Some Differential Subordination and Superordination Properties Of κ -Symmetric Functions

¹M.Elumalai and ²C.Selvaraj

^{1,2} Presidency College, Chennai-600 005, Tamilnadu, India.

Abstract

In this present paper, we study the interesting properties of differential subordination and superordination for the classes of k -symmetric multivalent analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also we derive sandwich results on the basis of this theory.

Keywords: Multivalent function, Starlike functions, Subordination, superordination, k -symmetric point.

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1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a fixed positive integer m , let $\mathcal{H}[a, m]$ be the subset of $\mathcal{H}$ consisting of functions f of the form $f(z) = a + a_m z^m + a_{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots$.

Also, let $\mathcal{A}_p$ denote the class of analytic functions in $\mathbb{U}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}). \quad (1.1)$$

and let $\mathcal{A} = \mathcal{A}_1$.

For the functions $f(z)$ of the form (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Suppose that h and l are two analytic functions in $\mathbb{U}$, and let

$$\varphi(r, s, t; z) : \mathbb{D}^3 \times \mathbb{U} \rightarrow \mathbb{D}.$$

If h and $\varphi(h(z), zh'(z), z^2 h''(z); z)$, are univalent functions in $\mathbb{U}$ and if h satisfies the second order superordination

$$l(z) \prec \varphi(h(z), zh'(z), z^2 h''(z); z), \quad (1.2)$$

then l is said to be a solution of the differential superordination (1.2). A function q in $\mathcal{H}(\mathbb{U})$ is called a subordinant to (1.2) if $q(z) \prec h(z)$ for all the functions h satisfying condition (1.2).

A univalent subordinant $\tilde{q}$ that satisfies $q(z) \prec \tilde{q}(z)$, for all subordinants q of (1.2), is said to be the best subordinant.

Recently, Miller and Mocanu obtained sufficient conditions on functions l , q and φ for which the following implication holds:

$$l(z) \prec \varphi(h(z), zh'(z), z^2 h''(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, the authors of consider certain classes of first-order differential subordinations (see also [4]), as well as superordination-preserving integral operators [3]. Aouf et al. [2,1] obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent normalized functions in $\mathbb{U}$. Very recently, shanmugam et al. [15, 16] obtained the so-called sandwich results for certain classes of analytic functions. We refer interested readers to the work done by the authors of [2, 10, 11, 14].

For fixed positive integers k , let $f_k(z)$ be defined by the following equality

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{f(\varepsilon^\nu z)}{\varepsilon^\nu}, \quad (\varepsilon = \exp(2\pi i / k); z \in \mathbb{U}). \quad (1.3)$$

The class $S_s^{(k)}$ of functions starlike with respect to k -symmetric points was studied by Chand and Singh [6].

Motivated by Muhammad [9], in the present paper, we investigate properties of differential subordination and superordination for the classes of k -symmetric functions analytic in the unit disc and sandwich results on the basis of this theory.

2. DEFINITIONS AND PRELIMINARIES

We define the following:

Definition: 2.1. A function $f \in \mathcal{A}_p$ is said to be starlike with respect to k -symmetric points in $\mathbb{U}$ if, and only if, it satisfies the condition

$$\operatorname{Re} \left(\frac{zf'(z)}{f_k(z)} \right) > 0, \quad z \in \mathbb{U}. \quad (2.1)$$

We denote by $S_s^{(k)}$ the subclass of $\mathcal{A}_p$ consisting of all functions starlike with respect to k -symmetric points in $\mathbb{U}$.

Definition: 2.2. [8] Denote by $\mathcal{L}$ the set of all functions f that are analytic and injective on $\bar{U} \setminus E(q)$, where $E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$, and such that $f'(\zeta) \neq 0$ for $\zeta \in \bar{U} \setminus E(q)$.

