CONGRUUM PROBLEM

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Abstract

The ternary quadratic Diophantine equation $X^2 + Z^2 = 2Y^2$ is considered and a few interesting properties among the solutions are presented.

Keywords – Arithmetic progression, Diophantine equation, Integral solution, Congruum.

NOTATIONS USED:

- (i) $P(n) = n(n + 1)$ Pronic number of rank n .
- (ii) $S_n = 6n^2 - 6n + 1$ Star number of rank n .
- (iii) $\mathbf{N}^* = \{0\} \cup \mathbf{N}$ Non negative integers.

INTRODUCTION

Number theory is the branch of Mathematics concerned with studying the properties and relations of integers. In Number Theory, a *congruum* (plural congrua) is the difference between successive square numbers in an arithmetic progression of three squares equally spaced apart from each other. Then the space between them, $Z^2 - Y^2 = Y^2 - X^2$ is called a congruum.

For instance, the number 96 is a *congruum* since it is the difference between each pair of three squares 4, 100 & 196. The first few *congrua* are 24, 96, 120, 216, 240,

336, 384, 480,... If X, Y, Z represents the sides of the Pythagorean triangle then the congruum is itself four times the area of the Pythagorean triangle.

The two right triangles with leg and hypotenuse (7, 13) and (13, 17) have equal third sides of length $\sqrt{120}$. The square of this side, 120, is a congruum: it is the difference between consecutive values in the Arithmetic progression of squares $7^2, 13^2, 17^2$.

Figure 1: A geometrical example of a Congruum

Many mathematicians of the ancient years Fibonacci, Fermat, Oystein Ore etc [1, 4, 6] studied the *congruum* problem. This problem can be formalized as a Diophantine equation $X^2 + Z^2 = 2Y^2$.

Diophantine problems are important because of their rich variety. In this communication, the ternary quadratic Diophantine equation $X^2 + Z^2 = 2Y^2$ is analyzed for various patterns of integer solutions.

METHOD OF ANALYSIS

In *congruum* problem, we want to find a number h such that $Y^2 + h = X^2$ and $Y^2 - h = Z^2$ which gives $2h = X^2 - Z^2 = (X + Z)(X - Z)$. since left hand side is even, $X - Z$ must be both odd or both even. Therefore, $X - Z$ is even, $X - Z = 2k$. Hence $X + Z = \frac{h}{k}$. Therefore $X = \frac{h}{2k} + k$ and $Z = \frac{h}{2k} - k$. which can be written as $Y^2 = \left(\frac{h}{2k}\right)^2 + k^2$.

i.e., $\left(Y, \frac{h}{2k}, k\right)$ forms a Pythagorean triangle with sides $t(m^2 + n^2), 2mnt, t(m^2 - n^2)$.

In this communication, the ternary quadratic Diophantine equation $X^2 + Z^2 = 2Y^2$ is analyzed for various patterns of integer solutions.

Figure 2: Pictorial representation of the equation

Considering the *congruum* problem as a ternary quadratic Diophantine equation with 3 unknown

$$X^2 + Z^2 = 2Y^2 \quad (1)$$

We present below different patterns of integral solution of (1):

A. Pattern

The linear transformation

$$X = u + v, \quad Z = u - v \quad \text{in (1)}$$

leads to the Pythagorean equation

$$u^2 + v^2 = Y^2 \quad (2)$$

with solutions

$$u = p^2 - q^2$$

$$v = 2pq$$

$$Y = p^2 + q^2$$

Thus the solutions of (1) are

$$X = p^2 + 2pq - q^2$$

$$Y = p^2 + q^2$$

$$Z = p^2 - 2pq - q^2$$

Some interesting properties of the above solutions are

- (i) If $p = q$, then $X - Z$ is a perfect square.
- (ii) $3[Y(1, n) - X(1, n)] + 1$ is Star number of rank n .
- (iii) $\frac{X(n,1)+Y(n,1)}{2}$ is a Pronic number of rank n .
- (iv) If $p = q$, then $6(X - Z)$ is a Congruum.
- (v) $3(X + Y + Z)$ is a Nasty number, when $q = p$.

