On the analysis of Hilbert module-valued functions on LCA groups

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Abstract

Let $\mathcal{M}$ be a Hilbert module over a C^* -algebra $\mathcal{A}$. This paper addresses some facts about functions in the space $L_2(G, \mathcal{M})$ of Bochner-square integrable Hilbert module-valued functions on the locally compact abelian group G . We construct an action of $\mathcal{A}$ over $L_2(G, \mathcal{M})$ and endow the latter with a pre-Hilbert module structure. We also prove some results of Parseval type.

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1. Introduction

First introduced by I. Kaplansky in 1953 [5], Hilbert C^* -modules are natural generalization of Hilbert spaces. They are linear spaces endowed with an inner product which values occur in a C^* -algebra. The subject has been deeply explored and extended by W. L. Paschke [10], M. Rieffel [11], Kasparov [6]. Nowadays, Hilbert C^* -module is intensely present in research [1], [4], [14], [15] and applications particularly in harmonic analysis [9]. The main purpose of this paper is to contribute to consolidate the link between Hilbert C^* -modules and Harmonic analysis. We are mainly interested in some properties of Hilbert C^* -module valued functions on locally compact abelian groups in connection with the Fourier transform.

The paper is organized as follow. The next section recalls some definitions and properties of notions like Hilbert module and Pontryagin duality. The section 3 is devoted to our main results.

2. Preliminaries

2.1. Hilbert modules

If $\mathcal{M}$ is a module over $\mathcal{A}$, the action of an element $a \in \mathcal{A}$ on an element x of $\mathcal{M}$ is denoted by $x \cdot a$.

Definition 2.1. A pre-Hilbert module over a C^* -algebra $\mathcal{A}$ is a (right) $\mathcal{A}$ -module $\mathcal{M}$ equipped with a sesquilinear map $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$ with the following properties:

1. $\forall x, y, z \in \mathcal{M}, \forall \alpha, \beta \in \mathbb{C}, \langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle,$
2. $\forall x, y \in \mathcal{M}, \forall a \in \mathcal{A}, \langle x, y \cdot a \rangle = \langle x, y \rangle a,$
3. $\forall x, y \in \mathcal{M}, \langle y, x \rangle = \langle x, y \rangle^*,$
4. $\forall x \in \mathcal{M}, \langle x, x \rangle \geq 0$ and if $\langle x, x \rangle = 0$ then $x = 0$.

The map $\langle \cdot, \cdot \rangle$ is called an $\mathcal{A}$ -valued inner product.

Let $\mathcal{M}$ be a pre-Hilbert module over $\mathcal{A}$ endowed with a norm $\|\cdot\|_{\mathcal{A}}$. For $x \in \mathcal{M}$, set $\|x\|_{\mathcal{M}} = \|\langle x, x \rangle\|_{\mathcal{A}}^{\frac{1}{2}}$.

Proposition 2.2. The map $x \mapsto \|x\|_{\mathcal{M}}$ is a norm on $\mathcal{M}$ and satisfies the following properties:

1. $\|x \cdot a\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \|a\|_{\mathcal{A}}$ for all $x \in \mathcal{M}, a \in \mathcal{A}$;
2. $\|\langle x, y \rangle\|_{\mathcal{A}} \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ for all $x, y \in \mathcal{M}$.

If $\mathcal{M}$ is a Banach space under the norm $\|\cdot\|_{\mathcal{M}}$, then $\mathcal{M}$ is called a Hilbert $\mathcal{A}$ -module (or Hilbert C^* -module over $\mathcal{A}$).

Here are some simple examples of Hilbert modules.

Example 2.3.

1. Every complex Hilbert space is a Hilbert C^* -module over $\mathbb{C}$.
2. Every C^* -algebra $\mathcal{A}$ is a Hilbert module over $\mathcal{A}$. The $\mathcal{A}$ -valued inner product is given by $\langle a, b \rangle = a^*b, \forall a, b \in \mathcal{A}$. A concrete example is $\mathcal{B}(H)$ the set of bounded operators on a Hilbert space H .

