2-Distance Strong b -coloring of Perfect Δ -ary Tree

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ABSTRACT

A 2-distance b -coloring is a 2-distance coloring in which every color class contains a vertex which has a neighbor in every other color class. A 2-distance strong b -coloring (2sb-coloring) is a 2-distance coloring in which every color class contains a vertex u such that there is a vertex v in every other color class satisfying the condition that the distance between u and v is at most 2. The 2-distance b -chromatic number $\chi_{2b}(G)$ (2b-number) is the largest integer k such that G admits a 2-distance b -coloring with k colors and the 2-distance strong b -chromatic number $\chi_{2sb}(G)$ (2sb-number) is the maximum k such that G admits a 2sb-coloring with k colors. A tree with a special vertex called the root is called a rooted tree. A perfect Δ -ary tree, is a rooted tree in which all internal vertices are of degree Δ and all pendant vertices are at the same level. In this paper, the exact bound of the 2sb-number of perfect Δ -ary tree are obtained.

Keywords: 2-distance b -coloring, perfect Δ -ary tree, 2-distance strong b -coloring, 2-distance strong b -chromatic number, strong color dominating vertex.

1. Introduction

In this paper, only finite, simple, and undirected graphs are considered. The reader may refer to [1] for those terminologies not defined here. The concept of distance coloring was introduced by F. Kramer and H. Kramer [4], [5] in 1969. A 2-distance k -coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, 2, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. The 2-distance chromatic number of G , denoted $\chi_2(G)$ is the smallest integer k for which G has a 2-distance k -coloring. R.W. Irving and D.F. Manlove [2] introduced the concept of b -coloring in 1991. A b -coloring of G is a proper coloring such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class. Such a vertex is called a color dominating vertex. The b -chromatic number $\chi_b(G)$ of G is the largest integer k such that G has a b -coloring by k colors. The m -degree $m(G)$ of a graph was defined as $m(G) = \max\{i: G \text{ has at least } i \text{ vertices of degree at least } i - 1\}$. A 2-distance b -coloring and a 2-distance strong b -coloring were introduced by M. Poobalaranjani and S. Saraswathi [6], [7] in 2019. A 2-distance b -coloring is a 2-distance coloring in which every color class contains a vertex which has a neighbor in every other color class. A 2-distance strong b -coloring (2sb-coloring) is a 2-distance coloring in which every color contains a vertex

u such that there is a vertex v in every other color class satisfying the condition that the distance between u and v is at most 2. The 2-distance b -chromatic number $\chi_{2b}(G)$ ($2b$ -number) is the largest integer k such that G admits a 2-distance b -coloring with k colors and the 2-distance strong b -chromatic number $\chi_{2sb}(G)$ ($2sb$ -number) is the maximum k such that G admits a $2sb$ -coloring with k colors. A tree with a special vertex called the root is called a rooted tree. Now we define another rooted tree called a perfect Δ -ary tree which is based on degree of its vertices. A perfect Δ -ary tree, is a rooted tree in which all internal vertices are of degree Δ and all pendant vertices are at the same level. Also, the exact bound of $2sb$ -number of perfect Δ -ary tree are obtained.

2. Definitions and Prior results

In this section, definition of the 2-distance strong b -coloring and is formally given. Some of the prior results obtained in [7] which are necessary for the proof of the main result are also given.

Definition 2.1. In a graph G , for $u \in V$, a vertex v is said to be a k -neighbor, if $d(u, v) = k$.

Definition 2.2. A set S of vertices is said to be an $(k \geq 2)$ k -neighborhood (or k -open neighborhood) of a vertex u , if S consists of all the vertices which are at distance k from u . It is denoted by $N_k(u)$.

i.e., $N_k(u) = \{v \in V : d(u, v) = k\}$.

A k -closed neighborhood of u is the set $N_k[u] = N_k(u) \cup \{u\}$.

Definition 2.3. In a graph G , for $u \in V$, a vertex v is said to be an at most k -neighbor, if $d(u, v) \leq k$.

Definition 2.4. A set S of vertices is said to be an $(k \geq 2)$ at most k -neighborhood (or at most k -open neighborhood) of a vertex u , if S consists of all the vertices which are at distance at most k from u . It is denoted by $N_{\leq k}(u)$.

i.e., $N_{\leq k}(u) = \{v \in V : d(u, v) \leq k\}$.

An at most k -closed neighborhood of u is the set $N_{\leq k}[u] = N_{\leq k}(u) \cup \{u\}$.

Definition 2.5. In a graph $G = (V, E)$, for $u \in V$, the at most k -degree ($k \geq 2$) of u is defined as the number of vertices which are at distance at most k from u and is denoted by $d_{\leq k}(u)$.

The at most k -degree $d_{\leq k}(u)$ of a vertex u is also called the $d_{\leq k}$ -degree of u .

