On Neighbourhood and Partial Sums of a Certain Subclasses of Analytic Functions with Respect to β -Symmetric Points

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Abstract

In the present investigation, we introduce a new class $k\text{-}\tilde{UC}^{(\beta)}(\lambda; \gamma)$ of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine coefficients estimates, neighbourhoods and partial sums for functions $f(z)$ belonging to this class.

Keywords: Analytic function; Uniformly starlike function; Quasiconvex functions; Coefficient estimate; Neighbourhood; Partial sums; β -Symmetric points; 2010 Mathematics subject classification; 30C45

Introduction

Let A denote the family of functions f of the form [1]:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

that are analytic in the open unit disc $U = \{z: |z| < 1\}$.

For $f \in A$ given by (1.1) and $g(z)$ given [2] by:

$$g(z) = z + \sum_{n=2}^{\infty} d_n z^n \quad (1.2)$$

their convolution (or Hadamard product) denoted by $(f * g)$ is defined [3] as:

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n d_n z^n = (g * f)(z) \quad (z \in U) \quad (1.3)$$

Let S^* ; K ; C and QC be the usual subclasses [4-7] of A consisting of starlike, convex, close-to-convex and quasi-convex in U , respectively. Thus, by definition, we have [8-15],

$$S^* = \left\{ f : f \in A \text{ and } \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 (z \in U) \right\}$$

$$K = \left\{ f : f \in A \text{ and } \operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0 (z \in U) \right\}$$

$$C = \left\{ f : f \in A \exists g \in S^* \text{ such that } \operatorname{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0 (z \in U) \right\}$$

And

$$QC = \left\{ f : f \in A \exists g \in K \text{ such that } \operatorname{Re} \left\{ \frac{zf'(z)}{g'(z)} \right\} > 0 (z \in U) \right\}$$

Let S_s^* be the subclass of S consisting of functions of the [16-28] form eqn. (1.1) satisfying:

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0 (z \in U) \quad (1.4)$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [19] (see also Robertson [17]; Stankiewicz [25]; Wu [28] and Owa [14]. In El-Ashwah and Thomas [9], introduced and studied two other classes namely the class S_s^* consisting of functions starlike with respect to conjugate points

and S_s^* consisting of functions [18] starlike with respect to symmetric conjugate points. Recently several subclasses of analytic functions with respect to k -symmetric points were introduced and studied by various authors [5,16].

A function $f \in A$ is said to be in $k\text{-US}(\gamma)$, the class of k -uniformly starlike functions of order γ ; $0 \leq \gamma < 1$; if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (k \geq 0) \quad (1.5)$$

and a function $f \in A$ is said to be $k\text{-UC}(\gamma)$, the class [19] of k -uniformly convex functions of order γ ; $0 \leq \gamma < 1$; if it satisfies the condition:

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| \quad (k \geq 0) \quad (1.6)$$

Uniformly starlike and uniformly convex functions were [20] first introduced by Goodman [11] and then studied by various authors. It follows from (1.5) and (1.6) that:

$$f(z) \in k\text{-UC}(\gamma) \Leftrightarrow zf'(z) \in k\text{-US}(\gamma)$$

Let $S_s^{(\beta)}(\gamma)$ denote the class of functions in A [21] satisfying the following inequality:

$$\operatorname{Re} \frac{zf'(z)}{f_{(\beta)}(z)} > \gamma (z \in U)$$

where $0 \leq \gamma < 1$; $\beta \geq 2$ is a fixed positive integer and $f_{(\beta)}(z)$ is [22] defined by the following equality:

$$f_{(\beta)}(z) = \frac{1}{\beta} \sum_{v=2}^{\beta-1} \epsilon^{-v} f(\epsilon^v z) \quad (\epsilon = \exp(2\pi i / \beta), \quad z \in U) \quad (1.7)$$

And a function $f(z) \in A$ is in the class $C_s^{(\beta)}(\gamma)$ if and only if $zf'(z) \in S_s^{(\beta)}(\gamma)$: The classes $S_s^{(\beta)}(\gamma)$ of starlike functions with respect to

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β -symmetric points of order [23] and $C_s^{(\beta)}(\gamma)$ of convex functions with respect to β - symmetric points of order were considered recently by Singh [4] and Wang et al. [26] respectively. We now introduce and investigate the following subclasses of A with respect [24-28] to β -symmetric points and obtain some interesting results.

