Exact 2-Distance b-Coloring and Exact 2-Distance b-Continuity of Helm Graph H_n

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ABSTRACT

An exact 2-distance coloring of a graph G is a coloring of vertices of G such that any two vertices which are at distance exactly 2 receive distinct colors. An exact 2-distance chromatic number $\chi_{e2}(G)$ of G is the minimum k for which G admits an exact 2-distance coloring with k colors. A b -coloring of G by k colors is a proper k -vertex coloring such that in each color class, there exists a vertex called a color dominating vertex which has a neighbor in every other color class. A vertex that has a 2-neighbor in all other color classes is called an exact 2-distance color dominating vertex (or an e2-cdv). Exact 2-distance b -coloring (or an e2b-coloring) of G is an exact 2-distance coloring such that each color class contains an e2-cdv. An exact 2-distance b -chromatic number (or an e2b-number) $\chi_{e2b}(G)$ of G is the largest integer k such that G has an e2b-coloring with k colors. If for each integer k , $\chi_{e2}(G) \leq k \leq \chi_{e2b}(G)$, G has an e2b-coloring by k colors, then G is said to be an exact 2-distance b -continuous graph. In this paper, the e2b-number $\chi_{e2b}(H_n)$ of the helm graph H_n is obtained and e2b-continuity of H_n is discussed.

Keywords: Exact 2-distance b -coloring, exact 2-distance b -chromatic number, exact 2-distance color dominating vertex, exact 2-distance b -continuity.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For those terminologies not defined in this paper, the reader may refer to [1]. In this paper, exact 2-distance b -coloring of a helm graph is studied. As the name indicates, this coloring is a combination of two types of coloring, viz., exact 2-distance coloring and b -coloring. An exact 2-distance coloring is a distance coloring which need not be proper and b -coloring is a proper coloring. Distance coloring was introduced by F. Kramer and H. Kramer [4], [5] in 1969. As the name suggests it is based on distance between two vertices. The definition of distance coloring requires the definition of 2-neighbors. If u and v are vertices in G such that $d(u, v) = 2$, then u is said to be a 2-neighbor of v and vice versa. The set of all 2 neighbors of u is denoted by $N_2(u)$ and is called open 2-neighborhood of u and $N_2[u] = N_2(u) \cup \{u\}$ is called the closed 2-neighborhood of u . The d_2 -degree of a vertex u denotes the number of 2-neighbors of u and $\Delta_2(G)$ denotes the maximum d_2 -degree.

A 2-distance coloring of G is an assignment of colors to vertices such that every pair of vertices u and v such that $d(u, v) \leq 2$ receive distinct colors. The 2-distance chromatic number of G , denoted $\chi_2(G)$ is the smallest integer k for which G admits a 2-distance k -coloring. An exact 2-distance coloring (or $e2$ -coloring) is a distance coloring such that, every pair of vertices u and v such that $d(u, v) = 2$ receive distinct colors. Clearly, it need not be a proper coloring and every 2-distance coloring is an $e2$ -coloring. The exact 2-distance chromatic number $\chi_{e2}(G)$ (or $e2$ -number) of G , is the smallest integer k for which G admits an $e2$ -coloring with k colors. The other coloring of interest is b -coloring. The concept of b -coloring was introduced by Irving and Manlove [2] in 1991. A b -coloring of G by k -colors is a proper k -coloring such that in each color class, there exists a vertex called a color dominating vertex, which has a neighbor in every other color class. The b -chromatic number $\chi_b(G)$ of G is the largest integer k such that G has a b -coloring by k colors. Another interesting idea in b -coloring is b -continuity. Since, χ -coloring is a b -coloring and a χ_b -coloring tries to find the maximum number of colors a b -coloring can have, there can be intermediate values between $\chi(G)$ and $\chi_b(G)$. If for each intermediate values, there exists a b -coloring, then the graph G is said to be b -continuous.

We [7] in 2019, made a study on combining these two coloring which resulted as a new coloring called exact 2-distance b -coloring. In that paper, the $e2b$ -number of path, cycle and wheel were obtained. In this paper, the $e2b$ -number of a helm graph H_n is studied, where a helm H_n is obtained from an $(n - 1)$ -cycle such that all the $n - 1$ vertices of the cycle are adjacent to a central vertex say u , and each vertex on the cycle is adjacent to one pendant vertex. Analogous to b -continuity, we defined exact 2-distance b -continuity ($e2b$ -continuity) of graphs. When a new concept is introduced, there may be the case that, all graphs may fall into that category. Hence, it is important to prove that, the definition is not trivial. In this paper, we have proved that H_4 is not $e2b$ -continuous and $H_n, n \geq 5$ is $e2b$ -continuous.