Lemma: 2.1. [7] Let the function $q(z)$ be univalent in the unit disc $\mathbb{U}$, and let θ and φ be analytic in a domain D containing $q(E)$, with $\varphi(w) \neq 0$ when $w \in q(E)$. Set $Q(z) = zq'(z)(q(z))$, $h(z) = \theta(q(z) + Q(z))$ and suppose that

(i) Q is a starlike function in $\mathbb{U}$,

(ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$ for $z \in \mathbb{U}$.

If p is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \quad (2.2)$$

then $p(z) \prec q(z)$, and q is the best dominate of (2.2).

Lemma: 2.2. [16] Let $q(z)$ be a convex univalent function in $\mathbb{U}$ and let $\sigma \in \mathbb{C}$, $\eta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with

$$\mathcal{R}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathcal{R}\left(\frac{\sigma}{\varsigma}\right)\right\}. \quad (2.3)$$

If the function $g(z)$ is analytic in $\mathbb{U}$ and

$$\sigma g(z) + \varsigma z g'(z) \prec \sigma q(z) + \varsigma z q'(z),$$

then $g(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma: 2.3. [16] Let the function $q(z)$ be univalent function in the unit disc $\mathbb{U}$, and let $\mathcal{G}$ and φ be analytic in a domain D containing $q(E)$. Suppose that

- (i) $\operatorname{Re} \frac{\mathcal{G}'(q(z))}{\varphi(q(z))} > 0$ for $z \in \mathbb{U}$,
- (ii) $h(z) = zq'(z)\varphi(q(z))$ is starlike in $\mathbb{U}$.

If $g \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ with $g(\mathbb{U}) \subseteq D$, $\mathcal{G}(g(z) + zg'(z))\varphi(g(z))$ is univalent in $\mathbb{U}$ and

$$\mathcal{G}(q(z)) + zq'(z)\varphi(q(z)) \prec \mathcal{G}(g(z)) + zg'(z)\varphi(g(z)) \quad (2.4)$$

then $g(z) \prec q(z)$, and q is the best dominate of (2.2).

Lemma: 2.4. [8] Let $q(z)$ be a convex function in $\mathbb{U}$ and $\varsigma \in \mathbb{C}$, with $\mathcal{R}(\varsigma) > 0$. If $g(z) \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$ and $g(z) + \varsigma z g'(z)$

is univalent in $\mathbb{U}$, then

$$q(z) + \varsigma z q'(z) \prec g(z) + \varsigma z g'(z) \quad (2.5)$$

implies $q(z) \prec g(z)$ and $q(z)$ is the best subordinate.

Lemma: 2.5. [12] Let function $q(z) = \frac{1}{(1-z)^{2ab}}$ is univalent in $\mathbb{U}$ if and only if $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$.

3. MAIN RESULTS

Theorem: 3.1. Let q be univalent in $\mathbb{U}$, with $q(0) = 1$, and suppose that

$$\mathcal{R}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re} \frac{p}{\delta}\right\}, \quad z \in \mathbb{U}, \quad (3.1)$$

where $\delta \in \mathbb{C}^* = \mathbb{C} - \{0\}$. If $f \in \mathcal{A}_p$ satisfies the subordination

$$(1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right) \prec q(z) + \frac{\delta}{p}zq'(z), \quad (3.2)$$

then

$$\frac{f_k(z)}{z^p} \prec q(z),$$

and $q(z)$ is the best dominant of (3.2).

Proof. Set

$$\frac{f_k(z)}{z^p} = h(z),$$

where $h(z)$ is analytic in $\mathbb{U}$ with $h(0)=1$. A simple computation shows that

$$(1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right) \prec h(z) + \frac{\delta}{p}zh'(z),$$

hence the subordination (3.2) is equivalent to

$$h(z) + \frac{\delta}{p}zh'(z) \prec q(z) + \frac{\delta}{p}zq'(z)$$

Combining this last relation together with Lemma 2.2 for special case $\gamma = \frac{\delta}{p}$ and $\mu = 1$, we obtain our result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.1, where $-1 \leq B < A \leq 1$, the condition (3.1) reduces to

$$\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\operatorname{Re}\frac{p}{\delta}\right\}, \quad z \in \mathbb{U}. \quad (3.4)$$