Definition 1.1

We say that $f \in A$ belongs to the class $k\text{-UC}^{(\beta)}(\lambda; \gamma)$ if for all $z \in U$:

$$\operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - \gamma \right\} \geq \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right|$$

for $0 \leq \gamma < 1$, $0 \leq \lambda \leq 1$ and $k \geq 0$.

Definition 1.2

We say that $f \in A$ belongs to the class $k\text{-UQC}^{(\beta)}(\lambda; \gamma)$ if for all $z \in U$.

$$\operatorname{Re} \left\{ \frac{zf''(z) + (2\lambda+1)z^2 f'''(z) + zf'(z)}{\lambda z^2 f''_\beta(z) + zf'_\beta(z)} - \gamma \right\} \geq \left| \frac{\lambda z^2 f''(z) + (2\lambda+1)z^2 f'''(z) + zf'(z)}{\lambda z^2 f''_\beta(z) + zf'_\beta(z)} - 1 \right|$$

for $0 \leq \gamma < 1$; $0 \leq \lambda \leq 1$ and $k \geq 0$:

Denote by T the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0, n \in \mathbb{N}) \quad (1.9)$$

For convenience, we write $k\text{-UC}^{(\beta)}(\lambda; \gamma) \cap T$ simple as $k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$; and $k\text{-UQC}^{(\beta)}(\lambda; \gamma) \cap T$ simple as $k\text{-}\tilde{\text{UQC}}^{(\beta)}(\lambda; \gamma)$.

Remark 1.1

(i) For $k=0$; the classes $k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$ and $k\text{-}\tilde{\text{UQC}}^{(\beta)}(\lambda; \gamma)$ reduce to the classes $C^{(\beta)}(\lambda; \gamma)$ and $QC^{(\beta)}(\lambda; \gamma)$ respectively introduced by Wang et al. [27].

The aim of the present paper to study the coefficient bounds, partial sums and certain neighbourhood results of the classes $k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$ and $k\text{-}\tilde{\text{UQC}}^{(\beta)}(\lambda; \gamma)$.

(ii) Coefficient bounds of the function classes $k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$ if and only if:

$$\sum_{n=2}^{\infty} [1 + \lambda(n-1)] [n(k+1) - b_n(k+\gamma)] a_n \leq 1 - \gamma \quad (2.1)$$

Where,

$$b_n = \frac{1}{\beta} \sum_{\nu=0}^{\beta-1} \epsilon^{(n-1)\nu} = \begin{cases} 1, & n = t\beta + 1 \\ 0, & n \neq t\beta + 1 \end{cases} \quad (2.2)$$

The result is sharp for the function $f(z)$ given by:

$$f(z) = z - \frac{1-\gamma}{[1 + \lambda(n-1)][n(k+1) - b_n(k+\gamma)]} z^n$$

Proof:

Assume that the inequality (2.1) holds true. Then we have:

$$\begin{aligned} & k \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right\} \\ & \leq (1+k) \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right| \leq \\ & \frac{(1+k) \sum_{n=2}^{\infty} [1 + \lambda(n-1)] [n - b_n] a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + \lambda(n-1)] a_n b_n z^{n-1}} \leq 1 - \gamma \end{aligned}$$

Conversely, if $f(z) \in k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$ and z is real, then

$$\frac{1 - \sum_{n=2}^{\infty} [1 + \lambda(n-1)] [n - b_n] a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + \lambda(n-1)] a_n b_n z^{n-1}} - \gamma \geq k \left| \frac{\sum_{n=2}^{\infty} [1 + \lambda(n-1)] [n - b_n] a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [1 + \lambda(n-1)] a_n b_n z^{n-1}} - 1 \right|$$

Letting $z \rightarrow 1$ - along the real axis, we obtain:

$$\sum_{n=2}^{\infty} [1 + \lambda(n-1)] [n(k+1)] a_n \leq 1 - \gamma$$

This completes the proof of Theorem 2.1.