2. Definitions and some prior results

In this section, the definition of the new coloring and the corresponding parameter are stated.

Definition 2.1 [7]. A vertex which has a 2-neighbor in all other color classes is called an exact 2-distance color dominating vertex (or an $e2-cdv$). If u is an $e2-cdv$ of color i , then it is called an i - $e2-cdv$.

Definition 2.2 [7]. An exact 2-distance b -coloring (or an $e2b$ -coloring) of a graph G is an exact 2-distance coloring ($e2$ -coloring) of G such that each color class contains an $e2-cdv$.

An exact 2-distance b -chromatic number (or an $e2b$ -number) $\chi_{e2b}(G)$ of G is the largest integer k such that G has an $e2b$ -coloring with k colors.

$$i.e., \chi_{e2b}(G) = \max \{k: G \text{ has an } e2b\text{-coloring with } k \text{ colors}\}.$$

Definition 2.3 [7]. The term m_2 -degree denoted by $m_2(G)$ of G is defined as the largest integer k such that G has at least k vertices having at least $(k - 1)$ 2-neighbors.

i. e., $m_2(G) = \max \{k: G \text{ has at least } k \text{ vertices whose } d_2\text{-degree is at least } k - 1 \}$.

Observation 2.4 [7].

- (i) $e2b$ -coloring exists for all graphs;
- (ii) For a graph G , $\chi_{e2}(G) \leq \chi_{e2b}(G) \leq m_2(G) \leq \Delta_2(G) + 1$;

Proposition 2.5 [7]. Any minimum $e2$ -coloring of a graph is an $e2b$ -coloring.

Since, the $e2b$ -number is the maximum number of colors an $e2b$ -coloring can have, to find the maximum, minimum number is required. From proposition 2.5, χ_{e2} is the minimum number required to have an $e2b$ -coloring. Hence, the $e2$ -number of H_n is given in the next proposition.

Proposition 2.6. For $n \geq 4$

$$\chi_{e2}(H_n) = \begin{cases} 2, & \text{if } n = 4 \\ 3, & \text{if } n = 5 \\ 4, & \text{if } n = 6, 7 \\ \lfloor \frac{n}{2} \rfloor, & \text{if } n \geq 8 \end{cases}$$

3. Exact 2-distance b -coloring of H_n

In this section, the bound $\chi_{e2b}(H_n)$ is obtained.

Theorem 3.1. For $n \geq 4$,

$$\chi_{e2b}(H_n) = \begin{cases} 4, & \text{if } n = 4 \\ n - 2, & \text{if } n \geq 5 \end{cases}$$

Proof: The d_2 -degree of the vertices of H_n are given by

$$d_2(u) = n - 1, d_2(v_i) = n - 2 \text{ and } d_2(w_i) = 3 \quad \text{----- (1)}$$

Claim 1: $m_2(H_n) = \begin{cases} 4, & \text{if } n = 4 \\ n - 1, & \text{if } n \geq 5 \end{cases}$

Let $n = 4$. Then from (1), u and w_i constitute a set of n vertices of d_2 -degree $= n - 1 = 3$. Hence, $m_2(H_n) = n = 4$.

If $n = 5$, then both v_i and w_i are of d_2 -degree $n - 2$. Hence, $m_2(H_n) = n - 1$.

Suppose $n \geq 6$. Then w_i does not contribute to $m_2(H_n)$. In this case, u and v_i constitute a set of n vertices of d_2 -degree at least $n - 2$. Hence, $m_2(H_n) = n - 1$. Thus, summarizing the m_2 -values, the claim is proved.

Throughout the proof, $c: V \rightarrow \{1, 2, \dots, m\}$ is a coloring of H_n where $m \geq \chi_{e2}(H_n)$ and $c(u) = 1$.

Suppose $n = 4$. Then from claim 1, $m_2(H_n) = 4$. Let $c(v_i) = c(w_i) = i, 2 \leq i \leq 4$. Then each w_i is an i - $e2$ -cdv (Refer figure 1). Hence, $\chi_{e2b}(H_4) = 4$.

Now let $n \geq 5$. The proof is done in 2 stages.

- (a) Proving H_n cannot have an $e2b$ -coloring with $n - 1$ colors;
- (b) Proving H_n has an $e2b$ -coloring with $n - 2$ colors.

Let D denote the set of all $e2$ -color dominating vertices of H_n .

Claim 2 : For $n \geq 5$, $\chi_{e2b}(H_n) \leq n - 2$. i.e, H_n cannot have an $e2b$ -coloring with $n - 1$ colors

Suppose c is an $e2b$ -coloring of H_n with $n - 1$ colors. We discuss 2 cases.