Notation 2.6. If c is any coloring of a graph $G = (V, E)$ and $S \subseteq V$, then $C(S)$ denotes the set of all colors used by the vertices of S .

i.e., $C(S) = \{c(u) : u \in S\}$

Observation 2.7. For any k , $2 \leq k \leq \text{diam}(G)$, and for any vertex u of a graph G ,

- (i) if $2 \leq l \leq k$, then $N(u) \subseteq N_{\leq l}(u) \subseteq N_{\leq k}(u)$;
- (ii) any neighbor of u is an at most k -neighbor of u , and if $2 \leq l \leq k$, then any at most l -neighbor of u is an at most k -neighbor of u ;
- (iii) $N_{\leq k}(u) = N_1(u) \cup N_2(u) \cup N_3(u) \cup \dots \cup N_k(u)$;
- (iv) $d_{\leq k}(u) = |N_{\leq k}(u)|$;

Definition 2.8 [7]. A vertex in a color class partition is said to be a strong color dominating vertex (or simply s -cdv) if it has at least a neighbor or a 2-neighbor (or simply at least an at most 2-neighbor) in every other color class.

Observation 2.9 [7]. Every color dominating vertex is a s -cdv and the converse is not true.

Definition 2.10 [7]. A 2-distance strong b -coloring (or simply a $2sb$ -coloring) of a graph G is a 2-distance coloring in which each color class contains a vertex u such that there is v in every other color class satisfying the condition that $d(u, v) \leq 2$.

Definition 2.11 [7]. The 2-distance strong b -chromatic number or simply $2sb$ -number $\chi_{2sb}(G)$ of a graph G is the largest k for which G has a 2-distance strong b -coloring with k colors.

Observation 2.12 [7].

- (i) Any 2-distance coloring with χ_2 colors is a $2sb$ -coloring;
- (ii) $2sb$ -coloring exists for all graphs.

Definition 2.13 [7]. The $m_{\leq 2}$ -degree of G , denoted by $m_{\leq 2}(G)$, is defined by $m_{\leq 2}(G) = \max\{i : 1 \leq i \leq |V(G)|, G \text{ has at least } i \text{ vertices whose } d_{\leq 2}\text{-degree is at least } i - 1\}$.

Observation 2.14 [7]. If G is any graph, then $\chi_2(G) \leq \chi_{2sb}(G) \leq m_{\leq 2}(G) \leq \Delta^2 + 1$.

Proposition 2.15 [7]. For $n \geq 3$, $\chi_{2sb}(P_n) = \begin{cases} 3, & \text{if } 3 \leq n \leq 5 \\ 4, & \text{if } 6 \leq n \leq 8 \\ 5, & \text{if } n \geq 9 \end{cases}$

3. Perfect Δ -ary tree and some Observations

In this section, we introduce a new rooted tree called a perfect Δ -ary tree. Hence, some fundamental definitions used in a rooted tree are given. This new tree definition was motivated by two types of rooted trees viz., full m -ary tree and a perfect binary tree. Hence, to understand the new tree and how the new tree differs from the other, definitions of various rooted trees relevant to our perfect Δ -ary tree are also given in this section.

Definition 3.1. A tree with a special vertex called the root is called a rooted tree.

Definition 3.2. If u is the root of a tree T , and v is any vertex in T , then level of v is the distance between u and v . Level of a vertex v is denoted by $l(v)$.

Definition 3.3. If T is a rooted tree and u is the root of T , then height of $T = \max\{d(u, x) : x \text{ is a pendant vertex in } T\}$.

Definition 3.4. In a rooted tree, let P be the path from the root to a vertex v . Then any vertex on P other than v is called an ancestor of v . A vertex v is a descendant of a vertex u if and only if u is an ancestor of v .

Definition 3.5. If u is an ancestor of v and adjacent to v , then u is called the parent of v and v is called a child of u .

Definition 3.6. Vertices that have children are called internal vertices.

Definition 3.7. A rooted tree is called an m -ary tree, if every internal vertex has no more than m children.

Definition 3.8. An m -ary tree is called full m -ary tree, if every internal vertex has exactly m children.

Definition 3.9. An m -ary tree with $m = 2$ is called a binary tree.

Definition 3.10. A Binary tree is a Perfect Binary Tree in which all the internal vertices have two children and all the pendant vertices are at the same level.

Definition 3.11. A perfect Δ -ary tree, is a rooted tree in which all internal vertices are of degree Δ and all pendant vertices are at the same level.

Example 3.12.

Figure 1: Perfect Δ -ary tree T with $k = 3$ & $\Delta = 3$

Notation 3.13. Let T be a perfect Δ -ary tree T of height $k \geq 2$. Then the following notations are observed.