Corollary 2.1:

If $f(z) \in k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$; then,

$$a_n \leq \frac{1-\gamma}{[1 + \lambda(n-1)][n(k+1) - b_n(k+\gamma)]},$$

where b_n is defined by (2.2) and $0 \leq \gamma < 1$, $0 \leq \lambda \leq 1$; $\beta \geq 2$; and $k \geq 0$.

Theorem 2.2:

A function $f(z) \in T$ is in the class $k\text{-}\tilde{\text{UQC}}^{(\beta)}(\lambda; \gamma)$ if and only if

$$\sum_{n=2}^{\infty} n [1 + \lambda(n-1)] [n(k+1) - b_n(k+\gamma)] a_n \leq 1 - \gamma \quad (2.3)$$

where b_n is defined by (2.2), and $0 \leq \gamma < 1$, $0 \leq \lambda < 1$; $\beta \geq 2$, and $k \geq 0$: The result is sharp for the function $f(z)$ given by:

$$f(z) = z - \frac{1-\gamma}{n[1 + \lambda(n-1)][n(k+1) - b_n(k+\gamma)]} z^n$$

Proof:

By Denition 1.2, the proof is similar to that of Theorem 2.1.

Neighbourhood of the Function Class $k\text{-}\tilde{\text{UQC}}^{(\beta)}(\lambda; \gamma)$

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [10], Ruscheweyh [18], Altintas et al. [1,2] and others including Srivastava et al. [22,23], Orhan [13], Deniz et al. [7], Catas [3,6], we dene the neighborhood of a function $f \in T$ by:

Definition 3.1

Let $0 < 1$; $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$, $\alpha \geq 0$ and b_n is dened by (2.2). We define the α -neighbourhood of a function $f \in T$ and denote by $N_\alpha(f)$ consisting of all functions $g(z) = z - \sum_{n=2}^{\infty} d_n z^n \in S$ ($d_n \geq 0, n \in \mathbb{N}$) satisfying:

$$\sum_{n=2}^{\infty} \frac{[1 + \lambda(n-1)][n(k+1) - b_n(k+\gamma)]}{1-\gamma} |a_n - d_n| \leq \alpha$$

Theorem 3.1

Let $f \in k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$ and for all real θ we have $\gamma(e^{i\theta} - 1) - 2e^{i\theta} \neq 0$: For and complex number ϵ with $|\epsilon| < \alpha$ ($\alpha \geq 0$); if it satisfies the following condition:

$$\frac{f(z) + \epsilon z}{1 + \epsilon} \in k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda, \gamma) \quad (3.1)$$

then $N_\alpha(f) \subset k\text{-}\tilde{\text{UC}}^{(\beta)}(\lambda; \gamma)$

Proof:

It is easy to see that

$$\operatorname{Re} w > k|w-1| + \gamma \Leftrightarrow \operatorname{Re}\{w(1+ke^{i\theta}) - ke^{i\theta}\} > \gamma; \quad (-\pi < \theta < \pi; 0 \leq \gamma < 1; k \geq 0);$$

By definition 1.1, $f \in k\text{-}\tilde{UC}^{(\beta)}(\gamma, \lambda)$ if and only if

$$-\operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right\} \geq k \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} - 1 \right|$$

the above inequality may be rewritten as follows:

$$\operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} (1 + ke^{i\theta}) - ke^{i\theta} \right\} > \gamma$$

or equivalently,

$$\operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z) (1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta + \lambda zf'_\beta(z)]}{(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z)} \right\} > \gamma$$

Now, by setting

$$F(z) = [zf'(z) + \gamma z^2 f''(z)](1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta(z) + \lambda zf'_\beta(z) + \lambda zf'_\beta(z)] \quad (3.4)$$