Case (i): $n = 5$

Then c is an $e2b$ -coloring of H_n with 4 colors and $d_2(u) = 4$ & $d_2(v_i) = d_2(w_i) = 3$. Hence, each vertex of V can be an $e2$ -cdv. Then if $v \in D, v \neq u$, then $N_2(v)$ receives distinct colors. Since $c(u) = 1, c(w_i) \neq 1$ for any i . Hence, D cannot be a set of w_i 's.

Sub claim 1: If $w_i \in D$, then $w_{i+2} \in D$ and $c(w_i) = c(w_{i+2})$

Since $w_i \in D, N_2[w_i]$ receives 4 distinct colors. Hence, as $N_2(w_{i+2}) = N_2(w_i) = \{u, v_{i-1}, v_{i+1}\}, N_2(w_{i+2})$ receives 3 distinct colors. Since c is an $e2$ -coloring, $c(w_{i+2}) \notin C(N_2(w_{i+2}))$ and as c is a 4-coloring, $c(w_{i+2}) \in \{1, 2, 3, 4\} - C(N_2(w_{i+2})) = \{1, 2, 3, 4\} - C(N_2(w_i)) = \{c(w_i)\}$. Hence, $c(w_{i+2}) = c(w_i)$ and $N_2[w_{i+2}]$ receives 4 distinct colors. This gives $w_{i+2} \in D$.

Sub claim 2: If $v_i \in D$ of color $r \neq 1$, then the following hold.

- (i) If $v \neq v_i$ and $N_2[v_i] = N_2[v]$, then $v \in D$;
- (ii) $c(v_{i+2}) = 1$ and v_{i+2} is a 1- $e2$ -cdv;
- (iii) w_{i-1} and w_{i+1} cannot be $e2$ -color dominating vertices;
- (iv) If $c(v_j) \neq 1$ and $j \neq i$, then $v_j \notin D$.

Proof of (i): Since $v_i \in D, C(N_2[v_i]) = \{1, 2, 3, 4\}$. If $N_2[v_i] = N_2[v]$, then $C(N_2[v]) = \{1, 2, 3, 4\}$. Hence, v is an $e2$ -cdv.

Proof of (ii): Since $v_i \in D, v_i$ has a 2-neighbor in each color class and hence in S_1 .

Since $u \in S_1$ is not a 2-neighbor of v_i and $w_j \notin S_1$ for any $j, c(v_j) = 1$ for some j . Since, $v_{i+2} = v_{i-2}$ is the only 2-neighbor of v_i among v_j 's, $c(v_{i+2}) = 1$. As $N_2[v_{i+2}] = \{v_i, v_{i+2}, w_{i-1}, w_{i+1}\} = N_2[v_i]$, by (i), $v_{i+2} \in D$.

Proof of (iii): Now $N_2(w_{i-1}) = N_2(w_{i+1}) = \{u, v_i, v_{i+2}\}$ and $c(u) = c(v_{i+2}) = 1$ implies that the 2-neighbors of both w_{i-1} and w_{i+1} are not receiving distinct colors. Hence, they cannot be $e2$ -color dominating vertices.

Proof of (iv): Suppose $v_j \in D$ for $j \neq i$ and $c(v_j) \neq 1$. Since $j \neq i, v_j \neq v_i$ and as $c(v_j) \neq 1$, from (ii), $v_j \neq v_{i+2}$. Therefore, either $v_j = v_{i-1}$ or $v_j = v_{i+1}$. Without loss of generality, let $v_j = v_{i-1}$. Then by (ii), $v_{j+2} = v_{i+1}$ is a 1- $e2$ -cdv. From (iii), $w_{i-2} = w_{i+2}$ and w_i cannot be $e2$ -color dominating vertices. Hence, none of the w_i is an $e2$ -cdv and hence they cannot be members of D . Since v_i & v_{i-1} are the only two $e2$ color dominating vertices of color other than 1, D contains $e2$ color dominating vertices for at most 3 colors, a contradiction to c is a 4- $e2b$ -coloring. Thus, (iv) holds and hence the proof of sub claim 1 is complete.

From sub claim 1, it is clear than D contains only even number of w_i 's. If D does not contain any w_i , then D must contain v_i 's for at least 3 distinct colors, and hence contain at least three v_i 's, a contradiction to (iv) of sub claim 2. Suppose D contains two w_i 's. Then from sub claim 1, they are of same color. Then D must contain v_i 's for at least 2 distinct colors, and hence

contain at least two v_i 's, again a contradiction to (iv) of sub claim 2. Suppose, four w_i 's. Then from sub claim 1, they are of 2 distinct colors. Then D must contain at least one v_i for a color other than 1. If $v_i \in D$, then by (ii) of sub claim 2, $w_{i-1}, w_{i+1} \notin D$, a contradiction to the choice of D . Hence, H_n cannot have an $e2b$ -coloring with $n - 1 = 4$ colors.