It is easy to verify that the function

$$\Phi(\xi) = \frac{1-\xi}{1+\xi}, \quad |\xi| \in B,$$

is convex in $\mathbb{U}$, and since $\Phi(\bar{\xi}) = \overline{\Phi(\xi)}$ for all $|\xi| \in B$, it follows that $\Phi(\mathbb{U})$ is a convex domain k -symmetric with respect to the real axis, hence

$$\inf\left\{\operatorname{Re}\left(\frac{1-Bz}{1+Bz}\right) : z \in \mathbb{U}\right\} = \frac{1-|B|}{1+|B|} > 0. \quad (3.4)$$

Then, the inequality (3.4) is equivalent to

$$\operatorname{Re} \frac{p}{\delta} \geq \frac{|B|-1}{|B|+1},$$

hence we have the following result.

Corollary: 3.2. Let $\delta \in \square^*$, $-1 \leq B < A \leq 1$. Suppose also that

$$\max \left\{ 0, -\operatorname{Re} \frac{p}{\delta} \right\} \leq \frac{1-|B|}{1+|B|}$$

If $f \in \mathcal{A}_p$, and satisfies the following subordination:

$$(1-\delta) \left(\frac{f_k(z)}{z^p} \right) + \delta \left(\frac{f_k'(z)}{p z^{p-1}} \right) \prec \frac{1+Az}{1+Bz} + \frac{\delta(A-B)z}{p(1+Bz)^2}, \quad (3.5)$$

then

$$\left(\frac{f_k(z)}{z^p} \right) \prec \frac{1+Az}{1+Bz}$$

and the function $\frac{1+Az}{1+Bz}$ is the best dominant (3.5).

Example: 3.1. Let $\delta \in \square^*$ with $\operatorname{Re} \frac{p}{\delta} \geq 0$. If $f \in \mathcal{A}_p$, and

$$(1-\delta) \left(\frac{f_k(z)}{z^p} \right) + \delta \left(\frac{f_k'(z)}{p z^{p-1}} \right) \prec \frac{1+z}{1+z} + \frac{\delta 2z}{p(1-z)^2}, \quad (3.6)$$

then

$$\left(\frac{f_k(z)}{z^p} \right) \prec \frac{1+z}{1-z}$$

and the function $\frac{1+z}{1-z}$ is the best dominant (3.6).

Theorem: 3.3. Let q be univalent in $\mathbb{U}$, with $q(0)=1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. Let $\varsigma, \sigma \in \square^*$ and $\alpha, \beta \in \square$, with $\alpha + \beta \neq 0$. Let $f \in \mathcal{A}_p$ and suppose that f and q satisfy the following conditions:

$$\frac{2[\alpha z f_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \neq 0, \quad z \in \mathbb{U}, \quad (3.7)$$

and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0, \quad z \in \mathbb{U}. \quad (3.8)$$

If

$$1 + \varsigma \sigma \left[\frac{\alpha z(zf_k'(z))' + \beta zf_k'(z)}{\alpha zf_k'(z) + \beta f_k(z)} - p \right] \prec 1 + \gamma \frac{zq'(z)}{q(z)}, \quad (3.9)$$

then

$$\left[\frac{\alpha zf_k'(z) + \beta zf_k'(z)}{(\alpha + \beta)z} \right]^\sigma \prec q(z),$$

and q is the best dominant of (3.9). The power is the principal one.