2.2. LCA group, Pontryagin duality and Fourier transform

Let G be a locally compact abelian group and λ its Haar measure. We denote by $\widehat{G}$ the dual group of G , which is group of all continuous group homomorphisms $\chi : G \rightarrow \mathbb{T}$ where $\mathbb{T}$ is the circle group, [2]. These group homomorphisms are called characters of G . The dual group $\widehat{G}$ is an abelian group under the point-wise multiplication

$$(\chi \cdot \eta)(g) = \chi(g) \cdot \eta(g), \text{ for all } g \in G.$$

The inverse element to a given character $\chi \in \widehat{G}$ is defined by

$$\chi^{-1}(g) = (\chi(g))^{-1} = \overline{\chi(g)}, \text{ for all } g \in G.$$

It is well-known that $\widehat{G}$ carries a natural topology called the compact-open topology [7], [13] under which $\widehat{G}$ is a locally compact group. Thereby, it admits a Haar measure which we denote Λ .

We denote $\widehat{\widehat{G}}$ the dual group of $\widehat{G}$. The Pontryagin duality states that there exists a canonical and topological isomorphism of groups between G and $\widehat{\widehat{G}}$. This isomorphism is given by

$$\delta : G \rightarrow \widehat{\widehat{G}}, \quad g \mapsto \delta_g$$

where

$$\delta_g : \widehat{G} \rightarrow \mathbb{C}, \quad \chi \mapsto \delta_g(\chi) = \chi(g).$$

If G is a compact abelian group, [3] we have for all $\chi, \eta \in \widehat{G}$,

$$\langle \chi, \eta \rangle_{\widehat{G}} = \int_G \chi(g) \overline{\eta(g)} dg = \begin{cases} 1 & \text{if } \chi = \eta \\ 0 & \text{otherwise} \end{cases}$$

If $\widehat{G}$ is a compact abelian group, we have for all $u, v \in G$,

$$\langle \delta_u, \delta_v \rangle_{\widehat{\widehat{G}}} = \int_{\widehat{G}} \delta_u(\chi) \overline{\delta_v(\chi)} d\Lambda(\chi) = \delta_{u,v}$$

where $\delta_{u,v}$ denotes Kronecker delta.

We denote respectively $L_1(G, \mathcal{M})$, $L_2(G, \mathcal{M})$ and $L_2(\widehat{G}, \mathcal{M})$ the space of the Bochner-integrable $\mathcal{M}$ -valued functions on G , the space of Bochner-square integrable $\mathcal{M}$ -valued functions on G and the space of Bochner-square integrable $\mathcal{M}$ -valued functions on $\widehat{G}$.

Let $f \in L_1(G, \mathcal{M})$, the Fourier transform of f is defined by

$$\widehat{f}(\chi) = \int_G f(u) \overline{\chi(u)} d\lambda(u) \text{ for all } \chi \in \widehat{G}.$$

3. Main results

In this section G is a locally compact abelian group unless otherwise stated. For $f, g \in L_2(G, \mathcal{M})$. Set

$$\langle f, g \rangle_{L_2(G, \mathcal{M})} = \int_G \langle f(t), g(t) \rangle d\lambda(t).$$

Likewise, for $\psi, \phi \in L_2(\widehat{G}, \mathcal{M})$, set

$$\langle \phi, \psi \rangle_{L_2(\widehat{G}, \mathcal{M})} = \int_{\widehat{G}} \langle \phi(\chi), \psi(\chi) \rangle d\Lambda(\chi).$$

On the other hand, for all $f \in L_1(G, \mathcal{M})$ and $a \in \mathcal{A}$, set

$$f \cdot a : G \rightarrow \mathcal{M}, \quad t \mapsto f(t)a.$$

And for all $\phi \in L_1(\widehat{G}, \mathcal{M})$ and $a \in \mathcal{A}$, set

$$\phi \cdot a : \widehat{G} \rightarrow \mathcal{M}, \quad \chi \mapsto \phi(\chi)a.$$

Proposition 3.1. The map $\mathcal{A} \times L_2(G, \mathcal{M}) \rightarrow L_2(G, \mathcal{M})$, $(a, f) \mapsto f \cdot a$ is an action of $\mathcal{A}$ on $L_2(G, \mathcal{M})$.