Case (ii): $n \geq 6$

Then $m_2(H_n) = \chi_{e2b}(H_n) = n - 1 \geq 5$ and as $d_2(w_i) = 3$, w_i cannot be an $e2$ -cdv, for any i . Since, c is an $(n - 1)$ -coloring and $c(u) = 1$, at least $(n - 2)$ v_i 's are $e2$ -color dominating vertices of distinct colors other than 1. Hence, at most one v_i can be of color 1. As an r - $e2$ -cdv, $r \neq 1$ requires a 2-neighbor of color 1, as in case (i), at least a v_j must be of color 1. Since $(n - 2)$ v_i 's are of distinct colors other than 1, only one v_j can be of color 1. Without loss of generality, let $c(v_j) = 1$. Then v_{j-1} and v_{j+1} cannot have a 2-neighbor of color 1, a contradiction.

Hence from subcases (i) and (ii), H_n cannot have an $e2b$ -coloring with $n - 1$ colors and hence, claim 2 is completed.

Hence, the first stage (a) of the proof that, proving H_n cannot have an $e2b$ -coloring with $n - 1$ colors is completed.

Now let us prove (b) that H_n has an $e2b$ -coloring with $n - 2$ colors.

Case (i): $n = 5, 6$

From proposition 2.6, $\chi_{e2}(H_n) = n - 2$. Then by claim 2, $n - 2 \leq \chi_{e2b}(H_n) \leq n - 2$ and the equality holds.

Case (ii): $n = 7$

Then by the same propositions of case (i), $\chi_{e2}(H_n) = 4$ and by claim 2, $\chi_{e2b}(H_n) \leq n - 2 = 5$, $4 \leq \chi_{e2b}(H_n) \leq 5$. From figure 2, H_7 has an $e2b$ -coloring with 5 colors and hence, $\chi_{e2b}(H_7) = 5$.

Figure 1: A 4- $e2b$ -coloring of H_4

Figure 2: A 5- $e2b$ -coloring of H_7

Figure 3: An $(n - 2)$ -coloring of H_n

Case (iii): $n \geq 8$

Let c be defined with $(n - 2)$ -colors as in (2).

$$\left. \begin{aligned} c(u) &= 1 \\ c(v_i) &= c(w_i) = i, \quad 2 \leq i \leq n - 2 \\ c(v_1) &= c(v_{n-1}) = 1 \\ c(w_1) &= c(w_{n-1}) = 3 \end{aligned} \right\} \text{----- (2)}$$

(Refer figure 4). First, let us verify this is an $e2$ -coloring. On the $(n - 1)$ -cycle, other than v_1 and v_{n-1} all are of distinct colors. Hence, on W_n $e2$ -coloring holds. Hence, it is enough to

check for each $i, c(v_i) \neq c(w_{i-1}), c(w_{i+1})$. For $3 \leq i \leq n-3, 4 \leq i+1 \leq n-2$ and $2 \leq i-1 \leq n-4$. Hence, $c(v_i) = i, c(w_{i-1}) = i-1$ & $c(w_{i+1}) = i+1$.

∴ for $3 \leq i \leq n-3, c(v_i) \neq c(w_{i-1}), c(w_{i+1})$ ----- (3)

For $i = 1, 2, n-2, n-1, c(v_i), c(w_{i-1})$ & $c(w_{i+1})$ are tabulated in table 1.

Table 1: Colors of v_i, w_{i-1} & w_{i+1}

i	$c(v_i)$	$c(w_{i-1})$	$c(w_{i+1})$
1	1	3	2
2	2	3	3
$n-2$	$n-2$	$n-3$	3
$n-1$	1	$n-2$	3

From table 1, $c(v_i) \neq c(w_{i-1}), c(w_{i+1})$ for the specified values of i . Hence, from (3) and from table 1, c is an $e2$ -coloring. Clearly, u is a 1- $e2$ -cdv and for $i, 2 \leq i \leq n-2, v_i$ is an i - $e2$ -cdv. Hence, H_n admits an $e2b$ -coloring with $n-2$ colors and hence, $\chi_{e2b}(H_n) = n-2$.

4. Exact 2-distance b -continuity of Graphs

In this section, analogous to the definition of b -continuity, we introduce the concept of exact 2-distance b -continuity of graphs. In connection with the concept, some more definitions are in need. Necessary new definitions are defined. As a main result, the exact 2-distance b -continuity of a Helm graph H_n is discussed.

Definition 4.1. If for each integer $k, \chi_{e2}(G) \leq k \leq \chi_{e2b}(G), G$ has a k - $e2b$ -coloring then G is said to be an exact 2-distance b -continuous graph (or $e2b$ -continuous graph) or we say G is $e2b$ -continuous.