- (i) u is the root of T ;
- (ii) $N(u) = \{u_i : 1 \leq i \leq \Delta\}$;
- (iii) For a vertex $v \in V$, if $l(v) = i$, $0 \leq i \leq k - 1$, then $N'(v)$ denotes its neighbors in the $(i + 1)^{th}$ level;
- (iv) n_i denotes the number of vertices in the i^{th} level.

Observation 3.14. Let T be a perfect Δ -ary tree of height $k \geq 2$ and v be any arbitrary vertex in T with $l(v) = i$, $0 \leq i \leq k$. Then the following hold.

- (i) If $\Delta = 2$, then T is a path of odd order;
- (ii) Adjacency between two vertices can occur only between a parent and a child. In other words, adjacency can occur only in two consecutive levels.
 - (a) If $i = 0$, then $v = u$ and, v has Δ neighbors in level 1;
 - (b) If $1 \leq i \leq k - 1$, then v has one neighbor in the $(i - 1)^{th}$ level and $(\Delta - 1)$ -neighbors in the $(i + 1)^{th}$ level;
 - (c) If $i = k$, then v has one neighbor in the $(i - 1)^{th}$ level.
 - (d) If $0 \leq i \leq k - 1$, then $d(v) = \Delta$ and if $i = k$, then $d(v) = 1$.
- (iii) If vertices x, y, z induce P_3 in T , then either
 - (a) x and z are in some i^{th} level and y is in the $(i - 1)^{th}$ level or
 - (b) x, y, z are respectively in the $i, i + 1, (i + 2)^{th}$ level.

Observation 3.15. Let T be a perfect Δ -ary tree of height $k \geq 2$ and v be any arbitrary vertex in T with $l(v) = i$, $0 \leq i \leq k$. Then the following hold.

- (i) Vertex v can have 2-neighbors in $(i - 2)^{th}$ level, i^{th} level and $(i + 2)^{th}$ level depending on i and k .
- (ii) If $i = 0$ and $k \geq 2$, then $v = u$ and v has $\Delta(\Delta - 1)$ 2-neighbors in level 2 and $d_2(v) = \Delta(\Delta - 1)$;
- (iii) If $i = 1$ and $k = 2$, then v has $(\Delta - 1)$ 2-neighbors in level 1 and $d_2(v) = (\Delta - 1)$;
- (iv) If $i = 1$ and $k \geq 3$, then v has $(\Delta - 1)$ 2-neighbors in level 1 and $(\Delta - 1)(\Delta - 1)$ 2-neighbors in level 3 and $d_2(v) = (\Delta - 1) + (\Delta - 1)(\Delta - 1) = \Delta^2 - \Delta$;

- (v) If $2 \leq i \leq k-2$ and $k \geq 4$, then v has one 2-neighbor in level $i-2$, $(\Delta-2)$ 2-neighbors in level i and $(\Delta-1)(\Delta-1)$ 2-neighbors in level $i+2$ and $d_2(v) = 1 + (\Delta-2) + (\Delta-1)(\Delta-1) = \Delta^2 - \Delta$;
- (vi) If $i = k-1$ and $k \geq 3$, then v has one 2-neighbor in level $i-2$ and $(\Delta-2)$ 2-neighbors in level i and $d_2(v) = 1 + (\Delta-2) = \Delta - 1$;
- (vii) If $i = k$ and $k \geq 2$, then v has one 2-neighbor in level $i-2$ and has $(\Delta-2)$ 2-neighbors in level i and $d_2(v) = 1 + (\Delta-2) = \Delta - 1$;
- (viii) $n_0 = 1$, $n_1 = \Delta$ and for $2 \leq i \leq k$, $n_i = \Delta(\Delta-1)^{i-1}$.

4. On the bound of 2-Distance strong b -chromatic number of a Perfect Δ -ary Tree

In this section, the bound of 2-distance strong b -chromatic number of a Perfect Δ -ary tree of different heights are obtained.

Theorem 4.1. Let T be a perfect Δ -ary tree of height $k \geq 2$. Then

$$\chi_{2sb}(T) = \begin{cases} \Delta + 1, & \text{if } k = 2 \\ 2\Delta, & \text{if } k = 3 \\ \Delta^2 + 1, & \text{if } k \geq 4 \end{cases}$$

Proof: Let u be the root of T .

Suppose $\Delta = 2$. Then T is a path. As in each level i , $1 \leq i \leq k$, there are exactly 2 vertices, $T = P_{2r+1}$, $r \geq 2$. i.e., $T = P_n$, $n \geq 5$. Then, from Proposition 2.15, $\chi_{2sb}(T)$ known and the values are listed in Table 1 for various k and n .

Table 1

k	n	$\chi_{2sb}(T)$
2	5	$3 = \Delta + 1$
3	7	$4 = 2\Delta$
≥ 4	≥ 9	$5 = \Delta^2 + 1$

From the last column of the table, it is clear that the theorem holds if $\Delta = 2$.

