Generalized biderivations on prime and semiprime rings

Asma Ali, Faiza Shujat and Kapil Kumar

*Department of Mathematics,
Aligarh Muslim University, Aligarh-202002 (India).*

Abstract

The purpose of our paper is to prove some results concerning symmetric generalized biderivation on prime (semiprime) rings, which are of independent interest. Moreover, we generalize the results of Argac [1, Theorem 3.1, Theorem 3.5].

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1. Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = 0$ implies that $a = 0$). We shall write $[x, y]$ the commutator $xy - yx$. We make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. An additive mapping $d : R \rightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists an element $a \in R$ such that $d(x) = [a, x]$, for all $x \in R$. Gy. Maksa [2] introduced the concept of a symmetric biderivation (see also [3], where an example can be found). It was shown in [2] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [7], [8] and [9]. The notion of commuting mappings is closely connected with the notion of biderivations. Every commuting mapping $f : R \rightarrow R$ gives rise to a biderivation on R . Namely linearizing $[x, f(x)] = 0$, for all $x, y \in R$, $(x, y) \rightarrow [f(x), y]$ is a biderivation (moreover, all derivations appearing are inner). There has been ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations.

A mapping $D(., .) : R \times R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$, where $D(., .) : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of D . It is obvious that in the case $D(., .) : R \times R \longrightarrow R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x + y) = f(x) + f(y) + 2D(x, y)$, for all $x, y \in R$. A biadditive symmetric mapping $D(., .) : R \times R \longrightarrow R$ is called a symmetric biderivation if $D(xy, z) = D(x, z)y + xD(y, z)$, for all $x, y, z \in R$. Obviously, in this case the relation $D(x, yz) = D(x, y)z + yD(x, z)$ is also satisfied for all $x, y, z \in R$. The notion of generalized symmetric biderivations introduced by Nurcan [1], which is defined as follows:

Let R be a ring and $D(., .) : R \times R \longrightarrow R$ be a biadditive map. A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be generalized biderivation if for every $x \in R$, the map $y \longrightarrow \Delta(x, y)$ is a generalized derivation of R associated with function $y \longrightarrow D(x, y)$ as well as if for every $y \in R$, the map $x \longrightarrow \Delta(x, y)$ is a generalized derivation of R associated with function $x \longrightarrow D(x, y)$, for all $x, y \in R$. It also satisfies $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$, for all $x, y, z \in R$. The trace g of Δ is defined as $\Delta(x, x) = g(x)$, which satisfies $g(x + y) = g(x) + g(y) + 2\Delta(x, y)$, for all $x, y \in R$.

2. Main Result

We shall frequently use the following identities and several well known fact about the prime ring without specific mention.

$$[xy, z] = x[y, z] + [x, z]y$$

$$[x, yz] = y[x, z] + [x, y]z$$

$$x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$$

Lemma 2.1. [4] Suppose that

$$\sum_{i=1}^n f_i(z)xa_i + \sum_{j=1}^n c_jzh_j(x) = 0 \text{ for all } x, z \in R,$$

where $a_i, c_j \in R$, $f_i : R \longrightarrow R$ and $h_j : R \longrightarrow RC$ are any maps. If the sets $\{a_1, a_2, \dots\}$ and $\{c_1, c_2, \dots\}$ are C-independent, then there exists $q_{ij} \in Q(S = RC)$

for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n$ such that $f_i(z) = -\sum_{j=1}^n c_jzq_{ij}$, $h_j(x) =$

$$\sum_{i=1}^n q_{ij}xa_i \text{ for all } x, z \in R.$$

Lemma 2.2. [4] If $a_i, b_i \in Q(S)$ satisfy $\sum_{i=1}^n a_i x b_i = 0$ for all $x \in R$, then $a_i's$ and $b_i's$ are C -dependent, unless $a_i = 0$ or all $b_i = 0$.

Lemma 2.3. [3] If $q \in Q(S)$ and q centralizes R , then $q \in C$.