Definition 4.2. The exact 2-distance b -spectrum (or $e2b$ -spectrum) $S_{e2b}(G)$ of G is defined by the set of all k such that G has an $e2b$ -coloring by k colors.

In other words, $S_{e2b}(G) = \{k: G \text{ has an } k\text{-}e2b\text{-coloring}\}$.

Thus, if $S_{e2b}(G)$ contains all the integers from $\chi_{e2}(G)$ to $\chi_{e2b}(G)$, then G is $e2b$ -continuous.

Theorem 4.3.

- (i) For $n = 4, H_n$ is not $e2b$ -continuous and $S_{e2}(H_n) = \{2, 4\}$.
- (ii) For $n \geq 5, H_n$ is $e2b$ -continuous and

$$S_{e2b}(H_n) = \begin{cases} \{3\}, & \text{if } n = 5 \\ \{4\}, & \text{if } n = 6 \\ \{4, 5\}, & \text{if } n = 7 \\ \left\{ \left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor + 1, \dots, n-2 \right\}, & \text{if } n \geq 8 \end{cases}$$

Proof:

(i) From proposition 2.6 and theorem 3.1, $\chi_{e2}(H_4) = 2$ & $\chi_{e2b}(H_n) = 4$. Hence, to check the $e2b$ -continuity, the existence of an $e2b$ -coloring with 3 colors has to be checked.

So suppose c is an $e2b$ -coloring with 3 colors. From equation (1), $d_2(u) = d_2(w_i) = 3$ and $d_2(v_i) = 2$. Hence, any $v \in V$ can be an $e2$ -cdv. As always let $c(u) = 1$. Since, $N_2(v_i) =$

$\{w_{i-1}, w_{i+1}\}$, and $c(w_j) \neq 1$ for each j , a v_i of color $r \neq 1$, cannot have 2-neighbor of color 1. Hence, a v_i can neither be a 2- $e2$ -cdv nor a 3- $e2$ -cdv. Hence, a 2- $e2$ -cdv and a 3- $e2$ -cdv occur among w_i 's only. Since, w_i can be given either color 2 or 3, and there are only 3 w_i 's, one color appears twice and the other only once.

Without loss of generality, let $c(w_1) = c(w_2) = 2$ and $c(w_3) = 3$. Now, $N_2(w_1) = \{u, v_2, v_3\}$, $N_2(w_2) = \{u, v_1, v_3\}$ & $N_2(w_3) = \{u, v_1, v_2\}$. Hence, $c(v_1), c(v_2) \neq 2, 3$ and therefore, $c(v_1) = c(v_2) = 1$. Then all the three neighbors of w_3 are of color 1 and hence, w_3 cannot be an $e2$ -cdv. Since, w_3 is the only vertex of color 3, there does not exist a 3- $e2$ -cdv. Hence, H_4 cannot have a 3- $e2b$ -coloring. Then $S_{e2b}(H_4) = \{2, 4\}$ and H_4 is not $e2b$ -continuous.

(ii) For $n \geq 5$, the $e2b$ -continuity is discussed in three cases.

Case (i): $5 \leq n \leq 7$

If $n = 5$, then $\chi_{e2}(H_n) = 3$ and $\chi_{e2b}(H_n) = n - 2 = 3$. Hence, $\chi_{e2b}(H_n) = \chi_{e2}(H_n)$ and $S_{e2b}(H_n) = \{3\}$. If $n = 6$, then $\chi_{e2}(H_n) = 4$ and $\chi_{e2b}(H_n) = n - 2 = 4$. Hence, $\chi_{e2b}(H_n) = \chi_{e2}(H_n)$ and $S_{e2b}(H_n) = \{4\}$. If $n = 7$, then $\chi_{e2}(H_n) = 4$ and $\chi_{e2b}(H_n) = n - 2 = 5$. Hence, $\chi_{e2b}(H_n) = \chi_{e2}(H_n) + 1$ and $S_{e2b}(H_n) = \{4, 5\}$. It is clear that, $\chi_{e2b}(H_n)$ is either $\chi_{e2}(H_n)$ or $\chi_{e2}(H_n) + 1$. Hence, H_n is $e2b$ -continuous.