Proof. Let $f \in L_2(G, \mathcal{M})$ and $a \in \mathcal{A}$, we have $f \cdot a \in L_2(G, \mathcal{M})$ since

$$\begin{aligned} \int_G \|(f \cdot a)(t)\|_{\mathcal{M}}^2 d\lambda(t) &= \int_G \|f(t)a\|_{\mathcal{M}}^2 d\lambda(t) \\ &= \int_G \|\langle f(t)a, f(t)a \rangle\|_{\mathcal{A}} d\lambda(t) \\ &= \int_G \|a^* \langle f(t), f(t) \rangle a\|_{\mathcal{A}} d\lambda(t) \\ &\leq \int_G \|a^*\|_{\mathcal{A}} \|f(t)\|_{\mathcal{M}}^2 \|a\|_{\mathcal{A}} d\lambda(t) \\ &= \int_G \|f(t)\|_{\mathcal{M}}^2 \|a\|_{\mathcal{A}}^2 d\lambda(t) \\ &= \left(\int_G \|f(t)\|_{\mathcal{M}}^2 d\lambda(t) \right) \|a\|_{\mathcal{A}}^2 < +\infty. \end{aligned}$$

Furthermore, for any $t \in G$ and $a, b \in \mathcal{A}$ we have

$$((f \cdot a) \cdot b)(t) = (f \cdot a)(t)b = (f(t)a)b = f(t)(ab) = (f \cdot (ab))(t).$$

Hence $(f \cdot a) \cdot b = f \cdot (ab)$. ■

Proposition 3.2. The map $L_2(G, \mathcal{M}) \times L_2(G, \mathcal{M}) \rightarrow \mathcal{A}$, $(f, g) \mapsto \langle f, g \rangle_{L_2(G, \mathcal{M})}$ is an $\mathcal{A}$ -valued inner product.

Proof. Let $f, g \in L_2(G, \mathcal{M})$ and $a \in \mathcal{A}$.

1. Then $\|f(\cdot)\|_{\mathcal{M}}$ and $\|g(\cdot)\|_{\mathcal{M}}$ belong to $L_2(G, \mathbb{C})$, so $\|f(\cdot)\|_{\mathcal{M}}\|g(\cdot)\|_{\mathcal{M}} \in L_1(G, \mathbb{C})$.

We have

$$\int_G \|\langle f(t), g(t) \rangle\|_{\mathcal{A}} d\lambda(t) \leq \int_G \|f(t)\|_{\mathcal{M}} \|g(t)\|_{\mathcal{M}} d\lambda(t) < +\infty. \text{ Hence } \langle \cdot, \cdot \rangle_{L_2(G, \mathcal{M})} \text{ is well defined.}$$

2. Let $\alpha, \beta \in \mathbb{C}$. We have

$$\langle f, \alpha g + \beta h \rangle_{L_2(G, \mathcal{M})} = \int_G \langle f(t), \alpha g(t) + \beta h(t) \rangle d\lambda(t) = \alpha \langle f, g \rangle_{L_2(G, \mathcal{M})} + \beta \langle f, h \rangle_{L_2(G, \mathcal{M})}.$$

3. We have

$$\langle f, g \cdot a \rangle_{L_2(G, \mathcal{M})} = \int_G \langle f(t), g(t)a \rangle d\lambda(t) = \int_G \langle f(t), g(t) \rangle a d\lambda(t) = \langle f, g \rangle_{L_2(G, \mathcal{M})} a.$$

4. We have

$$\begin{aligned} \langle f, g \rangle_{L_2(G, \mathcal{M})} &= \int_G \langle f(t), g(t) \rangle d\lambda(t) = \int_G \langle g(t), f(t) \rangle^* d\lambda(t) \\ &= \left(\int_G \langle g(t), f(t) \rangle d\lambda(t) \right)^* = \langle g, f \rangle_{L_2(G, \mathcal{M})}^*. \end{aligned}$$

5. We have $\langle f, f \rangle_{L_2(G, \mathcal{M})} = \int_G \langle f(t), f(t) \rangle d\lambda(t) \geq 0$ since $\forall t \in G, \langle f(t), f(t) \rangle \geq 0$.

0. Furthermore; let $f \in L_2(G, \mathcal{M})$ such that $\int_G \langle f(t), f(t) \rangle d\lambda(t) = 0$, then $t \mapsto \langle f(t), f(t) \rangle$ is null λ -a.e., Therefore $f = 0$ λ -a.e.. So $\langle f, f \rangle_{L_2(G, \mathcal{M})} = 0$ gives $f = 0$ for $f \in L_2(G, \mathcal{M})$. ■

Corollary 3.3. The space $L_2(G, \mathcal{M})$ equipped with the above $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle_{L_2(G, \mathcal{M})}$ is a pre-Hilbert module over $\mathcal{A}$.

Proof. We only need to show that $L_2(G, \mathcal{M})$ is a complex vector space.