Proof: We begin by setting

$$\left[\frac{2[\alpha zf_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \right]^\sigma = h(z), \quad z \in \mathbb{U}, \quad (3.10)$$

where $h(z)$ is analytic in $\mathbb{U}$ with $h(0) = 1$. Differentiating (3.10) logarithmically with respect to z , we have

$$\sigma \left[\frac{\alpha z(zf_k'(z))' + \beta zf_k'(z)}{\alpha zf_k'(z) + \beta f_k(z)} - p \right] \prec \frac{zh'(z)}{h(z)}.$$

To prove our result we use Lemma 2.2. Consider in this lemma

$$\mathcal{G}(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{\sigma}{w},$$

then θ is analytic in $\square$ and $\varphi(w) \neq 0$ is analytic in $\square^*$. Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \sigma \frac{zq'(z)}{q(z)}.$$

and

$$g(z) = \mathcal{G}(q(z)) + Q(z) = 1 + \sigma \frac{zq'(z)}{q(z)},$$

then, since $Q(0) = 0$ and $Q'(0) \neq 0$, the assumption (3.8) would yield that Q is a starlike function in $\mathbb{U}$. From (3.8) we have

$$\operatorname{Re} \frac{zg'(z)}{Q(z)} = \operatorname{Re} \left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0, \quad z \in \mathbb{U},$$

and by using Lemma 2.2 we deduce that the subordination (3.9) implies $h(z) \prec q(z)$, and the function q is the best dominate of (3.9).

In particular, with $\alpha = 0$, $\beta = \sigma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.3, it is easy to see that the assumption (3.8) holds whenever $-1 \leq B < A \leq 1$, which leads to the next result:

Corollary: 3.4. Let $-1 \leq B < A \leq 1$ and $\sigma \in \mathbb{R}^*$. Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \neq 0$ for $z \in \mathbb{U}$. If

$$1 + \sigma \left[\frac{zf_k'(z)}{f_k(z)} - 1 \right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \quad (3.11)$$

then

$$\left(\frac{zf_k(z)}{z^p} \right)^\sigma \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.11). The power is the principal one.

Putting $\alpha = 0$, $\beta = 1$, $\varsigma = \frac{1}{ab}$ with $a, b \in \mathbb{R}^*$, $\sigma = a$, and $q(z) = \frac{1}{(1-z)^{2ab}}$ in Theorem 3.3, then combining this together with Lemma 2.5, we have the next result.

Corollary: 3.5. Let $a, b \in \mathbb{R}^*$ such that $|2ab-1| \leq 1$ or $|2ab+1| \leq 1$. Let $f \in \mathcal{A}_p$ and let $\frac{f_k(z)}{z^p} \neq 0$ for all $z \in \mathbb{U}$. If

$$1 + \frac{1}{b} \left(\frac{zf_k'(z)}{f_k(z)} - 1 \right) \prec \frac{(1+z)}{(1-z)}, \quad (3.12)$$

then

$$\left(\frac{zf_k(z)}{z^p} \right)^a \prec \frac{1}{(1-z)^{2ab}},$$

and $\frac{1}{(1-z)^{2ab}}$ is the best dominant of (3.12). The power is the principal one.

Putting $\alpha = 0$, $\beta = \varsigma = 1$ and $q(z) = (1+Bz)^{\sigma(A-B)/B}$, $-1 \leq B < A \leq 1$, $B \neq 0$ in Theorem 3.3, and using Lemma 2.5, we have the next result.

Corollary: 3.6. Let $-1 \leq B < A \leq 1$ with $B \neq 0$, and suppose that

$$\left| \frac{\sigma(A-B)}{B-1} \right| \leq 1 \quad \text{or} \quad \left| \frac{\sigma(A-B)}{B+1} \right| \leq 1.$$

Let $f \in \mathcal{A}_p$ such that $\frac{f_k(z)}{z^p} \neq 0$ for all $z \in \mathbb{U}$, and let $\sigma \in \mathbb{C}^*$. If

$$1 + \sigma \left(\frac{zf_k'(z)}{f_k(z)} - p \right) \prec 1 + \frac{1 + [B + \sigma(A-B)]z}{1 + Bz}, \quad (3.13)$$

then

$$\left(\frac{zf_k(z)}{z^p} \right)^\sigma \prec (1 + Bz)^{\sigma(A-B)/B},$$

and $(1 + Bz)^{\sigma(A-B)/B}$ is the best dominant of (3.12). The power is the principal one.