Now suppose $n \geq 8$. The $e2b$ -continuity of H_n is checked for each k , $\chi_{e2}(H_n) = \left\lfloor \frac{n}{2} \right\rfloor \leq k \leq n - 2 = \chi_{e2b}(H_n)$. Hence, it is enough to verify the $e2b$ -coloring with k colors only for the intermediate values. i.e., for k , $\left\lfloor \frac{n}{2} \right\rfloor < k < n - 2$, i.e., for k , $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n - 3$. We show all the $e2b$ -coloring of H_n for the intermediate values, if there are atmost 2 intermediate values. The general proof is given for higher values. Now consider the value of n satisfying $(n - 2) - \left\lfloor \frac{n}{2} \right\rfloor \leq 3$. Suppose n is even. Then

$$(n - 2) - \left\lfloor \frac{n}{2} \right\rfloor \leq 3 \Leftrightarrow n - 2 - \frac{n}{2} \leq 3 \\ \Leftrightarrow n \leq 10$$

Hence, for H_8 and H_{10} , there are at most 2 intermediate values between $\chi_{e2}(H_n)$ and $\chi_{e2b}(H_n)$.

$$\text{Suppose } n \text{ is odd. Then } n - 2 - \left\lfloor \frac{n}{2} \right\rfloor \leq 3 \Leftrightarrow n - 2 - \left(\frac{n-1}{2} \right) \leq 3 \\ \Leftrightarrow \frac{n-3}{2} \leq 3 \\ \Leftrightarrow n \leq 9$$

Since $n \geq 8$, H_9 has atmost 2 intermediate values between $\chi_{e2}(H_9)$ and $\chi_{e2b}(H_9)$. Hence case (ii) follows.

Case (ii): $8 \leq n \leq 10$

If $n = 8$, then $\chi_{e2}(H_n) = \left\lfloor \frac{n}{2} \right\rfloor = 4$, $\chi_{e2b}(H_n) = n - 2 = 6$ and hence, the intermediate value is 5. From figure 4, H_8 has a 5- $e2b$ -coloring.

If $n = 9$, then $\chi_{e2}(H_n) = \left\lfloor \frac{n}{2} \right\rfloor = 4$, $\chi_{e2b}(H_n) = n - 2 = 7$ and hence, the intermediate values

Figure 4: 5-e2b-coloring of H_8

are 5 and 6. From figure 5, H_9 has a 5-e2b-coloring and from figure 6, H_9 has a 6-e2b-coloring. If $n = 10$, then $\chi_{e2}(H_n) = \left\lfloor \frac{n}{2} \right\rfloor = 5$, $\chi_{e2b}(H_n) = n - 2 = 8$ and hence, the intermediate values are 6 and 7. From figures 7 and 8, H_{10} has a 6-e2b-coloring and a 7-e2b-coloring.

Figure 5: 5-e2b-coloring of H_9

Figure 6: 6-e2b-coloring of H_9

Figure 7: 6-e2b-coloring of H_{10}

Figure 8: 7-e2b-coloring of H_{10}

Case (ii): $n \geq 11$

In the $(n - 2)$ -e2b-coloring of H_n on the $(n - 1)$ -cycle, v_1 and v_{n-1} are of color 1 and the remaining $n - 3$ vertices from v_2 to v_{n-2} are of distinct colors. Hence, in an e2b-coloring of H_n with k colors, $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n - 3$, on the $(n - 1)$ -cycle, leaving v_1 and v_{n-1} , the remaining $n - 3$ vertices from v_2 to v_{n-2} receive $k - 1$ colors. Then

$$\text{number of colors repeated} = (n - 3) - (k - 1) = n - k - 2 \quad \text{----- (4)}$$

When a color is repeated, it occurs in two adjacent vertices. We combine pairs of vertices in the sequences $v_2, v_3, \dots, v_{n-2}$ with higher suffices. Let vertices v_i to v_{n-2} be combined in a k -coloring. Clearly v_i is the vertex of smallest index to receive the same color. Hence, it is necessary to find v_i in each coloring in the specified interval. If k colors are given, then from (4), $n - k - 2$ colors are repeated. In other words, $2(n - k - 2)$ vertices are combined pairwise. Hence, i of v_i is computed as follows

$$i = (n - 2) - [2(n - k - 2) - 1] = 2k - n + 3 \quad \text{----- (5)}$$

Now to compute v_i for the maximum and minimum values of k . When k is maximum, $k = n - 3$, and from (5), $i = n - 3$ and hence $v_i = v_{n-3}$. Suppose k is minimum. Then $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

$$\therefore \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n - 3 \quad \text{----- (6)}$$

Hence, there are 2 values for i depending on n whether it is even or odd.

If n is even, then $k = \frac{n}{2} + 1$ and $i = 2\left(\frac{n}{2} + 1\right) - n + 3 = 5$. Hence, vertices are combined from v_5 onwards.

If n is odd, then $k = \frac{n-1}{2} + 1$ and $i = 2\left(\frac{n-1}{2} + 1\right) - n + 3 = 4$. Hence, vertices are combined from v_4 onwards.