Now let $\Delta \geq 3$ and let $N(u) = \{u_i : 1 \leq i \leq \Delta\}$. For any vertex v , $d_{\leq 2}(v) = d(v) + d_2(v)$. Hence, from observation 3.14(ii) (d) & 3.15, $d_2(v)$ is known. Using the value of $d_2(v)$ and the fact that, for non pendant vertices, $d(v) = \Delta$, $d_{\leq 2}(v)$ is computed as follows.

- (i) If $i = 0$ and $k = 2$, then $v = u$ and $d_{\leq 2}(u) = \Delta + \Delta(\Delta-1) = \Delta^2$.
- (ii) If $i = 1$ and $k = 2$, then $d_{\leq 2}(v) = \Delta + (\Delta-1) = 2\Delta - 1$.
- (iii) If $i = 1$ and $k \geq 3$, then $d_{\leq 2}(v) = \Delta + \Delta^2 - \Delta = \Delta^2$.
- (iv) If $2 \leq i \leq k-2$ and $k \geq 4$, then $d_{\leq 2}(v) = \Delta + \Delta^2 - \Delta = \Delta^2$.
- (v) If $i = k-1$ and $k \geq 3$, then $d_{\leq 2}(v) = \Delta + (\Delta-1) = 2\Delta - 1$.
- (vi) If $i = k$ and $k \geq 2$, then $d_{\leq 2}(v) = 1 + (\Delta-1) = \Delta$.

Hence, the above can be summarized as,

For a vertex v in the i^{th} level, $0 \leq i \leq k$, $k \geq 2$,

- (i)' if $0 \leq i \leq k-2$, then $d_{\leq 2}(v) = \Delta^2$;
- (ii)' if $i = k-1$, then $d_{\leq 2}(v) = 2\Delta - 1$;
- (iii)' if $i = k$, then $d_{\leq 2}(v) = \Delta$.

From (i)' to (iii)', it is clear that vertices are of 3 types.

- (a) Vertex with $d_{\leq 2}$ -degree is Δ^2 ;
- (b) Vertex with $d_{\leq 2}$ -degree is $2\Delta - 1$;
- (c) Vertex with $d_{\leq 2}$ -degree is Δ .

Let r_1 , r_2 and r_3 be the number of vertices of $d_{\leq 2}$ -degree is Δ^2 , $2\Delta - 1$ and Δ respectively.

From notation (3.13), n_i = number of vertices in the i^{th} level. Then

$$r_i = \begin{cases} \sum_{j=0}^{j=k-2} n_j, & \text{if } i = 1 \\ \sum_{j=0}^{j=k-1} n_j, & \text{if } i = 2 \\ \sum_{j=0}^{j=k} n_j, & \text{if } i = 3 \end{cases} \quad \text{-----}(1)$$

Using the values of r_i , $m_{\leq 2}(T)$ can be computed as below.

$$\left. \begin{array}{l} T_1: \text{If } r_1 \geq \Delta^2 + 1, \text{ then } m_{\leq 2}(T) = \Delta^2 + 1 \\ T_2: \text{If } r_1 < \Delta^2 + 1 \text{ and } r_2 \geq 2\Delta, \text{ then } m_{\leq 2}(T) = 2\Delta \\ T_3: \text{If } r_2 < 2\Delta \text{ and } r_3 \geq \Delta + 1, \text{ then } m_{\leq 2}(T) = \Delta + 1 \end{array} \right\} \quad \text{-----}(2)$$

Using (2), the following claim is proved.

$$\textbf{Claim 1 : } m_{\leq 2}(T) = \begin{cases} \Delta + 1, & \text{if } k = 2 \\ 2\Delta, & \text{if } k = 3 \\ \Delta^2 + 1, & \text{if } k \geq 4 \end{cases}$$

Three cases are discussed.

Case (i): $k = 2$

From (1) and (2), $r_1 = n_0 = 1 < \Delta^2 + 1$, $r_2 = n_0 + n_1 = \Delta + 1 < 2\Delta$ and

$$r_3 = n_0 + n_1 + n_2 = 1 + \Delta + \Delta(\Delta - 1) = \Delta^2 + 1 > \Delta + 1.$$

Hence from (2), $m_{\leq 2}(T) = \Delta + 1$.

Case (ii): $k = 3$

From (1) and simplifying, $r_1 = \sum_{j=0}^{j=1} n_j = \Delta + 1 < \Delta^2 + 1$; $r_2 = \sum_{j=0}^{j=2} n_j = \Delta^2 + 1 \geq 2\Delta$.

From (2), $m_{\leq 2}(T) = 2\Delta$.