By taking $\alpha = 0$, $\beta = 1$, $\varsigma = \frac{e^{i\lambda}}{ab \cos \lambda}$ with $a, b \in \mathbb{C}^*$, $|\lambda| < \pi/2$, $\sigma = a$, and $q(z) = (1 - z)^{2abe^{-i\lambda} \cos \lambda}$ in Theorem 3.3, we obtain the following result.

Corollary: 3.7. Let $a, b \in \mathbb{C}^*$ and $|\lambda| < \pi/2$, and suppose that $|abe^{-i\lambda} \cos \lambda - 1| \leq 1$ or $|abe^{-i\lambda} \cos \lambda + 1| \leq 1$. Let $f \in \mathcal{A}_p$ be such that $\frac{f_k(z)}{z^p} \neq 0$ for all $z \in \mathbb{U}$. If

$$1 + \frac{e^{i\lambda}}{b \cos \lambda} \left(\frac{zf_k'(z)}{f_k(z)} - p \right) \prec \frac{(1+z)}{(1-z)}, \quad (3.14)$$

then

$$\left(\frac{zf_k(z)}{z^p} \right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}},$$

and $\frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}}$ is the best dominant of (3.14). The power is the principal one.

Theorem: 3.8. Let q be univalent in $\mathbb{U}$, with $q(0) = 1$. Let $\varsigma, \sigma \in \mathbb{C}^*$ and let $\delta, \Omega, \alpha, \beta \in \mathbb{C}$ with $\alpha + \beta \neq 0$. Let $f \in \mathcal{A}_p$ and suppose that f and q satisfy the next two conditions:

$$\frac{2[\alpha zf_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \neq 0, \quad z \in \mathbb{U}, \quad (3.15)$$

and

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\frac{\delta}{\varsigma}\right\}, \quad z \in \mathbb{U}. \quad (3.16)$$

If

$$\Psi(z) = \left[\frac{2[\alpha zf_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \right]^\sigma \times \left[\delta + \varsigma \sigma \left(\frac{\alpha z(zf_k'(z))' + \beta zf_k'(z)}{\alpha zf_k'(z) + \beta f_k(z)} - p \right) \right] + \Omega, \quad (3.17)$$

and

$$\Phi(z) \prec \delta q(z) + \varsigma zq'(z) + \Omega, \quad (3.18)$$

then

$$\left[\frac{\alpha zf_k'(z) + \beta f_k(z)}{(\alpha + \beta)z} \right]^\sigma \prec q(z),$$

and q is the best dominant of (3.18). The power is the principal one.

Proof: We begin by setting

$$\left[\frac{2[\alpha zf_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \right]^\sigma = h(z), \quad z \in \mathbb{U}, \quad (3.19)$$

where $h(z)$ is analytic in $\mathbb{U}$ with $h(0) = 1$. Differentiating (3.10) logarithmically with respect to z , we have

$$\sigma \left[\frac{\alpha z(zf_k'(z))' + \beta zf_k'(z)}{\alpha zf_k'(z) + \beta f_k(z)} - p \right] \prec \frac{zh'(z)}{h(z)}.$$

and hence

$$\sigma h(z) \left[\frac{\alpha z(zf_k'(z))' + \beta zf_k'(z)}{\alpha zf_k'(z) + \beta f_k(z)} - p \right] \prec zh'(z).$$

Let us consider the functions:

$$\mathcal{G}(w) = \delta w + \Omega, \quad \varphi(w) = \varsigma, \quad w \in \square,$$

$$Q(z) = zq'(z)\varphi(q(z)) = \varsigma zq'(z), \quad z \in \mathbb{U},$$

and

$$g(z) = \mathcal{G}(q(z)) + Q(z) = \delta q(z) + \varsigma zq'(z) + \Omega, \quad z \in \mathbb{U}.$$