In general, from (5), in a k -coloring of H_n , v_2 to v_{2k-n+2} are of distinct colors and vertices v_{2k-n+3} to v_{n-2} are combined pairwise and given colors from $2k - n + 3$ to k . Now let $c: V \rightarrow \{1, 2, \dots, k\}$, $\left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k \leq n - 3$. First assign colors to u, v_1 and v_{n-1} as in (7).

$$c(u) = c(v_1) = c(v_{n-1}) = 1 \quad \text{----- (7)}$$

In a k -coloring, leaving color 1, $k - 1$ colors are available for the $n - 3$ vertices from v_2 to v_{n-2} . From (5), vertices from v_{2k-n+3} to v_{n-2} are combined pairwise. Hence, v_2 to v_{2k-n+2} receive distinct colors. Let

$$c(v_i) = c(w_i) = i, \quad 2 \leq i \leq 2k - n + 2 \quad \text{----- (8)}$$

From (8), colors used from v_2 to $v_{2k-n+2} = 2k - n + 1$

$$\therefore \text{Colors available for } v_{2k-n+3} \text{ to } v_{n-2} = k - 1 - (2k - n + 1) = n - k - 2 \quad \text{----- (9)}$$

Number of vertices from v_{2k-n+3} to $v_{n-2} = n - 2 - (2k - n + 2) = 2n - 2k - 4$

$$\therefore \text{Number of pairs from } v_{2k-n+3} \text{ to } v_{n-2} = n - k - 2 \quad \text{----- (10)}$$

Hence, from (9) & (10), the $2n - 2k - 4$ vertices from v_{2k-n+3} to v_{n-2} can receive $n - k - 2$ colors from $2k - n + 3$ to $k - 1$ pairwise. Now to color v_i , $2k - n + 3 \leq i \leq n - 2$. For $1 \leq i \leq n - k - 2$, $2k - n + 3 \leq 2k - n + 2i + 1 \leq n - 3$ and $2k - n + 4 \leq 2k - n + 2i + 2 \leq n - 2$. Hence,

(i) if n is even, then $2k - n + 2i + 1$ takes all odd values and $2k - n + 2i + 2$ even values between $2k - n + 3$ & $n - 2$ and

(ii) if n is odd, then $2k - n + 2i + 1$ takes all even values and $2k - n + 2i + 2$ odd values between $2k - n + 3$ & $n - 2$.

Hence, from (i) and (ii), $2k - n + 2i + 1$ and $2k - n + 2i + 2$ assume all the values between $2k - n + 3$ & $n - 2$. Hence, the following fact.

Fact 1: Coloring v_i , $2k - n + 3 \leq i \leq n - 2$ is same as coloring $v_{2k-n+2i+1}$ & $v_{2k-n+2i+2}$, $1 \leq i \leq n - k - 2$.

Now color v_i 's from v_{2k-n+3} to v_{n-2} , pairwise as described below.

$$c(v_{2k-n+2i+1}) = c(v_{2k-n+2i+2}) = 2k - n + 1 + i, \quad 1 \leq i \leq n - k - 2$$

$$\text{i.e., } c(v_{2k-n+2i+1}) = c(v_{2k-n+2i+2}) = 2k - n + i + 1, \quad 1 \leq i \leq n - k - 2 \quad \text{----- (11)}$$

Thus, all the vertices of H_n but w_i 's from w_{2k-n+3} to w_{n-2} are colored and they are colored based on whether n is even or odd. Before assigning colors to w_i 's, c defined in (7), (8) & (11) is collectively given in (12)

$$\left. \begin{aligned} c(u) &= c(v_1) = c(v_{n-1}) = 1; \\ c(v_i) &= c(w_i) = i, \quad 2 \leq i \leq 2k - n + 2; \\ c(v_{2k-n+2i+1}) &= c(v_{2k-n+2i+2}) = 2k - n + i + 2, \quad 1 \leq i \leq n - k - 2; \\ c(w_1) &= c(w_{n-1}) = 3. \end{aligned} \right\} \text{-----} (12)$$

The remaining w'_i s, $2k - n + 3 \leq i \leq n - 2$ are assigned colors as described below.

(i) If n is even, let $c(w_i) = 3$, $2k - n + 3 \leq i \leq n - 2$.

(ii) If n is odd, then for $\left\lfloor \frac{n}{2} \right\rfloor + 2 \leq k \leq n - 3$ and $2k - n + 3 \leq i \leq n - 2$, let $c(w_i) = 3$.

If $k = \left\lfloor \frac{n}{2} \right\rfloor + 1$, then let $c(w_i) = 3$, for $5 \leq i \leq n - 2$ and $c(w_4) = 5$.