Theorem 2.4. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta([x, y], [u, v]) = 0$ for all $x, y, u, v \in R$, then either R is commutative or Δ acts as a bimultiplier.

Proof. We have

$$\Delta([x, y], [u, v]) = 0 \text{ for all } x, y, u, v \in R. \quad (1)$$

Replacing y by yx in (1), and use (1) to get

$$[x, y]D(x, [u, v]) = 0 \text{ for all } x, y, u, v \in R. \quad (2)$$

Replacing u by uv in (2), we have

$$[x, y][u, v]D(x, v) = 0 \text{ for all } x, y, u, v \in R. \quad (3)$$

Substitute ry for y in (3) to find

$$[x, y]y[u, v]D(x, v) = 0 \text{ for all } x, y, u, v \in R. \quad (4)$$

This implies that $[x, r]D(w, z)y[u, v]D(x, v) = 0$ for all $x, y, u, v, w \in R$. Since R is prime, we have $[x, r]D(w, z) = 0$ for all $x, r, w, z \in R$. Replacing r by rs , we obtain

$$[x, r]sD(w, z) = 0 \text{ for all } x, r, w, z, s \in R. \quad (5)$$

Using primeness of R , we have either $[x, r] = 0$ or $D(w, z) = 0$ for all $x, r, w, z \in R$. If $[x, r] = 0$, then R is commutative. In the later case we get Δ acts as a bimultiplier. ■

Corollary 2.5. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta([x, y], z) = 0$ for all $x, y, z \in R$, then either R is commutative or Δ acts as a bimultiplier.

Theorem 2.6. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta([x, y], u \circ v) = 0$ for all $x, y, u, v \in R$, then either R is commutative or Δ acts as a bimultiplier.

Proof. Suppose that

$$\Delta([x, y], u \circ v) = 0 \text{ for all } x, y, u, v \in R \quad (6)$$

Replacing y by yx , in (6), we get

$$\Delta([x, y], u \circ v)x + [x, y]D(x, u \circ v) = 0 \text{ for all } x, y, u, v \in R \quad (7)$$

Now using (6), we obtain

$$[x, y]D(x, u \circ v) = 0 \text{ for all } x, y, u, v \in R. \quad (8)$$

Again replacing u by uv in (8), we have

$$[x, y]D(x, u \circ v)v + [x, y](u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v \in R.$$

By equation (8), we get

$$[x, y](u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v \in R. \quad (9)$$

Now substituting yr instead of y in (9) and using (9), we find that

$$[x, y]r(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v \in R. \quad (10)$$

Replacing r by $D(w, t)r$ in (10), we have

$$[x, y]D(w, t)r(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v, w, t \in R.$$

Using primeness of R , we have either $[x, y]D(w, t) = 0$ or $(u \circ v)D(x, v) = 0$. Now, first suppose that

$$[x, y]D(w, t) = 0 \text{ for all } x, y, w, t \in R. \quad (11)$$

Substituting ys instead of y in (11) and using (11), we have

$$[x, y]sD(w, t) = 0 \text{ for all } x, y, w, t \in R.$$

As R is prime, then either $[x, y] = 0$ or $D(w, t) = 0$. If $[x, y] = 0$, then R is commutative. In the later case Δ acts as a bimultiplier. In second case suppose

$$(u \circ v)D(x, v) = 0 \text{ for all } x, u, v \in R. \quad (12)$$

Substituting $(u \circ v)t$ instead of $u \circ v$, we obtain

$$(u \circ v)tD(x, v) = 0 \text{ for all } x, u, v, t \in R. \quad (13)$$

Using primeness of R , either $u \circ v = 0$ or $D(x, v) = 0$. If $u \circ v = 0$, then R is commutative. In the later case we get Δ acts as a bimultiplier. ■

Theorem 2.7. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta(x \circ y, u \circ v) = 0$ for all $x, y, u, v \in R$, then either R is commutative or Δ acts as a bimultiplier.