From the assumption (3.16), we see that Q is a starlike function in $\mathbb{U}$ that

$$\operatorname{Re} \frac{zg'(z)}{Q(z)} = \operatorname{Re} \left(\frac{\delta}{\varsigma} + 1 + \frac{zq''(z)}{q'(z)} \right) > 0, \quad z \in \mathbb{U}.$$

Thus, applying Lemma 2.2, this completes the proof.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.3, where $-1 \leq B < A \leq 1$ and according to (3.4) the condition (3.16) becomes

$$\max \left\{ 0, -\operatorname{Re} \frac{\delta}{\varsigma} \right\} \leq \frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\alpha = 1 = \varsigma$, $\beta = 0$, we have the following result:

Corollary: 3.9. Let $-1 \leq B < A \leq 1$ and $\delta \in \mathbb{C}$ with

$$\max \{0, -\operatorname{Re} \delta\} \leq \frac{1-|B|}{1+|B|}.$$

Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \neq 0$ for $z \in \mathbb{U}$ and let $\sigma \in \mathbb{C}^*$. If

$$[f_k'(z)]^\sigma \left[\delta + \sigma \left(\frac{zf_k'(z)}{f_k(z)} - p \right) \right] + \Omega \prec \delta \frac{1+Az}{1+Bz} + \Omega + \frac{z(A-B)}{(1+Bz)^2}, \quad (3.20)$$

then

$$[f_k'(z)]^\sigma \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.20). All the powers are the principal ones.

Taking $\beta = 1 = \varsigma$, $\alpha = 0$, and $q(z) = \frac{1+z}{1-z}$ in Theorem 3.3, we obtain:

Corollary: 3.10. Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \neq 0$ for $z \in \mathbb{U}$ and let $\sigma \in \mathbb{C}^*$.

If

$$\left[\frac{f_k(z)}{z^p} \right]^\sigma \left[\delta + \sigma \left(\frac{zf_k'(z)}{f_k(z)} - p \right) \right] + \Omega \prec \delta \frac{1+z}{1-z} + \Omega + \frac{2z}{(1-z)^2}, \quad (3.21)$$

then

$$\left[\frac{f_k(z)}{z^p} \right]^\sigma \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.21). All the powers are the principal ones.

Theorem: 3.11. Let q be convex in $\mathbb{U}$, with $q(0)=1$, let $\lambda \in \mathbb{C}^*$ with $\operatorname{Re} \lambda > 0$. Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$. If the function

$$(1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right)$$

is p -valent in the unit disc $\mathbb{U}$, and

$$q(z) + \frac{\delta}{p} z q'(z) \prec (1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right), \quad (3.22)$$

then

$$q(z) \prec \frac{f_k(z)}{z^p},$$

and $q(z)$ is the best subordinate of (3.22).

Proof: Set

$$\frac{f_k(z)}{z^p} = h(z), \quad z \in \mathbb{U}.$$

Then $h(z)$ is analytic in $\mathbb{U}$ with $h(0)=1$. Taking logarithmic differentiation with respect z , we have

$$z\left(\frac{f_k'(z)}{f_k(z)}\right) - 1 = \frac{zh'(z)}{h(z)}. \quad (3.23)$$

A simple computations show that

$$h(z) + \frac{\delta}{p} zh'(z) \prec (1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right),$$

and now, by using Lemma 2.5, we obtain the desired result.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 3.11, where $-1 \leq B < A \leq 1$, we obtain the next result.