Since, v_2 to v_{2k-n+2} receive colors from 2 to $2k - n + 2$, v_{2k-n+2} to v_{n-2} receive colors from $2k - n + 3$ to k . Hence,

$$2k - n + 3 \leq i \leq n - 2 \Rightarrow 2k - n + 3 \leq c(v_i) \leq k \text{-----} (13)$$

Now to verify $c(v_i) \neq c(w_{i-1}), c(w_{i+1})$.

Suppose n is even. Then from (6), $k \geq \frac{n}{2} + 1$. Hence,

$$2k - n + 3 \geq 4 \text{-----} (14)$$

If $i = 1$, then $c(v_i) = 1, c(w_{i-1}) = c(w_{n-1}) = 3$ & $c(w_{i+1}) = c(w_2) = 2$.

If $i = 2$, then $c(v_i) = 2, c(w_{i-1}) = c(w_1) = 3$ & $c(w_{i+1}) = c(w_3) = 3$.

If $3 \leq i \leq 2k - n + 1$, then $2 \leq i - 1 \leq 2k - n$ and $4 \leq i + 1 \leq 2k - n + 2$. Hence,

$$c(v_i) = i, c(w_{i-1}) = i - 1 \text{ & } c(w_{i+1}) = i + 1.$$

If $i = 2k - n + 2$, then $c(v_i) = i = 2k - n + 2$, $c(w_{i-1}) = c(w_{2k-n+1}) = 2k - n + 1$

and $c(w_{i+1}) = c(w_{2k-n+3}) = 3$

If $i = 2k - n + 3$, then $c(v_i) = c(v_{2k-n+3}) = c(v_{2k-n+2 \times 1 + 1}) = 2k - n + 1 + 2 = 2k - n + 3$ and $c(w_{i-1}) = c(w_{2k-n+2}) = 2k - n + 2$ & $c(w_{i+1}) = c(w_{2k-n+4}) = 3$.

If $2k - n + 4 \leq i \leq n - 3$, then from (13) & (14), $c(v_i) \geq 4$ and from (i), $c(w_{i-1}) = c(w_{i+1}) = 3$.

If $i = n - 2$, then $c(v_i) = k \geq 4$, $c(w_{i-1}) = c(w_{n-3}) = 3$ and $c(w_{i+1}) = c(w_{n-1}) = 3$.

If $i = n - 1$, then $c(v_i) = 1$, $c(w_{i-1}) = c(w_{n-2}) = 3$ and $c(w_{i+1}) = c(w_1) = 3$.

Hence, $c(v_i) \neq c(w_{i-1}), c(w_{i+1})$.

By similar discussion, $c(v_i) \neq c(w_{i-1}), c(w_{i+1})$ can be proved if n is odd. Thus the k -coloring c defined is a 2-distance coloring. Since, for distinct colors $c(v_i) = c(w_i)$ and for pairwise colors $c(v_i) = c(v_{i+1})$, each v_i is an $e2$ -cdv. Hence, H_n has an $e2b$ -coloring for each k between $\left\lfloor \frac{n}{2} \right\rfloor$ to $n - 2$ and hence, H_n is $e2b$ -continuous.

5. Conclusion

In this paper, we introduced exact 2-distance b -continuous graph and exact 2-distance b -spectrum. Also, the exact 2-distance b -chromatic number of Helm graph H_n was obtained. Further it is proved that for $n = 4$, H_n is not $e2b$ -continuous and for $n \geq 5$, H_n is $e2b$ -continuous and there by proving there exists $e2b$ -continuous graphs and graphs which are not $e2b$ -continuous.

REFERENCES

- [1] F. Harary, *Graph Theory*, Narosa /Addison Wesley, Indian Student Edition, 1988.
- [2] R.W. Irving and D.F. Manlove, *The b-chromatic number of a graph*, *Discrete. Appl. Math.* 91 pp. 127-141, 1991.
- [3] T. N. Janakiraman, M. Poobalaranjani and A. SenthilThilak, *Exact k-distance colouring of Graphs*, *Proceedings of the UGC sponsored National Seminar on Applications in Graph Theory*
- [4] F. Kramer and H. Kramer, *Un probl`eme de coloration des sommets d'un graphe*. *C. R. Acad. Sci. Paris A* 268(7), pp. 46–48, 1969.
- [5] F. Kramer and H. Kramer, *EinFarbungsproblem der Knotenpunkte eines Graphen bezuglich der Distanz p*, *Rev. Roumaine Math. Pures Appl.*, 14(2), pp. 1031-1038, 1969.
- [6] Nesetril J, Ossona de Mendez, *Sparsity, Graphs, Structures, and Algorithms*, Springer Verlag, Berlin, Heidelberg, 2012.
- [7] S. Saraswathi and M. Poobalaranjani, *Exact 2-distance b-coloring of some classes of graphs*, *Malaya Journal of Matematik*, Vol. 8, No. 1, 195-200, 2020 MJM 08010033