Let $f, g \in L_2(G, \mathcal{M})$, we have

$$\begin{aligned} \int_G \|(f+g)(t)\|_{\mathcal{M}}^2 d\lambda(t) &= \int_G \|\langle f(t) + g(t), f(t) + g(t) \rangle\|_{\mathcal{A}} d\lambda(t) \\ &\leq \int_G \|\langle f(t), f(t) \rangle\|_{\mathcal{A}} d\lambda(t) + \int_G \|\langle f(t), g(t) \rangle\|_{\mathcal{A}} d\lambda(t) \\ &\quad + \int_G \|\langle g(t), f(t) \rangle\|_{\mathcal{A}} d\lambda(t) + \int_G \|\langle g(t), g(t) \rangle\|_{\mathcal{A}} d\lambda(t). \end{aligned}$$

The integral

$$\int_G \|\langle f(t), f(t) \rangle\|_{\mathcal{A}} d\lambda(t) = \int_G \|f(t)\|_{\mathcal{M}}^2 d\lambda(t)$$

is finite. And it is the same to

$$\int_G \|\langle g(t), g(t) \rangle\|_{\mathcal{A}} d\lambda(t).$$

By Hölder's inequality,

$$\begin{aligned} \int_G \|\langle f(t), g(t) \rangle\|_{\mathcal{A}} d\lambda(t) &\leq \int_G \|f(t)\|_{\mathcal{M}} \|g(t)\|_{\mathcal{M}} d\lambda(t) \\ &\leq \left(\int_G \|f(t)\|_{\mathcal{M}}^2 d\lambda(t) \right)^{\frac{1}{2}} \left(\int_G \|g(t)\|_{\mathcal{M}}^2 d\lambda(t) \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

Similary

$$\int_G \|\langle g(t), f(t) \rangle\|_{\mathcal{A}} d\lambda(t) < +\infty.$$

Hence

$$\int_G \|(f+g)(t)\|_{\mathcal{M}}^2 d\lambda(t) < +\infty \text{ and } f+g \in L_2(G, \mathcal{M}).$$

Moreover for all $f \in L_2(G, \mathcal{M})$ and $\alpha \in \mathbb{C}$, $\alpha f \in L_2(G, \mathcal{M})$. ■

Proposition 3.4. The map $\mathcal{A} \times L_2(\widehat{G}, \mathcal{M}) \rightarrow L_2(\widehat{G}, \mathcal{M})$, $(a, \phi) \mapsto \phi \cdot a$ is an action of $\mathcal{A}$ on $L_2(\widehat{G}, \mathcal{M})$.

Proof. Let $\phi \in L_2(\widehat{G}, \mathcal{M})$ and $a \in \mathcal{A}$. We have

$$\begin{aligned} \int_{\widehat{G}} \|(\phi \cdot a)(\chi)\|_{\mathcal{M}}^2 d\Lambda(\chi) &= \int_{\widehat{G}} \|\phi(\chi)a\|_{\mathcal{M}}^2 d\Lambda(\chi) \\ &\leq \int_{\widehat{G}} \|\phi(\chi)\|_{\mathcal{M}}^2 \|a\|_{\mathcal{A}}^2 d\Lambda(\chi) \\ &= \left(\int_{\widehat{G}} \|\phi(\chi)\|_{\mathcal{M}}^2 d\Lambda(\chi) \right) \|a\|_{\mathcal{A}}^2 < +\infty \end{aligned}$$

Moreover, if $\phi \in L_2(\widehat{G}, \mathcal{M})$ and $a, b \in \mathcal{A}$, then $((\phi \cdot a) \cdot b)(\chi) = ((\phi \cdot a)(\chi))b = ((\phi(\chi)a)b = \phi(\chi)(ab) = (\phi \cdot (ab))(\chi)$ for all $\chi \in \widehat{G}$. ■

Proposition 3.5. The map $L_2(\widehat{G}, \mathcal{M}) \times L_2(\widehat{G}, \mathcal{M}) \rightarrow \mathcal{A}$, $(\phi, \psi) \mapsto \langle \phi, \psi \rangle_{L_2(\widehat{G}, \mathcal{M})}$ is an $\mathcal{A}$ -valued inner product.

Proof. It is similar to the proof of the Proposition 3.2. ■

Corollary 3.6. The space $L_2(\widehat{G}, \mathcal{M})$ equipped with the above $\mathcal{A}$ -valued inner product $\langle \cdot, \cdot \rangle_{L_2(\widehat{G}, \mathcal{M})}$ is a pre-Hilbert module over $\mathcal{A}$.