And

$$E(z) = (1-\lambda)f_\beta(z) + \lambda zf'_\beta(z) \quad (3.5)$$

the inequality (3.3) becomes equivalent to

$$|F(z) + (1-\gamma)E(z)| > |F(z) - (1+\gamma)E(z)| \quad \text{for } 0 \leq \gamma < 1$$

Then

$$\left| \frac{[zf'(z) + \lambda z^2 f''(z) (1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta + \lambda zf'_\beta(z)]]}{[zf'(z) + \lambda z^2 f''(z) (1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta + \lambda zf'_\beta(z)]]} \right| < 1$$

($-\pi < \theta < \pi$); for any complex number s with $|s|=1$, we have

$$\left| \frac{[zf'(z) + \lambda z^2 f''(z) (1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta + \lambda zf'_\beta(z)]]}{[zf'(z) + \lambda z^2 f''(z) (1 + ke^{i\theta}) - ke^{i\theta} [(1-\lambda)f_\beta + \lambda zf'_\beta(z)]]} \right| \neq s$$

which is equivalent to

$$\frac{(f * h)(z)}{z} \neq 0 \quad (z \in U) \quad (3.6)$$

Where

$$h(z) = z - \sum_{n=2}^{\infty} c_n z^n \quad (3.7)$$

$$C_n := \frac{[1 + \lambda(n-1)][n - b_n(1 + ke^{i\theta} - ske^{i\theta}) - s(n + b_n) - b_n \gamma(1-s)]}{\gamma(s-1) - 2s}$$

It follows from (3.7) that

$$|C_n| = \frac{[1 + \lambda(n-1)][n - b_n(1 + ke^{i\theta} - ske^{i\theta}) - s(n + b_n) - b_n \gamma(1-s)]}{\gamma(s-1) - 2s} \leq \frac{[1 + \lambda(n-1)][n(1+k) - (k-\gamma)b_n]}{1-\lambda}$$

If $f(z) \in k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$ given by (1.9), satisfies the inclusion property (3.1), then (3.6) yields

$$\left| \frac{f(z) * h(z)}{z} \right| \geq \alpha \quad (z \in \mathbb{U}) \quad (3.8)$$

Now, if we suppose that

$$g(z) = z - \sum_{n=2}^{\infty} d_n z^n \in N_\alpha(f) \quad (3.9)$$

We easily see that

$$\begin{aligned} \left| \frac{(g-f)(z) * h(z)}{z} \right| &= \left| \sum_{n=2}^{\infty} (d_n - a_n) c_n z^n \right| \\ &\leq \sum_{n=2}^{\infty} |d_n - a_n| |c_n| |z|^n \\ &\leq \sum_{n=2}^{\infty} \frac{[1 + \lambda(n-1)][n(1+k) - b_n(k-\gamma)]}{1-\gamma} |a_n - d_n| \leq \alpha \end{aligned}$$

Then,

$$\begin{aligned} \left| \frac{(g)(z) * h(z)}{z} \right| &= \left| \frac{(f + (g-f))(z) * h(z)}{z} \right| \\ &\geq \left| \frac{f(z) * h(z)}{z} \right| - \left| \frac{(g-f)(z) * h(z)}{z} \right| > 0 \end{aligned}$$

Thus, for any complex number s such that $|s|=1$, we have

$$\frac{(g)(z) * h(z)}{z} \neq 0 \quad (z \in \mathbb{U})$$

which implies that $g(z) \in k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$ so $N_\alpha(f) \subset k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$.