Case (iii): $k \geq 4$

From (1) and simplifying, $r_1 = \sum_{j=0}^{j=k-2} n_j \geq \sum_{j=0}^{j=2} n_j = \Delta^2 + 1$. Hence, $m_{\leq 2}(T) = \Delta^2 + 1$

From case (i) to (iii), claim 1 is proved.

From observations 3.14 and 3.15, consecutive 3 levels have to be checked for 2-distance coloring and consecutive 5 levels have to be checked for strong color domination. Further, from level 1 to level k , the number of vertices in each level form a geometric progression with first term Δ and common ratio $\Delta - 1$. Hence, labelling the vertices is a must.

Labelling the vertices

We have $N(u) = \{u_i : 1 \leq i \leq \Delta\}$. Let us call the vertex u_i as the i^{th} neighbor of u in level 1. With the existing notation we see, u is in level 0 and u has 0 suffix. Its neighbor u_i is in level 1 and can be interpreted as u with 1 suffix. Analogous to this, let us label the vertices such that, a vertex v in level i be denoted by u with i suffices separated by commas such as $v = u_{s_1, s_2, \dots, s_{i-1}, s_i}$. Since, a vertex has exactly one ancestor in each level, we write these i suffices such a way that $s_1, s_2, \dots, s_{i-1}, s_i$ denote the ancestral path of v in the sense that first suffix s_1 indicates that v 's ancestor in level 1 is u_{s_1} , the s_1^{th} neighbor of u , the second suffix s_2

indicates that v 's ancestor in level 2 is u_{s_1, s_2} , the s_2^{th} neighbor of u_{s_1} and so on. Finally, $u_{s_1, s_2, \dots, s_{i-1}, s_i}$ is the s_i^{th} neighbor of $u_{s_1, s_2, \dots, s_{i-1}}$. In short, the sequence of i suffices $s_1, s_2, \dots, s_{i-1}, s_i$ indicates that $u_{s_1}, u_{s_1, s_2}, u_{s_1, s_2, s_3}, \dots, u_{s_1, s_2, \dots, s_{i-1}}$ are the ancestors of $v = u_{s_1, s_2, \dots, s_{i-1}, s_i}$.

The labelling of T with $k = 3$ and $\Delta = 3$ is illustrated in figure 2.

Figure 2: Labelling of a perfect Δ -ary tree with $k = \Delta = 3$

By the above notation, for $1 \leq i \leq \Delta$, $N'(u_i) = \{u_{i,j} : 1 \leq j \leq \Delta - 1\}$, where the vertex $u_{i,j}$ denotes the j^{th} neighbor of u_i in level 2.

Now the theorem is proved. There are 3 cases to be considered.

Case (i): $k = 2$

Since, $\chi_2(T) = \Delta + 1$, $\chi_2(T) = \Delta + 1 = m_{\leq 2}(T)$. Hence $\chi_{2sb}(T) = \Delta + 1$.

Case (ii): $k = 3$

From claim 2, $m_{\leq 2}(T) = 2\Delta$. Let $c: V \rightarrow \{1, 2, \dots, 2\Delta\}$ be a coloring of T such that vertices from level 0 to level 2 are colored as described in (3).

$$\left. \begin{aligned} c(u) &= \Delta + 1, \\ c(u_i) &= i, 1 \leq i \leq \Delta, \\ c(u_{i,j}) &= \Delta + 1 + j, 1 \leq i \leq \Delta, 1 \leq j \leq \Delta - 1 \end{aligned} \right\} \text{-----(3)}$$

From the 3rd equation of (3), for each i , $C(N'(u_i)) = \{\Delta + 2, \Delta + 3, \dots, 2\Delta\}$. Hence, vertices up to level 2 are colored with 2Δ colors.

From equation (3), the following are observed.

(a) In level 1, u has neighbors u_i in the color classes i , $1 \leq i \leq \Delta$ and in level 2, 2-neighbors $u_{i,j}$ in the color classes S_r , $\Delta + 2 \leq r \leq 2\Delta$. Hence, u has an at most 2-neighbor in S_i , $1 \leq i \leq 2\Delta$, $i \neq \Delta + 1$. Hence, u is a s -cdv of the color class $S_{\Delta+1}$.

(b) For each i , $1 \leq i \leq \Delta$, in level 0, u_i has a neighbor u in $S_{\Delta+1}$, in level 1, 2-neighbors u_j in S_j , $1 \leq j \leq \Delta$, $j \neq i$ and in level 2, neighbours $u_{i,j}$ in S_r , $\Delta + 2 \leq r \leq 2\Delta$. Hence, u_i has an at most 2-neighbors in the color classes r , $1 \leq r \leq 2\Delta$, $r \neq i$. Hence, u_i is a s -cdv of the color class S_i .