Proof. The proof is the same as that of the corollary 3.3. ■

Proposition 3.7. Let $f \in L_1(G, \mathcal{M})$, then $\sup_{\chi \in \widehat{G}} \|\widehat{f}(\chi)\|_{\mathcal{M}} \leq \|f\|_1$.

Proof. Let $f \in L_1(G, \mathcal{M})$. We have

$$\begin{aligned} \|\widehat{f}(\chi)\|_{\mathcal{M}} &= \left\| \int_G \overline{\chi(t)} f(t) d\lambda(t) \right\|_{\mathcal{M}} \\ &\leq \int_G \|\overline{\chi(t)} f(t)\|_{\mathcal{M}} d\lambda(t) \\ &= \int_G |\overline{\chi(t)}| \|f(t)\|_{\mathcal{M}} d\lambda(t) \\ &= \int_G \|f(t)\|_{\mathcal{M}} d\lambda(t) \\ &= \|f\|_1. \end{aligned}$$

Hence

$$\sup_{\chi \in \widehat{G}} \|\widehat{f}(\chi)\|_{\mathcal{M}} \leq \|f\|_1. \quad \blacksquare$$

We designate by $\mathcal{C}_0(\widehat{G}, \mathcal{M})$ the space of continuous $\mathcal{M}$ -valued functions on $\widehat{G}$ which vanish at infinity and by $\mathcal{C}_c(G, \mathcal{M})$ and $\mathcal{C}_c(\widehat{G}, \mathcal{M})$ the spaces of continuous $\mathcal{M}$ -valued functions with compact support in G and $\widehat{G}$ respectively.

Let K be compact set of G and $r > 0$, put

$$U_r = \{z \in \mathbb{C}; |1 - z| < r\} \text{ and } N_K(r) = \{\chi \in \widehat{G}; \chi(t) \in U_r \text{ for all } t \in K\}.$$

The set $\{N_K(r); K \text{ compact of } G \text{ and } r > 0\}$ is a family of open sets of $\widehat{G}$ [12] and is better taken as a basis of neighbourhoods of the unit character.

Proposition 3.8. Let $f \in L_1(G, \mathcal{M})$ then $\widehat{f} \in \mathcal{C}_0(\widehat{G}, \mathcal{M})$.

Proof. Let $f \in \mathcal{C}_c(G, \mathcal{M})$. Let us show that $\widehat{f}$ is uniformly continuous. Let $\epsilon > 0$ and $\chi, \eta \in \widehat{G}$. Since the Haar measure λ is regular, there exists a compact set $K \subset G$ such that $|\lambda|(G - K) < \frac{\epsilon}{4 \sup_G \|f(t)\|_{\mathcal{M}}}$.

We have

$$\begin{aligned} \|\widehat{f}(\chi) - \widehat{f}(\eta)\|_{\mathcal{M}} &= \left\| \int_G (\overline{\chi(t)} - \overline{\eta(t)}) f(t) d\lambda(t) \right\|_{\mathcal{M}} \\ &\leq \int_G \|(\overline{\chi(t)} - \overline{\eta(t)}) f(t)\|_{\mathcal{M}} d\lambda(t) \\ &= \int_G \|f(t)\|_{\mathcal{M}} |\chi(t) - \eta(t)| d\lambda(t) \\ &= \int_K \|f(t)\|_{\mathcal{M}} |\chi(t) - \eta(t)| d\lambda(t) \\ &\quad + \int_{G-K} \|f(t)\|_{\mathcal{M}} |\chi(t) - \eta(t)| d\lambda(t) \\ &= \int_K \|f(t)\|_{\mathcal{M}} |1 - \chi^{-1}(t)\eta(t)| d\lambda(t) \\ &\quad + \int_{G-K} \|f(t)\|_{\mathcal{M}} |\chi(t) - \eta(t)| d\lambda(t). \end{aligned}$$

If $\chi^{-1}\eta \in N_K(\frac{\epsilon}{2\|f\|_1})$ then $|1 - \chi^{-1}(t)\eta(t)| < \frac{\epsilon}{2\|f\|_1}$. First, one has

$$\int_K \|f(t)\|_{\mathcal{M}} |1 - \chi^{-1}(t)\eta(t)| d\lambda(t) \leq \frac{\epsilon}{2\|f\|_1} \int_K \|f(t)\|_{\mathcal{M}} d\lambda(t) \leq \frac{\epsilon}{2}.$$