Corollary: 3.12. Let q be convex in $\mathbb{U}$, with $q(0)=1$, let $\lambda \in \mathbb{C}^*$ with $\operatorname{Re} \lambda > 0$. Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$. If the function

$$(1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right)$$

is p -valent in the unit disc $\mathbb{U}$, and

$$\frac{1+Az}{1+Bz} + \frac{\lambda(A-B)z}{(1+Bz)^2} \prec (1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right), \quad (3.24)$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{f_k(z)}{z^p},$$

and $\frac{1+Az}{1+Bz}$ is the best subordinate of (3.24), where $-1 \leq B < A \leq 1$.

Using the same techniques as we used in Theorem 3.8, and then applying Lemma 2.3, we have the following theorem.

Theorem: 3.13. Let q be convex in $\mathbb{U}$, with $q(0)=1$ suppose that $\zeta, \sigma \in \mathbb{C}^*$ and let $\delta, \Omega, \alpha, \beta \in \mathbb{C}$ with $\zeta + \sigma \neq 0$ and $\operatorname{Re} \frac{\delta}{\zeta} > 0$.

Let $f \in \mathcal{A}_p$ and suppose that f and q satisfy the next two conditions:

$$\frac{2[\alpha z f_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \neq 0, \quad z \in \mathbb{U},$$

and

$$\left[\frac{2[\alpha z f_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \right]^\sigma \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}.$$

If the function Ψ given by equation (3.17) is p -valent in $\mathbb{U}$, and

$$\delta q(z) + \zeta z q'(z) + \Omega \prec \Phi(z), \quad (3.25)$$

then

$$q(z) \prec \left[\frac{\alpha z f_k'(z) + \beta z f_k(z)}{(\alpha + \beta)z} \right]^\sigma,$$

and $q(z)$ is the best subordinate of (3.25). The power is the principal ones.

Observe that by combining Theorem 3.1 with Theorem 3.11 and Theorem 3.8 with Theorem 3.13, respectively, we have the next two sandwich theorems. In particular:

Theorem: 3.14. Let q_1, q_2 be two convex functions in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, let $\delta \in \mathbb{C}^*$ with $\operatorname{Re} \delta > 0$. Let $f \in \mathcal{A}_p$ and suppose that $\frac{f_k(z)}{z^p} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$. If the function

$$(1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right)$$

is p -valent in the unit disc $\mathbb{U}$, and

$$q_1(z) + \frac{\delta}{p} z q_1'(z) \prec (1-\delta)\left(\frac{f_k(z)}{z^p}\right) + \delta\left(\frac{f_k'(z)}{pz^{p-1}}\right) \prec q_2(z) + \frac{\delta}{p} z q_2'(z) \quad (3.26)$$

then

$$q_1(z) \prec \frac{f_k(z)}{z^p} \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinate and the best dominate of (3.26).

Theorem: 3.15. Let q_1, q_2 be convex in $\mathbb{U}$, with $q_1(0) = q_2(0) = 1$, suppose that $\varsigma, \sigma \in \mathbb{C}^*$ and let $\delta, \Omega, \alpha, \beta \in \mathbb{C}$ with $\varsigma + \sigma \neq 0$ and $\operatorname{Re} \frac{\delta}{\varsigma} > 0$.

Let $f \in \mathcal{A}_p$ satisfy the following conditions:

$$\frac{2[\alpha z f_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \neq 0, \quad z \in \mathbb{U},$$

and

$$\left[\frac{2[\alpha z f_k'(z) + \beta f_k(z)]}{(\alpha + \beta)z} \right]^\sigma \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}.$$

If the function Ψ given by equation (3.17) is p -valent in $\mathbb{U}$, and

$$\delta q_1(z) + \varsigma z q_1'(z) + \Omega \prec \Phi(z) \delta q_2(z) + \varsigma z q_2'(z) + \Omega, \quad (3.27)$$

then

$$q_1(z) \prec \left[\frac{\alpha z f_k'(z) + \beta z f_k(z)}{(\alpha + \beta)z} \right]^\sigma \prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinate and the best dominate of (3.25). The power is the principal ones.

Remark: 3.1. If we let $p = 1, k = 2$, in this paper reduce to the results of Muhammad [9].

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