Thus, there exist strong color dominating vertices in the color classes S_i , $1 \leq i \leq \Delta + 1$. Now the vertices $u_{i,j,r}$ of level 3 are to be colored and strong color dominating vertices from the color classes S_r , $\Delta + 2 \leq r \leq 2\Delta$ are to be determined. From (i)'' to (iii)'', $d_{\leq 2}(u_{i,j,r}) = \Delta$ and $d_{\leq 2}(u_{i,j}) = 2\Delta - 1$. Hence $u_{i,j,r}$ cannot be s -cdv. Hence, only $u_{i,j}$ can be s -cdv of the remaining colors from $\Delta + 2 \leq r \leq 2\Delta$.

Consider the vertex $u_{i,j}$. It has a 2-neighbor u of color $\Delta + 1$, a neighbor u_i of color i , and 2-neighbors $u_{i,r}$, $r \neq j$, $1 \leq r \leq \Delta - 1$ of colors s , $\Delta + 2 \leq s \leq 2\Delta$, $s \neq \Delta + 1 + j$. Also $u_{i,j}$ has uncolored $(\Delta - 1)$ neighbors $u_{i,j,r}$, $1 \leq r \leq \Delta - 1$ in level 3. Since $d(u_{i,j}, u_r) = 3$, $1 \leq r \leq \Delta$, $r \neq i$, $u_{i,j}$ has no at most 2-neighbor of colors, $\Delta + 2 \leq s \leq 2\Delta$, $s \neq i$ from level 0 to level 2. Thus, there are $\Delta - 1$ colors available at $u_{i,j}$ and it has $(\Delta - 1)$ neighbors $u_{i,j,t}$ uncolored. Hence, for each i , $1 \leq i \leq \Delta$ and for each j , $1 \leq j \leq \Delta - 1$, the $\Delta - 1$ colors s , $1 \leq s \leq \Delta$, $s \neq i$ can be assigned to the $\Delta - 1$ neighbors $u_{i,j,t}$, $1 \leq t \leq \Delta - 1$ of $u_{i,j}$ in the following way:

$$c(u_{i,j,t}) = \begin{cases} t, & \text{if } t < i \\ t + 1, & \text{if } t \geq i \end{cases} \quad \text{-----(4)}$$

Hence, level 3 vertices are colored and vertex $u_{i,j}$, $1 \leq j \leq \Delta - 1$ is a s -cdv of the color class $\Delta + 1 + j$. Hence, c is a 2-distance strong b -coloring of T with 2Δ colors. From claim 2, $m_{\leq 2}(T) = 2\Delta$ and hence $\chi_{2sb}(T) = 2\Delta$.

Case (iii): $k \geq 4$

From claim 2, $m_{\leq 2}(T) = \Delta^2 + 1$. Let us show that T has a 2-distance strong b -coloring with $\Delta^2 + 1$ colors. Let $c: V \rightarrow \{1, 2, \dots, \Delta^2 + 1\}$ be a coloring of T defined such that $\Delta + 1$ vertices from level 0 and level 1 are colored as in $k = 3$. Thus, $\Delta + 1$ distinct colors are used to color these $\Delta + 1$ vertices and $\Delta^2 - \Delta$ colors are available for the vertices of level 2. From (1), number of vertices in level 3 = $n_2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$. Hence, each vertex of level 2 can be given a new color and thus making u a s -cdv of color $\Delta + 1$.

Method of assigning colors to vertices of level 2

Consider the $(\Delta^2 - \Delta)$ vertices of level 2 from left to right, i.e., the neighbors of u_1 to u_Δ as a vertex series such as $u_{1,1}, u_{1,2}, \dots, u_{1,\Delta-1}, u_{2,1}, u_{2,2}, \dots, u_{2,\Delta-1}, \dots, u_{\Delta,1}, u_{\Delta,2}, \dots, u_{\Delta,\Delta-1}$.

Since, each u_i has $\Delta - 1$ neighbors, in the vertex series, $u_{i,1}$ is the $[(i - 1)(\Delta - 1) + 1]^{th}$ term. Hence, an arbitrary vertex $u_{i,j}$, $1 \leq i \leq \Delta$, $1 \leq j \leq \Delta - 1$, is the $[(i - 1)(\Delta - 1) + j]^{th}$ term in the vertex series. Now the $(\Delta^2 - \Delta)$ colors available for level 3 form the color series $\Delta + 2, \Delta + 3, \dots, \Delta^2 + 1$. Clearly, these colors are in an arithmetic progression with the first term as $\Delta + 2$ and common difference as 1. Let c map the vertex series to the color series such that it maps the i^{th} term of the vertex series to the i^{th} term of the color series, $1 \leq i \leq \Delta^2 - \Delta$. Hence, vertex $u_{i,j}$, the $[(i - 1)(\Delta - 1) + j]^{th}$ term in the vertex series is mapped to the $[(i - 1)(\Delta - 1) + j]^{th}$ term of the color series. Since, the $[(i - 1)(\Delta - 1) + j]^{th}$ term of the color series is $(i - 1)(\Delta - 1) + j - 1$, for $1 \leq i \leq \Delta$, $1 \leq j \leq \Delta - 1$, c is defined as

$$c(u_{i,j}) = \Delta + 2 + (i - 1)(\Delta - 1) + j - 1$$

i.e., $c(u_{i,j}) = \Delta + j + 1 + (i - 1)(\Delta - 1)$ -----(5)