Proof. Suppose that

$$\Delta(x \circ y, u \circ v) = 0 \text{ for all } x, y, u, v \in R. \quad (14)$$

Replacing yx instead of y in (14), we get

$$\Delta(x \circ y, u \circ v)x + (x \circ y)D(x, u \circ v) = 0 \text{ for all } x, y, u, v \in R.$$

Now using (14), we obtain

$$(x \circ y)D(x, u \circ v) = 0, \text{ for all } x, y, u, v \in R. \quad (15)$$

Again replacing uv instead of u in (15), we have

$$(x \circ y)(D(x, (u \circ v)v + (u \circ v)D(x, v)) = 0 \text{ for all } x, y, u, v \in R.$$

This implies that

$$(x \circ y)D(x, (u \circ v)v + (x \circ y)(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v \in R.$$

By equation (15), we get

$$(x \circ y)(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v \in R. \quad (16)$$

Replacing yr for y and using (16), we get

$$[x, y]r(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v, r \in R.$$

Again substituting $D(w, z)r$ instead of r , we have

$$[x, y]D(w, z)r(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v, t \in R.$$

This implies that

$$[x, y]D(w, z)R(u \circ v)D(x, v) = 0 \text{ for all } x, y, u, v, t \in R.$$

Using primeness of R , we have either $[x, y]D(w, z) = 0$ or $(u \circ v)D(x, v) = 0$ for all $x, y, u, v, z, w \in R$. First suppose

$$[x, y]D(w, z) = 0. \quad (17)$$

Now we replace y by ys , then we have $[x, y]sD(w, z) = 0$. Since R is prime then we have either $[x, y] = 0$ or $D(w, z) = 0$. If $[x, y] = 0$ then R is commutative. In the later case Δ acts as bimultiplier. Similarly, we consider second case $(u \circ v)D(x, v) = 0$. Now replacing $(u \circ v)r$ instead of $(u \circ v)$, we obtain

$$(u \circ v)rD(x, v) = 0.$$

Using primeness of R , we have, either $(u \circ v) = 0$ or $D(x, v) = 0$. If $(u \circ v) = 0$ then R is commutative. In the later case we get Δ acts as a bimultiplier. ■

Corollary 2.8. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta(x \circ y, z) = 0$ for all $x, y, z \in R$. then either R is commutative or Δ acts as a bimultiplier.

Theorem 2.9. Let R be a 2-torsion free prime ring and Δ_1, Δ_2 be symmetric generalized biderivations of R with associated biderivations D_1, D_2 respectively. If $a\Delta_1(x, x) = \Delta_2(x, x)a$ for all $x \in R$, then one of the following holds:

1. $a \in C$ then either $a = 0$ or $\Delta_1 = \Delta_2$.
2. There exist $p, q \in Q(S)$ such that $a\Delta_1(x, y) = xpa - axp$, $\Delta_2(x, y) = [x, p]$, $D_1(r, y) = \lambda[r, a]$, $D_2(r, y) = [r, p]$.

Proof. We have

$$a\Delta_1(x, x) = \Delta_2(x, x)a, \text{ for all } x \in R. \quad (18)$$

Linearizing (18) and using the symmetry, we find

$$a\Delta_1(x, y) = \Delta_2(x, y)a, \text{ for all } x, y \in R. \quad (19)$$

Substitute xr for x in (19) to get

$$a\Delta_1(x, y)r + axD_1(r, y) - \Delta_2(x, y)ra - xD_2(r, y)a = 0, \text{ for all } x, y, r \in R. \quad (20)$$

If setting in (20), keeping y as a constant

$\delta_1 = a\Delta_1(x, y)$, $\delta_2 = -\Delta_2(x, y)$, $\beta_1(r) = D_1(r, y)$ and $\beta_2(r) = -D_2(r, y)a$. Then (20) can be written as

$$\sum_{i=1}^2 \delta_i(x)ra_i + \sum_{j=1}^2 c_jx\beta_j(r) = 0 \text{ for all } x, r \in R, \quad (21)$$

where $a_1 = 1$, $a_2 = a$, $c_1 = a$, $c_2 = 1$. Hence we have the following possibilities:

1. if $a \in C$ it implies that either $a = 0$ or $\Delta_1 = \Delta_2$.
2. if $a \notin C$, then $\{1, a\}$ are C -independent. Hence by Lemma 2.1, there exists an element $q_{ij} \in Q(S)$, $i, j = 1, 2$ such that $\delta_j(x) = -\sum_{i=1}^2 c_i x q_{ij}$, $\beta_i(r) =$

$\sum_{j=1}^2 q_{ij}ra_j$ for all $x, r \in R$. Then we have for all $x, y, r \in R$

$$a\Delta_1(x, y) = -xq_{21} - axq_{11}. \quad (22)$$

$$\Delta_2(x, y) = axq_{12} + xq_{22}. \quad (23)$$

$$D_1(r, y) = q_{11}r + q_{12}ra. \quad (24)$$

$$-D_2(r, y)a = q_{21}r + q_{22}ra. \quad (25)$$

Replace x by xu in (22) and using (24) we get

$$x([u, q_{21}]) + ax(uq_{11} + q_{12}ua) = 0. \quad (26)$$

We conclude from (26) that

$$q_{21} \in C \text{ for all } u \in R, \quad (27)$$

and $(uq_{11} + q_{12}ua) = 0$. As $\{1, a\}$ are C -independent. Using Lemma 2.3 and Lemma 2.2, we arrive at

$$q_{12} \in C \quad (28)$$

and

$$q_{12}a = -q_{11} \quad (29)$$

Substitute xr for x in (23), we find

$$\Delta_2(xr, y) = axrq_{12} + xrq_{22} \text{ for all } x, y, r \in R.$$

This gives that

$$\Delta_2(x, y)r + xD_2(r, y) = axrq_{12} + xrq_{22} \text{ for all } x, y, r \in R.$$

By (24), we get

$$(axq_{12} + xq_{22})r + xD_2(r, y) = axrq_{12} + xrq_{22} \text{ for all } x, y, r \in R.$$

Now using (28), we obtain

$$x(D_2(r, y) + [q_{22}, r]) = 0 \text{ for all } r, x, y \in R.$$

Since R is prime, we have

$$D_2(r, y) = [r, q_{22}] \text{ for all } r, x, y \in R. \quad (30)$$

Now comparing (25) and (30), we obtain

$$q_{21} = -q_{22}a. \quad (31)$$

From (24) and (29), we have $D_1(r, y) = q_{12}([r, a])$. Now, combining (23) and (29), we get

$$\Delta_2(x, y) = xq_{22} - q_{11}x. \quad (32)$$

Similarly, we are combining (22) and (32), we get

$$a\Delta_1(x, y) = xq_{22}a - axq_{11}. \quad (33)$$

If we set $q_{12} = \lambda$, $q_{11} = q_{22} = p$ and for all $r, x, y \in R$, then we have the following form:

$$a\Delta_1(x, y) = xpa - axp. \quad (34)$$

$$\Delta_2(x, y) = [x, p]. \quad (35)$$

$$D_1(r, y) = \lambda[r, a]. \quad (36)$$

$$D_2(r, y) = [r, p]. \quad (37)$$

■

Corollary 2.10. Let R be a 2-torsion free prime ring and Δ be symmetric generalized biderivation of R with associated biderivation D . If $\Delta([x, y], [u, v]) = 0$ for all $x, y, u, v \in I$. then either Δ acts as a bimultiplier or $[I, I]I = 0$, where I is a left ideal of ring R .

The following example illustrates that R to be prime is essential in the hypothesis of the above results.

Example 2.11. Let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z}, \text{ ring of integers} \right\}$. Define maps $\Delta, D : R \times R \longrightarrow R$ by

$$D \left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Delta \left(\begin{pmatrix} 0 & a_1 & b_1 \\ 0 & 0 & c_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_2 & b_2 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & a_1 a_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then Δ is a generalized biderivation associated with a biderivation D satisfying hypothesis of the Theorems 2.1, 2.3, 2.4 and Corollaries 2.2 and 2.5. However, R is not commutative.

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