B. Pattern

(1) can be rewritten as

$$X^2 - Y^2 = Y^2 - Z^2 \tag{3}$$

$$\frac{X - Y}{Y - Z} = \frac{Y + Z}{X + Y} = \frac{p}{q} \text{ (say)}$$

Expressing this as a system of simultaneous equations:

$$qX - (p + q)Y + pZ = 0 \tag{4}$$

$$pX + (p - q)Y - qZ = 0 \tag{5}$$

(4) and (5) solving we get

$$X = -p^2 + q^2 + 2pq$$

$$Y = p^2 + q^2$$

$$Z = p^2 - q^2 + 2pq$$

Choice 1:

(3) can be also written as

$$\frac{X - Y}{Y + Z} = \frac{Y - Z}{X + Y} = \frac{p}{q} \text{ (say)}$$

we obtain the solution of (1) as

$$X = p^2 - q^2 - 2pq$$

$$Y = -p^2 - q^2$$

$$Z = p^2 - q^2 + 2pq$$

Choice 2:

(3) can be also written as

$$\frac{X + Y}{Y - Z} = \frac{Y + Z}{X - Y} = \frac{p}{q} \text{ (say)}$$

we obtain the solution of (1) as

$$X = p^2 - q^2 + 2pq$$

$$Y = p^2 + q^2$$

$$Z = p^2 - q^2 - 2pq$$

C. Pattern

The substitution

$$X = u + v, \quad Y = u - v \quad \text{in (1)}$$

gives

$$u = 3v \pm \sqrt{(8v^2 + Z^2)}$$

Assume

$$\alpha^2 = 8v^2 + Z^2 \tag{6}$$

Case (I)

Solving (6), we get

$$v = 2pq$$

$$Z = 8p^2 - q^2$$

$$\alpha = 8p^2 + q^2$$

For the corresponding two values of u , we get the solutions of (1) as

$$X = 8pq + 8p^2 + q^2$$

$$Y = 4pq + 8p^2 + q^2$$

$$Z = 8p^2 - q^2$$

and

$$X = 8pq - 8p^2 - q^2$$

$$Y = 4pq - 8p^2 - q^2$$

$$Z = 8p^2 - q^2$$

Case (II)

Proceeding as in the above case for the choice

$$v = 2pq$$

$$Z = p^2 - 8q^2$$

$$\alpha = p^2 + 8q^2$$

The corresponding solutions of (1) are

$$X = 8pq + p^2 + 8q^2$$

$$Y = 4pq + p^2 + 8q^2$$

$$Z = p^2 - 8q^2$$

and

$$X = 8pq - p^2 - 8q^2$$

$$Y = 4pq - p^2 - 8q^2$$

$$Z = p^2 - 8q^2$$

Case (III)

Letting $Z = 1$, in (6) we get

$$\alpha^2 = 8v^2 + 1 \tag{7}$$

Solving (7) as a Pell's equation, we get

$$u_n = 3v_n \pm \alpha_n$$

where

$$\alpha_n = \frac{1}{2} \left[(3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1} \right]$$

$$v_n = \frac{1}{2\sqrt{8}} \left[(3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1} \right]$$

Sub case I

For the choice of

$$u_n = \frac{3}{2\sqrt{8}} \left[(3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1} \right] + \frac{1}{2} \left[(3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1} \right]$$

We have the sequence of solutions of (1) as

$$X_n = \frac{1}{2\sqrt{8}} \left[(4 + \sqrt{8})(3 + \sqrt{8})^{n+1} - (4 - \sqrt{8})(3 - \sqrt{8})^{n+1} \right]$$

$$Y_n = \frac{1}{2\sqrt{8}} \left[(2 + \sqrt{8})(3 + \sqrt{8})^{n+1} - (2 - \sqrt{8})(3 - \sqrt{8})^{n+1} \right]$$

A few numerical examples are presented in the following table:

X	Y	$h = X^2 - Y^2$ $= Y^2 - Z^2$
$X_0 = 7$	$Y_0 = 5$	24
$X_1 = 41$	$Y_1 = 29$	840
$X_2 = 239$	$Y_2 = 169$	28560
$X_3 = 1393$	$Y_3 = 985$	970224
$X_4 = 8119$	$Y_4 = 5741$	32959080
$X_5 = 47321$	$Y_5 = 33461$	1119638520
$X_6 = 275807$	$Y_6 = 195025$	$3.803475062 \times 10^{10}$

From the above solutions we observe some interesting properties:

- (i) X_n, Y_n, Z_n all are odd .
- (ii) $(X_n, Y_n) = 1$, for all $n \in \mathbf{N}^*$.
- (iii) $X_n Y_n (X_n^2 - Y_n^2)$ is a area of a Pythagorean triangle.
- (iv) $4(X_n^3 Y_n - X_n Y_n^3)$ is a congruum.
- (v) (X_n, Y_n, Z_n) is any one of the solution then,
 $(-X_n, -Y_n, -Z_n), (X_n, Y_n, -Z_n), (X_n, -Y_n, Z_n), (-X_n, Y_n, Z_n), (X_n, -Y_n, -Z_n),$
 $(-X_n, -Y_n, Z_n), (-X_n, Y_n, -Z_n)$, is also solution of above equation.
- (vi) $X_{6(n+1)} - X_{6n} \equiv 0 \pmod{10}$, for all $n \in \mathbf{N}^*$.
- (vii) $X_{2(n+1)} - X_{2n} \equiv 0 \pmod{10}$, for all n such that n and $n + 1 \neq 3m$, for any integer m .
- (viii) $Y_{3n} \equiv 0 \pmod{5}$, for all $n \in \mathbf{N}^*$.
- (ix) $Y_{3(n+1)} - Y_{3n} \equiv 0 \pmod{10}$, for all $n \in \mathbf{N}^*$
- (x) $Y_{n+1} - Y_n \equiv 0 \pmod{10}$, for all n such that $n \equiv 1 \pmod{3}$.

Sub Case (II)

Substituting these values in

$$u_n = 3v_n - \alpha_n$$

Giving the solutions of (1) as,

$$X_n = \frac{1}{2\sqrt{8}} [(4 - \sqrt{8})(3 + \sqrt{8})^{n+1} - (4 + \sqrt{8})(3 - \sqrt{8})^{n+1}]$$

$$Y_n = \frac{1}{2\sqrt{8}} [(2 - \sqrt{8})(3 + \sqrt{8})^{n+1} - (2 + \sqrt{8})(3 - \sqrt{8})^{n+1}]$$

The solutions in the above two sub cases satisfy the following recurrence relations.

- (i) $X_{n+2} - 6X_{n+1} + X_n = 0$
- (ii) $Y_{n+2} - 6Y_{n+1} + Y_n = 0$
- (iii) $Z_n = 1$.

OBSERVATION

Congrua is closely related to congruent numbers which is defined as the area of a Pythagorean triangle with rational sides. Every congruent number is a *congruum* multiplied by a square of a rational number. The problem of three squares in A.P is also closely related to congruent numbers.

CONCLUSION

To Conclude, One may search for other patterns of solutions.

REFERENCES

- [1] L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing Company, New York, 1952.
- [2] Dr. Manju somanath, J. Kannan and Mr. K. Raja, "Gaussian Pythagorean Triples $X^2 + Y^2 = Z^2$ ", International Journal of Engineering Research and Management, Vol. 03 Issue 04 (April 2016).
- [3] Carmichael. R. D, Theory of Numbers and Diophantine Analysis, Dover Publication Inc., New York, 1952.
- [4] Niven. J, Zuckerman and Montgomery, An introduction to the theory of numbers, fifth edition, John We lay, New York 1991.
- [5] M. A. Gopalan, Manju somanath and N.Vanitha, "On Space Pythagorean equation $X^2 + Y^2 + Z^2 = W^2$ ", International Journal of Mathematics, Computer Science and Information Technology Vol. 1, pp129-133, January-June 2007.
- [6] Oystein Ore, Number Theory and its History, Dover publication, New York.
- [7] Conway J H and Guy R K, "The book of Numbers", Springer Science and Business Media, 2006.