Thus each vertex from level 0 to level 2 has received a distinct color and u has at most 2-neighbors in the color classes S_i , $1 \leq i \leq \Delta$, $\Delta + 2, \Delta + 3, \dots, \Delta + 1 + j, \dots, \Delta^2 + 1$. This leads to the fact that **u is a $(\Delta + 1)$ - s -cdv**.

Color assigning method to make u_i s -cdv for each i

Let $v \neq u$ be a vertex in level $i < k$. Since, each such v has $\Delta - 1$ neighbors in level $i + 1$, instead of considering the color of a single vertex among the $\Delta - 1$ neighbors, we consider the set of colors assigned to the set $N'(v)$ of all neighbors of v in level $i + 1$. Let for $1 \leq i \leq \Delta$,

$$Y = \{1, 2, \dots, \Delta\} \text{ and } Y_i = Y - \{i\} \quad \text{-----(6)}$$

$$X_i = C(N'(u_i)) \quad \text{-----(7)}$$

As each vertex in level 2 receives a distinct color, X_i is a set of cardinality $\Delta - 1$ and $X_i \cap X_j = \emptyset, i \neq j$. Let

$$R = \{\Delta + 1\} \cup Y \cup (\cup_{j=1}^{j=\Delta} X_j) \quad \text{-----(8)}$$

Now if v to become a s -cdv, then it should have an at most 2-neighbor of color r for each $r \in R - c(v)$. Now consider the vertex u_i and all its descendants up to level 3. With vertices colored up to level 2, u_i has at most 2-neighbors in the following colors.

- (i) In level 0, it has a neighbor u of color $\Delta + 1$;
- (ii) In level 1, from (6), it has a 2-neighbor of color r , for each $r \in Y_i$;
- (iii) In level 2, from (7), it has a neighbor of color r , for each $r \in X_i$.

Since $c(u_i) = i \in Y$, if u_i to become a s -cdv, from (6), it should have an at most 2-neighbor of color r , for each $r \in \cup_{j=1, j \neq i}^{j=\Delta} X_j$. Hence, the 2-neighbors of u_i in level 3 should be colored

accordingly. Now color the vertices of level 3 as described in (9). Let for $1 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 1$,

$$C(N'(u_{i,j})) = \begin{cases} X_j, & \text{if } j < i \\ X_{j+1}, & \text{if } j \geq i \end{cases} \quad \text{-----(9)}$$

From (9), for any i, j , the vertices of $N'(u_{i,j})$ in level 3 are not assigned the colors of X_i , the color set of $N'(u_i)$. Hence, u_i has an at most 2-neighbor of color r for each $r \in R - c(u_i)$. Hence, u_i is an i - s -cdv for each i .

Color assigning method to make $u_{i,j}$ s -cdv for each i, j

Now to make each v of level 2, an r - s -cdv for each color $r \in \cup_{j=1}^{j=\Delta} X_j$. Consider vertex $u_{i,j}$ and its descendants up to level 4. From (9), there are 3 subcases to consider when coloring the vertices of level 4. In all the subcases the at most 2-neighbors up to level 2 are the same. Hence, the following hold for all the subcases.

- (i) In level 0, $u_{i,j}$ has a 2-neighbor u of color $\Delta + 1$;
- (ii) In level 1, it has a neighbor of color i ;
- (iii) In level 2, it has a 2-neighbor of color r , for each $r \in X_i - c(u_{i,j})$.

Subcase (i): $j < i$

From (9), $C(N'(u_{i,j})) = X_j$. Hence, $(iv)'$ in level 3, $u_{i,j}$ has a neighbor of color r , for each $r \in X_j$. If $u_{i,j}$ to become a s -cdv, then it should have an at most 2-neighbor of color r for each $r \in Y_i \cup (\cup_{s=1, s \neq i, j}^{s=\Delta} X_s)$. Color the $\Delta - 1$ groups of 2-neighbors of $u_{i,j}$ as described below.

$$C(N'(u_{i,j,t})) = \begin{cases} Y_i, & \text{if } t = 1 \\ X_{t-1}, & \text{if } 2 \leq t \leq j \\ X_t, & \text{if } j+1 \leq t \leq i-1 \\ X_{t+1}, & \text{if } i \leq t \leq \Delta-1 \end{cases} \quad \text{-----}(10)$$

From (10), for any i, j, t the vertices of $N'(u_{i,j,t})$ in level 4 are neither assigned the colors of X_i , the color set of $N'(u_i)$ nor the colors of X_j , the color set of $N'(u_{i,j})$. Hence, $u_{i,j}$ has an at most 2-neighbor of color r for each $r \in R - c(u_{i,j})$. Hence,

$u_{i,j}$ is a s-cdv for each $i > j, 1 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 1$.

