Utilization of Basic-Event Statistical Independence in Simplifying the Uncertainty Analysis of Fault Trees

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Abstract. Analysis of fault trees used in modelling computer system security is typically based on orthogonalization of a Boolean expression as well as preservation of existing statistical independencies. This note shows that careful utilization of statistical independencies is beneficial not only for evaluating the mean value of the top-event probability but also for quantifying its uncertainty. A demonstrative example indicates that such utilization leads to a dramatic reduction in the number of coefficients calculated, as well as a significant simplification of the calculations needed for each of them.

Keywords: Fault-tree analysis; Computer system security; Statistical independence; Uncertainty quantification; Doubly-stochastic modelling.

1. Introduction

The fault-tree model, a traditional reliability tool^[1-5], is now being used also in the analysis and design of security systems^[6-8]. Sometimes, fault trees are unnecessarily renamed as attack trees or attack-defense trees for this particular application^[9-11]. Analysis of fault trees is typically achieved in the Boolean domain followed by a transform to the probability domain that can be performed directly, on a one-to-one basis, if the success or failure expression obtained in the Boolean domain is a Probability-Ready Expression (PRE)^[12-16], i.e., if it is an expression in which

- a) All ANDed subexpressions are statistically independent,
- b) All ORed subexpressions are disjoint (orthogonal).

While there are literally hundreds of methods to introduce characteristic (b) of

orthogonality (disjointness) into a Boolean expression (see, e.g., Bennetts^[17, 18], Abrahams^[19], Dotson and Gobien^[20], and Rushdi^[21]) there is no way to induce characteristic (a) of statistical independence. The best that one can do is to observe statistical independence when it exists, and then take care to preserve it and take advantage of it. The aim of this note is to demonstrate that statistical independence is beneficial not only in the evaluation of the nominal value of top-event probability but also in computing the uncertainty in that probability.

2. Uncertainty Analysis of Fault Trees

Analysis of a fault tree results in a PRE expression for the indicator variable S^- of the top event in terms of the indicator variables X of the basic events. The expression $S^-(X)$ is transformed to an expression $Q(q)$ of the top-event probability in terms of the basic-event

probabilities q . To quantify uncertainty in Q , we use a doubly-stochastic model in which probabilities are treated as random variables [22-25]. We assume that the basic event probabilities q_i are random variables of mean values and variances given respectively by:

$$v_{i1} = E\{q_i\}, \quad 1 \leq i \leq n. \quad (1)$$

$$v_{i2} = E\{(q_i - v_{i1})^2\} = E\{q_i^2\} - v_{i1}^2, \quad 1 \leq i \leq n. \quad (2)$$

The mean value of the top-event probability is [22]

$$\mu_1 = E\{Q(\mathbf{q})\} = Q(\mathbf{q})|_{\mathbf{q}=\mathbf{v}_1}, \quad (3)$$

while its variance is given by [22]

$$\begin{aligned} \mu_2 = & \sum_{1 \leq i \leq n} C_i^2 v_{i2} \\ & + \sum_{1 \leq i < j \leq n} \sum C_{ij}^2 v_{i2} v_{j2} \\ & + \sum \sum \sum_{1 \leq i < j < k \leq n} C_{ijk}^2 v_{i2} v_{j2} v_{k2} + \dots \\ & + C_{12\dots n}^2 v_{12} v_{22} \dots v_{n2}. \end{aligned} \quad (4)$$

where

$$C_i = (\partial Q / \partial q_i)_{\mathbf{q}=\mathbf{v}_1} = (Q(\mathbf{q})|_{q_i=1} - Q(\mathbf{q})|_{q_i=0})_{\mathbf{q}=\mathbf{v}_1} \quad (5)$$

$$\begin{aligned} C_{ij} = & (\partial^2 Q / \partial q_i \partial q_j)_{\mathbf{q}=\mathbf{v}_1} \\ = & (Q(\mathbf{q})|_{q_i=q_j=1} - Q(\mathbf{q})|_{q_i=1, q_j=0} - \\ & Q(\mathbf{q})|_{q_i=0, q_j=1} + Q(\mathbf{q})|_{q_i=q_j=0})_{\mathbf{q}=\mathbf{v}_1} \end{aligned} \quad (6)$$

