or division, according to the condition that zero doesn't belong to $\tilde{B}[\alpha]$ for all α when $*$ denotes division. ^{[5], [6]}

4. FM/FM/1 queue

Assume that arrival rates varies from $\lambda_{\min}$ to $\lambda_{\max}$ during the period T , and that: $\bar{\lambda} = \lim_{T \rightarrow \infty} \frac{\lambda(T)}{T}$ so we can say that $\tilde{\lambda} = (\lambda_{\min} / \bar{\lambda} / \lambda_{\max})$ is the fuzzy arrival

rate, also we'll assume that departures varies from $\mu_{\min}$ to $\mu_{\max}$ during the same period T , and

$$\bar{\mu} = \lim_{T \rightarrow \infty} \frac{\mu(T)}{T} \quad \text{so we may also say that}$$

$\tilde{\mu} = (\mu_{\min} / \bar{\mu} / \mu_{\max})$ is the fuzzy departure rate.

Using (3.2) we can see that:

$$\begin{aligned} \tilde{\lambda}[\alpha] &= [\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha, \lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha] \\ \tilde{\mu}[\alpha] &= [\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha, \mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha] \end{aligned} \quad (4.1)$$

$$\tilde{\rho}[\alpha] = \left[\frac{\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha}{\mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha}, \frac{\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha}{\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha} \right]$$

Applying (4.1) on (2.1), (2.2), (2.3), (2.4) using rules of fuzzy arithmetic gives:

$$\tilde{L}_S[\alpha] = [L_{S_{\min}}, L_{S_{\max}}]$$

Where:

$$\begin{aligned} L_{S_{\min}} &= \frac{\frac{\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha}{\mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha}}{1 - \frac{\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha}{\mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha}} \\ L_{S_{\max}} &= \frac{\frac{\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha}{\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha}}{1 - \frac{\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha}{\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha}} \end{aligned} \quad (4.2)$$

$$\tilde{L}_q[\alpha] = \tilde{L}_S[\alpha] * \tilde{\rho}[\alpha] \dots (4.3)$$

$$\tilde{W}_S[\alpha] = [W_{S_{\min}}, W_{S_{\max}}]$$

Where:

$$\begin{aligned} W_{S_{\min}} &= \frac{\frac{\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha}{(\mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha)(\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha)}}{1 - \frac{\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha}{\mu_{\max} - (\mu_{\max} - \bar{\mu})\alpha}} \\ W_{S_{\max}} &= \frac{\frac{\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha}{(\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha)(\lambda_{\min} + (\bar{\lambda} - \lambda_{\min})\alpha)}}{1 - \frac{\lambda_{\max} - (\lambda_{\max} - \bar{\lambda})\alpha}{\mu_{\min} + (\bar{\mu} - \mu_{\min})\alpha}} \end{aligned} \quad (4.4)$$

$$\tilde{W}_q[\alpha] = \tilde{L}_q[\alpha] \cdot \left[\frac{1}{\lambda_{max} - (\lambda_{max} - \bar{\lambda})\alpha}, \frac{1}{\lambda_{min} + (\bar{\lambda} - \lambda_{min})\alpha} \right] \quad (4.5)$$

This system is stable if $\tilde{\rho} < 1$ i.e. $\lambda_{max} < \mu_{min}$.

5. Numerical Example

Consider a system which is with fuzzy arrivals and fuzzy departures: $\tilde{\lambda} = (3/7/12)$, $\tilde{\mu} = (13/15/18)$ then using equations (4.1) we have: $\tilde{\lambda}[\alpha] = [3 + 4\alpha, 12 - 5\alpha]$, $\tilde{\mu}[\alpha] = [13 + 2\alpha, 18 - 3\alpha]$; and using (4.2), (4.3), (4.4), (4.5) we get:

$$\begin{aligned} \tilde{\rho}[\alpha] &= \left[\frac{4\alpha + 3}{-3\alpha + 18}, \frac{-5\alpha + 12}{2\alpha + 13} \right] \\ \tilde{L}_s[\alpha] &= \left[\frac{-3 - 4\alpha}{-15 + 7\alpha}, \frac{12 - 5\alpha}{1 + 7\alpha} \right] \\ \tilde{W}_s[\alpha] &= \left[\frac{3 + 4\alpha}{35\alpha^2 - 159\alpha + 180}, \frac{12 - 5\alpha}{28\alpha^2 + 25\alpha + 3} \right] \\ \tilde{W}_q[\alpha] &= \left[\frac{-(3 + 4\alpha)^2}{105\alpha^3 - 1107\alpha^2 + 3402\alpha - 3240}, \frac{(12 - 5\alpha)^2}{56\alpha^3 + 414\alpha^2 + 331\alpha + 39} \right] \\ \tilde{L}_q[\alpha] &= \left[\frac{(3 + 4\alpha)^2}{21\alpha^2 - 171\alpha + 270}, \frac{(-12 + 5\alpha)^2}{14\alpha^2 + 93\alpha + 13} \right] \end{aligned}$$

α	$\tilde{L}_s[\alpha]$	$\tilde{W}_s[\alpha]$
0.0	[0.2000, 12.0000]	[0.0167, 4.0000]
0.1	[0.2378, 6.7646]	[0.0207, 1.9896]
0.2	[0.2794, 4.5835]	[0.0254, 1.2062]
0.3	[0.3256, 3.3871]	[0.0310, 0.8065]
0.4	[0.3771, 2.6316]	[0.0377, 0.5721]
0.5	[0.4348, 2.1111]	[0.0458, 0.4222]
0.6	[0.4999, 1.7307]	[0.0556, 0.3205]
0.7	[0.5743, 1.4407]	[0.0676, 0.2484]
0.8	[0.6596, 1.2121]	[0.0824, 0.1955]
0.9	[0.7586, 1.0274]	[0.1012, 0.1557]
1.0	[0.8750, 0.8750]	[0.1250, 0.1250]

α	$\tilde{L}_q[\alpha]$	$\tilde{W}_q[\alpha]$
0.0	[0.0333, 11.0777]	[0.0028, 3.6923]
0.1	[0.0457, 5.8934]	[0.0040, 1.7334]
0.2	[0.0610, 3.7625]	[0.0055, 0.9901]
0.3	[0.0799, 2.6150]	[0.0076, 0.6226]
0.4	[0.1032, 1.9070]	[0.0103, 0.4146]
0.5	[0.1318, 1.4325]	[0.0139, 0.2865]
0.6	[0.1667, 1.0970]	[0.0185, 0.2032]
0.7	[0.2095, 0.8504]	[0.0246, 0.1466]
0.8	[0.2622, 0.6642]	[0.0328, 0.1071]
0.9	[0.3272, 0.5206]	[0.0436, 0.0789]
1.0	[0.4083, 0.4083]	[0.0583, 0.0583]

Figure (1): Membership function of Ls

Figure (2): Membership function of Ws

Figure (3): Membership function of L_q

Figure (4): Membership function of W_q

6. Conclusions

In this paper, we discussed the performance measures of FM/FM/1 queueing model where both arrival and departure rates are fuzzy numbers, so the performance measures were also fuzzy numbers. Numerical examples were presented to show the efficiency of this new approach.

7. Recommendations

Find fuzzy approaches to performance measures of other queueing models other than FM/FM/1 and study its costs.

8. References

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الخلاصة: تمت مناقشة عدم التأكيد (الضبابية) في صف الانتظار الكلاسيكي M/M/1 بافتراض أن معدل الوصول هو عدد ضبابي $(\lambda = \lambda_{min} / \bar{\lambda} / \lambda_{max})$ وأن معدل الخدمة أيضاً عدد ضبابي $(\mu = \mu_{min} / \bar{\mu} / \mu_{max})$ وتمت مناقشة مقاييس أدائه والتي كانت أيضاً أعداد ضبابية وذلك بالاعتماد على مفهوم مستويات α ، وفي النهاية تم عرض مثال عددي لإظهار كفاءة هذا التقريب الجديد.

كلمات مفتاحية: صفوف الانتظار، أعداد ضبابية، الحساب المجالي، مستويات α ، مقاييس الأداء.