Partial Sums of the Function Class $k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$

In this section, applying methods used by Silverman [20] and Silvia [21], We investigate the ratio of the real part of $f(z)$ which dened by (1.9) to its sequence of partial sums dened by $f_1(z)=z$ and

$$f_m(z) = z - \sum_{n=2}^m a_n z^n \quad (m \in \mathbb{N}).$$

Theorem 4.1:

If $f(z) \in k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$ then

$$\operatorname{Re} \left\{ \frac{f(z)}{f_m(z)} \right\} \geq 1 - \frac{1}{1 + \delta_{m+1}} \quad (z \in U, m \in \mathbb{N}) \quad (4.1)$$

And

$$\operatorname{Re} \left\{ \frac{f_m(z)}{f(z)} \right\} \geq \frac{\delta_{m+1}}{1 + \delta_{m+1}} \quad (z \in U, m \in \mathbb{N}) \quad (4.2)$$

Where

$$\delta_n = \frac{[1 + \lambda(n-1)][n(1+k)] - (k+\gamma)b_n}{1-\gamma}$$

The bounds in (4.1) and (4.2) are the best possible for each $m \geq 2$. Proof. We employ the same technique used by Silverman [20]: The function $f(z) \in k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$, If and only if $\sum_{n=2}^{\infty} \delta_n a_n \leq 1$. It is not difficult to verify that $\delta_{n+1} > \delta_n > 1$ [29].

$$\sum_{n=2}^{\infty} a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} \delta_n a_n \leq 1 \quad (4.4)$$

We may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \delta_{m+1} \left\{ \frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{\delta_{m+1}} \right) \right\} \\ &= \left\{ \frac{1 - \sum_{n=2}^m a_n z^{n-1} - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}{1 - \sum_{n=2}^m a_n z^{n-1}} \right\} \end{aligned} \quad (4.5)$$

Then, from (4.5) we can obtain,

$$w(z) = \frac{-\delta_{m+1} \sum_{n=2}^m a_n z^{n-1}}{2 - 2 \sum_{n=2}^m a_n z^{n-1} - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n z^{n-1}}$$

And

$$|w(z)| \leq \frac{-\delta_{m+1} \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^m a_n z^{n-1} - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n}$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^m a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 1$$

which is true by (4.4). This readily yields the assertion (4.1) of Theorem 4.1. To see that:

$$f(z) = z - \frac{z^{m+1}}{\delta_{m+1}} \quad (4.6)$$

gives sharp results, we observe that for $z = \gamma e^{im/m}$

$$\frac{f(z)}{f_m(z)} = 1 - \frac{z^m}{\delta_{m+1}} \rightarrow 1 - \frac{1}{\delta_{m+1}} \quad \text{as } z \rightarrow 1^-$$

Similarly, we take

$$\frac{1+w(z)}{1-w(z)} = (1+\delta_{m+1}) \left\{ \frac{f_m(z)}{f(z)} - \frac{\delta_{m+1}}{1+\delta_{m+1}} \right\},$$

Where

$$|w(z)| \leq \frac{\delta_{m+1} \sum_{n=m+1}^{\infty} a_n}{2 - 2 \sum_{n=2}^m a_n - \delta_{m+1} \sum_{n=m+1}^{\infty} a_n} \leq 1$$

Now $|w(z)| \leq 1$ if and only if,

$$\sum_{n=2}^m a_n + \delta_{m+1} \sum_{n=m+1}^{\infty} a_n \leq 1$$

which is true by (4.4). This immediately leads to the assertion (4.2) of Theorem 4.1. The estimate in (4.2) is sharp with the extremal function $f(z)$ given by (4.6). This completes the proof of Theorem 4.1.

Theorem 4.2:

If $f(z) \in k\text{-}\tilde{UC}^{(\beta)}(\lambda, \gamma)$ then,

$$\operatorname{Re} \left\{ \frac{f'(z)}{f'(z)} \right\} \geq 1 - \frac{m+1}{\delta_{m+1}}, \quad (z \in U, m \in \mathbb{N}) \quad (4.7)$$

$$\operatorname{Re} \left\{ \frac{f'_m(z)}{f'(z)} \right\} \geq 1 - \frac{\delta_{m+1}}{1+m+\delta_{m+1}}, \quad (z \in U, m \in \mathbb{N}) \quad (4.8)$$

The estimates in (4.7) and (4.8) are sharp with the extremal function given by (4.6).

Proof: The proof is similar to that of Theorem 4.1.

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