$$\begin{aligned} C_{ijk} = & (\partial^3 Q / \partial q_i \partial q_j \partial q_k)_{\mathbf{q}=\mathbf{v}_1} \\ = & (Q(\mathbf{q})|_{q_i=q_j=q_k=1} - Q(\mathbf{q})|_{q_i=q_j=1, q_k=0} - \\ & Q(\mathbf{q})|_{q_i=q_k=1, q_j=0} - Q(\mathbf{q})|_{q_j=q_k=1, q_i=0} + \\ & Q(\mathbf{q})|_{q_i=1, q_j=q_k=0} + Q(\mathbf{q})|_{q_j=1, q_i=q_k=0} + \\ & Q(\mathbf{q})|_{q_k=1, q_i=q_j=0} - \\ & Q(\mathbf{q})|_{q_i=q_j=q_k=0})_{\mathbf{q}=\mathbf{v}_1}. \end{aligned} \quad (7)$$

and so on. The number of terms in (4) is

$$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n - 1. \quad (8)$$

which is exponential and hence undesirably large.

3. Utilization of Statistical Independence

In many problems of fault tree analysis, the assumption of statistical independence among basic events is valid. If care is taken not to spoil the characteristics of statistical independence, one might arrive at an expression for the top-event probability of the form

$$Q = Q(Q_1(\mathbf{q}_1), Q_2(\mathbf{q}_2), \dots, Q_m(\mathbf{q}_m)), \quad (9)$$

where $Q_1, Q_2, \dots, Q_m$ are statistically independent probabilities, thanks to being functions of disjoint subsets $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$ of the set of basic-event probabilities $\mathbf{q}$. These subsets are assumed to contain $n_1, n_2, \dots, n_m$ basic events, respectively, where $n_1 + n_2 + \dots + n_m = n$.

Equation (9) can be used to obtain the mean value and uncertainty of Q in terms of those of $Q_1, Q_2, \dots, Q_m$, which in turn are readily obtained in terms of those of $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m$. A great saving is achieved in the number of coefficients involved, since it reduces from that in (8) to

$$\begin{aligned} & \sum_{j=1}^m (2^{n_j} - 1) + (2^m - 1) \\ & = \left(\sum_{j=1}^m 2^{n_j} \right) + 2^m - (m + 1) \end{aligned} \quad (10)$$

The saving is even more substantial since the evaluation of coefficients becomes much simpler as it involves significantly fewer variables. The savings gained are now demonstrated via the following illustrative example.

4. Example

Consider the fault tree in Fig. 1, with its top and basic events interpreted in Table 1. The indicator variable $\bar{S}$ of the top event for this fault tree can be expressed in terms of those of the basic events as

$$\bar{S} = \bar{X}_1 \vee \bar{X}_2 \vee \bar{X}_3 \vee \bar{X}_4 \vee \bar{X}_5 \vee \bar{X}_6 \bar{X}_7 \bar{X}_8. \quad (11)$$

Henceforth, the basic events are assumed to be statistically independent. Expression (11) for $\bar{S}$ can be rewritten in the PRE form

$$\bar{S} = \bar{X}_1 \vee X_1 \bar{X}_2 \vee X_1 X_2 \bar{X}_3 \vee X_1 X_2 X_3 \bar{X}_4 \vee X_1 X_2 X_3 X_4 \bar{X}_5 \vee X_1 X_2 X_3 X_4 X_5 \bar{X}_6 \bar{X}_7 \bar{X}_8, \quad (12)$$

which corresponds to the following expression for the top-event probability:

$$Q(\mathbf{q}) = q_1 + p_1 q_2 + p_1 p_2 q_3 + p_1 p_2 p_3 q_4 + p_1 p_2 p_3 p_4 q_5 + p_1 p_2 p_3 p_4 p_5 q_6 q_7 q_8. \quad (13)$$

Alternatively, the complement of (11) is written as

$$S = X_1 X_2 X_3 X_4 X_5 (X_6 \vee X_7 \vee X_8), \quad (14)$$

which results in the following unusually simple expression for the top-event probability:

$$Q(\mathbf{q}) = 1 - p_1 p_2 p_3 p_4 p_5 (1 - q_6 q_7 q_8). \quad (15)$$

$$\begin{aligned} v_{a2} = & c_1^2 v_{12} + c_2^2 v_{22} + c_3^2 v_{32} + c_4^2 v_{42} + c_5^2 v_{52} + c_{12}^2 v_{12} v_{22} + c_{13}^2 v_{12} v_{32} + c_{14}^2 v_{12} v_{42} + \\ & c_{15}^2 v_{12} v_{52} + c_{23}^2 v_{22} v_{32} + c_{24}^2 v_{22} v_{42} + c_{25}^2 v_{22} v_{52} + c_{34}^2 v_{32} v_{42} + c_{35}^2 v_{32} v_{52} + c_{45}^2 v_{42} v_{52} + \\ & c_{123}^2 v_{12} v_{22} v_{32} + c_{124}^2 v_{12} v_{22} v_{42} + c_{125}^2 v_{12} v_{22} v_{52} + c_{134}^2 v_{12} v_{32} v_{42} + c_{135}^2 v_{12} v_{32} v_{52} + \\ & c_{145}^2 v_{12} v_{42} v_{52} + c_{234}^2 v_{22} v_{32} v_{42} + c_{235}^2 v_{22} v_{32} v_{52} + c_{245}^2 v_{22} v_{42} v_{52} + c_{345}^2 v_{32} v_{42} v_{52} + \\ & c_{1234}^2 v_{12} v_{22} v_{32} v_{42} + c_{1235}^2 v_{12} v_{22} v_{32} v_{52} + c_{1245}^2 v_{12} v_{22} v_{42} v_{52} + c_{1345}^2 v_{12} v_{32} v_{42} v_{52} + \\ & c_{2345}^2 v_{22} v_{32} v_{42} v_{52} + c_{12345}^2 v_{12} v_{22} v_{32} v_{42} v_{52}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} c_1 = & -(1 - v_{21})(1 - v_{31})(1 - v_{41})(1 - v_{51}), c_2 = -(1 - v_{11})(1 - v_{31})(1 - v_{41})(1 - v_{51}), \\ c_3 = & -(1 - v_{11})(1 - v_{21})(1 - v_{41})(1 - v_{51}), c_4 = -(1 - v_{11})(1 - v_{21})(1 - v_{31})(1 - v_{51}), \\ c_5 = & -(1 - v_{11})(1 - v_{21})(1 - v_{31})(1 - v_{41}), c_{12} = (1 - v_{31})(1 - v_{41})(1 - v_{51}), \\ c_{13} = & (1 - v_{21})(1 - v_{41})(1 - v_{51}), c_{14} = (1 - v_{21})(1 - v_{31})(1 - v_{51}), \\ c_{15} = & (1 - v_{21})(1 - v_{31})(1 - v_{41}), c_{23} = (1 - v_{11})(1 - v_{41})(1 - v_{51}), \end{aligned}$$

Further analysis via (13) demands the calculation of $2^8 - 1 = 255$ cumbersome coefficients. Instead, we continue the analysis by rewriting (15) as

$$Q(\mathbf{q}) = 1 - r_a r_b. \quad (16)$$

where r_a, r_b , are statistically-independent quantities given by:

$$r_a = p_1 p_2 p_3 p_4 p_5 = (1 - q_1)(1 - q_2)(1 - q_3)(1 - q_4)(1 - q_5) \quad (17a)$$

$$r_b = 1 - q_6 q_7 q_8, \quad (17b)$$

Now, the mean value of the top-event probability is

$$\mu_1 = 1 - v_{a1} v_{b1}, \quad (18)$$

Where

$$v_{a1} = (1 - v_{11})(1 - v_{21})(1 - v_{31})(1 - v_{41})(1 - v_{51}), \quad (19a)$$

$$v_{b1} = 1 - v_{61} v_{71} v_{81}, \quad (19b)$$

The uncertainty in the top-event probability is obtained via the following equations, which involves the evaluation of $(3+31+7) = 41$ much simpler coefficients rather than the aforementioned coefficients.

$$\mu_2 = c_a^2 v_{a2} + c_b^2 v_{b2} + c_{ab}^2 v_{a2} v_{b2}. \quad (20)$$

Where $c_a = -v_{b1}$, $c_b = -v_{a1}$, $c_{ab} = -1$.