Subcase (ii): $j = i$

From (9), $C(N'(u_{i,j})) = X_{i+1}$. Hence, $(iv)''$ in level 3, $u_{i,j}$ has a neighbor of colors r , for each $r \in X_{i+1}$. If $u_{i,j}$ to become a s-cdv, then it should have an at most 2- neighbor of color r for each $r \in Y_i \cup (\bigcup_{s=1}^{\Delta} X_s)$. Color the $\Delta - 1$ groups of 2-neighbors of $u_{i,j}$ as described below.

$$C(N'(u_{i,j,t})) = \begin{cases} Y_i, & \text{if } t = 1 \\ X_{t-1}, & \text{if } 2 \leq t \leq i \\ X_{t+1}, & \text{if } i+1 \leq t \leq \Delta-1 \end{cases} \quad \text{-----}(11)$$

As in subcase (i), from (11), for any i, j, t the vertices of $N'(u_{i,j,t})$ in level 4 are neither assigned the colors of X_i , nor the colors of X_{i+1} . Hence, $u_{i,j}$ has an at most 2-neighbor of color r for each $r \in R - c(u_{i,j})$. Hence,

$u_{i,j}$ is a s-cdv for each $i = j, 1 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 1$.

Subcase (iii): $j > i$

From (9), $C(N'(u_{i,j})) = X_{j+1}$. Hence, $(iv)'''$ in level 3, $u_{i,j}$ has a neighbor of colors r , for each $r \in X_{j+1}$. If $u_{i,j}$ to become a s-cdv, then it should have an at most 2- neighbor of color r for each $r \in Y_i \cup (\bigcup_{s=1}^{\Delta} X_s)$. Color the $\Delta - 1$ groups of 2-neighbors of $u_{i,j}$ as described below.

$$C(N'(u_{i,j,t})) = \begin{cases} Y_i, & \text{if } t = 1 \\ X_{t-1}, & \text{if } 2 \leq t \leq i \\ X_t, & \text{if } i+1 \leq t \leq j \\ X_{t+1}, & \text{if } j+1 \leq t \leq \Delta-1 \end{cases} \quad \text{-----}(12)$$

As before, from (12), for any i, j, t the vertices of $N'(u_{i,j,t})$ in level 4 are neither assigned the colors of X_i , nor the colors X_{j+1} . Hence, $u_{i,j}$ has an at most 2-neighbor of color r for each $r \in R - c(u_{i,j})$ and hence, **$u_{i,j}$ is a s-cdv** for each $i > j, 1 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 1$.

Then from subcases (i), (ii) and (iii), $u_{i,j}$ is a s-cdv for each $i, j, 1 \leq i \leq \Delta$ and $1 \leq j \leq \Delta - 1$.

Thus, there is an r -s-cdv for each $r \in \bigcup_{j=1}^{\Delta} X_j$ in level 3 and hence, there is a s-cdv for each $r \in R$.

If $k = 4$, then the coloring given to T by the above method is a 2-distance b -coloring with $\Delta^2 + 1$ colors and from claim 2, $\chi_{2sb}(T) = \Delta^2 + 1$.

Suppose $k > 4$. Color the vertices up to level 4 as described above. With the existing coloring procedure, s-cdv exists for each $r \in R$. Hence, it is enough to give a 2-distance coloring to the vertices in level $k \geq 5$. Let v be a vertex in level 4 having a 2-neighbor x in level 2 and a neighbor y in level 3. By the coloring method discussed above, $x \in X_i$ and $y \in X_j$ for some

$i, j, i \neq j$. Since, there are ΔX_r 's, and $\Delta \geq 3$, there is at least one $X_r, r \neq i, j$. Let $C(N'(y)) = X_r$. Repeat this for each vertex of level 5. In a similar way color vertices of the remaining levels. Hence, T is given a 2-distance b -coloring with $\Delta^2 + 1$ colors and the theorem is proved.

5. Conclusion

In this paper, a new rooted tree called a perfect Δ -ary tree is introduced and the exact bounds of the 2-distance strong b -chromatic number ($2sb$ -number) of this tree for height $k \geq 2$ is obtained. It is proved that, $2sb$ -number of this tree for any height $k \geq 2$, can assume the values, $\Delta + 1$ or 2Δ or $\Delta^2 + 1$.

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