$$\begin{aligned}
c_{24} &= (1 - v_{11})(1 - v_{31})(1 - v_{51}), c_{25} = (1 - v_{11})(1 - v_{31})(1 - v_{41}), \\
c_{34} &= (1 - v_{11})(1 - v_{21})(1 - v_{51}), c_{35} = (1 - v_{11})(1 - v_{21})(1 - v_{41}), \\
c_{45} &= (1 - v_{11})(1 - v_{21})(1 - v_{31}), c_{123} = -(1 - v_{41})(1 - v_{51}), c_{124} = -(1 - v_{31})(1 - v_{51}), \\
c_{125} &= -(1 - v_{31})(1 - v_{41}), c_{134} = -(1 - v_{21})(1 - v_{51}), c_{135} = -(1 - v_{21})(1 - v_{41}), \\
c_{145} &= -(1 - v_{21})(1 - v_{31}), c_{234} = -(1 - v_{11})(1 - v_{51}), c_{235} = -(1 - v_{11})(1 - v_{41}), \\
c_{245} &= -(1 - v_{11})(1 - v_{31}), c_{345} = -(1 - v_{11})(1 - v_{21}), c_{1234} = (1 - v_{51}), c_{1235} = (1 - v_{41}), \\
c_{1245} &= (1 - v_{31}), c_{1345} = (1 - v_{21}), c_{2345} = (1 - v_{11}), c_{12345} = 1. \\
v_{b2} &= c_6^2 v_{62} + c_7^2 v_{72} + c_8^2 v_{82} + c_{67}^2 v_{62} v_{72} + c_{68}^2 v_{62} v_{82} + c_{78}^2 v_{72} v_{82} + c_{678}^2 v_{62} v_{72} v_{82}, \quad (22)
\end{aligned}$$

where $c_6 = -v_{71} v_{81}$, $c_7 = -v_{61} v_{81}$, $c_8 = -v_{61} v_{71}$, $c_{67} = -v_{81}$, $c_{68} = -v_{71}$, $c_{78} = -v_{61}$ and $c_{678} = -1$.

The above calculations are for non-identical basic-event probabilities and allow a detailed exploration of the effect of their variation on the nature of top-event probability. For the identical case of $v_{i1} =$

10^{-1} , $v_{i2} = 10^{-4}$ for $1 \leq i \leq 8$, we obtain the following numerical results:

$$\begin{aligned}
v_{a1} &= \mathbf{0.59049}, v_{b1} = \mathbf{0.999}, \mu_1 = \mathbf{0.41010049}, \\
v_{a2} &= \mathbf{0.0002152868}, \quad v_{b2} = \mathbf{0.0000000303}, \\
\mu_2 &= \mathbf{0.000214867}.
\end{aligned}$$

Table 1. Meaning of variables indicating top and basic events of the fault tree.

	Indicator Variable	Indicated Event
Top Event	$\bar{S}$	The adversary illegally gains access to an industrial secret of the competitor.
Basic Events	$\bar{X}_1$	The adversary breaks into the computer system of the competitor.
	$\bar{X}_2$	The adversary secretly steals a written document containing the secret.
	$\bar{X}_3$	The adversary openly obtains the secret through an armed robbery of the competitor's headquarters.
	$\bar{X}_4$	The adversary obtains the secret by bribing an employee of the competitor.
	$\bar{X}_5$	The adversary acquires the secret by blackmailing an insider in the competitor's company.
	$\bar{X}_6$	The adversary intercepts a communicated message containing the secret in concealed encrypted form.
	$\bar{X}_7$	The adversary detects presence of the secret and extracts its cipher-text despite its concealment via steganography.
	$\bar{X}_8$	The adversary succeeds in his cryptanalysis attack and converts the cipher-text into a plaintext.

Fig. 1. A typical fault-tree model for a computer security application.

5. Conclusions

Careful utilization of statistical independencies in a probability expression is shown to achieve a substantial reduction not only in the computational cost for the mean value of probability but also in that of its variance. This note demonstrated the above reduction in a typical doubly-stochastic fault-tree model of computer system security.

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استخدام الاستقلال الإحصائي للأحداث الأساسية في تفسير تحليل الريبة لأشجار الأخطاء

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المستخلص. نقدم مقارنة تفصيلية بين مخ الإنسان والماكينات الحاسبة، كما نسرد روايات توضيحية يعتمد تحليل أشجار الأخطاء المستخدمة في نمذجة أمن نظم المحساب على استحداث التعامدية في تعبير بولاني مع المحافظة على ما يكون موجودا في هذا التعبير من استقلالات إحصائية. تظهر هذه العجالة أن العناية باستغلال هذه الاستقلالات الإحصائية لا يقتصر نفعه على حساب القيمة المتوسطة لاحتمال الحدث الأوجي، بل يمتد هذا النفع إلى تكمية الريبة في هذا الاحتمال. نظهر من خلال مثال توضيحي أن هذه العناية تؤدي إلى تقليل هائل في عدد المعاملات التي يلزم حسابها فضلا عن تبسيط ملحوظ في الحسابات المطلوبة لكل منها.

الكلمات المفتاحية: تحليل أشجار الأخطاء، أمن نظم المحساب، الاستقلال الإحصائي، تكمية الريبة، النمذجة ثنائية العشوائية.