And thereafter one has

$$\begin{aligned} \int_{G-K} \|f(t)\|_{\mathcal{M}} |\chi(t) - \eta(t)| d\lambda(t) &\leq \int_{G-K} \|f(t)\|_{\mathcal{M}} (|\chi(t)| + |\eta(t)|) d\lambda(t) \\ &= 2 \int_{G-K} \|f(t)\|_{\mathcal{M}} d\lambda(t) \\ &\leq 2 \int_{G-K} \|f(t)\|_{\mathcal{M}} d|\lambda|(t) \\ &\leq 2 \sup_G \|f(t)\|_{\mathcal{M}} |\lambda|(G - K) \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

Hence $\forall \chi, \eta \in \widehat{G}$, if $\chi^{-1}\eta \in N_K\left(\frac{\epsilon}{2\|f\|_1}\right)$ then

$$\|\widehat{f}(\chi) - \widehat{f}(\eta)\|_{\mathcal{M}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We deduce that $\widehat{f}$ is uniformly continuous.

Let $f \in L_1(G, \mathcal{M})$, since $\mathcal{C}_c(G, \mathcal{M})$ is dense in $L_1(G, \mathcal{M})$, for all $\epsilon > 0$, there exists $g \in \mathcal{C}_c(G, \mathcal{M})$ such that $\|f - g\|_1 < \frac{\epsilon}{3}$. Let $\chi, \eta \in \widehat{G}$ be such that $\chi^{-1}\eta \in N_K(\frac{\epsilon}{2\|f\|_1})$.

We have

$$\|\widehat{f}(\chi) - \widehat{f}(\eta)\|_{\mathcal{M}} \leq \|\widehat{f}(\chi) - \widehat{g}(\chi)\|_{\mathcal{M}} + \|\widehat{g}(\chi) - \widehat{g}(\eta)\|_{\mathcal{M}} + \|\widehat{f}(\eta) - \widehat{g}(\eta)\|_{\mathcal{M}}.$$

From the proposition 3.7, we have $\|\widehat{f}(\theta) - \widehat{g}(\theta)\|_{\mathcal{M}} \leq \|f - g\|_1 \leq \frac{\epsilon}{3}$ for all $\theta \in \widehat{G}$.

On the other hand we have $\|\widehat{g}(\chi) - \widehat{g}(\eta)\|_{\mathcal{M}} \leq \frac{\epsilon}{3}$. Therefore $\|\widehat{f}(\chi) - \widehat{f}(\eta)\|_{\mathcal{M}} \leq \epsilon$.

Hence $\widehat{f}$ is uniformly continuous on $\widehat{G}$.

Finally, since $\widehat{f}$ is continuous map, there exists a sequence ϕ_n in $\mathcal{C}_c(\widehat{G}, \mathcal{M})$ which converges to $\widehat{f}$. For all $\chi \in \widehat{G}$, one has $\|\phi_n(\chi) - \widehat{f}(\chi)\|_{\mathcal{M}} \leq \|\phi_n - \widehat{f}\|_1$. Hence ϕ_n converges uniformly to $\widehat{f}$. Therefore $\widehat{f} \in \mathcal{C}_0(\widehat{G}, \mathcal{M})$. $\blacksquare$

Proposition 3.9. Let G be a compact abelian group, then for $f, g \in L_2(G, \mathcal{M})$, one has

$$\langle \widehat{f}, \widehat{g} \rangle_{L_2(\widehat{G}, \mathcal{M})} = \langle f, g \rangle_{L_2(G, \mathcal{M})}.$$

Proof. We have:

$$\begin{aligned} \langle \widehat{f}, \widehat{g} \rangle_{L_2(\widehat{G}, \mathcal{M})} &= \int_{\widehat{G}} \langle \widehat{f}(\chi), \widehat{g}(\chi) \rangle d\Lambda(\chi) \\ &= \int_{\widehat{G}} \left\langle \int_G f(u) \overline{\chi(u)} d\lambda(u), \int_G g(v) \overline{\chi(v)} d\lambda(v) \right\rangle d\Lambda(\chi) \\ &= \int_{\widehat{G}} \left(\int_G \int_G \langle f(u) \overline{\chi(u)}, g(v) \overline{\chi(v)} \rangle d\lambda(u) d\lambda(v) \right) d\Lambda(\chi) \\ &= \int_G \int_G \left(\langle f(u), g(v) \rangle \int_{\widehat{G}} \chi(u) \overline{\chi(v)} d\Lambda(\chi) \right) d\lambda(u) d\lambda(v) \\ &= \int_G \int_G \left(\langle f(u), g(v) \rangle \int_{\widehat{G}} \delta_u(\chi) \overline{\delta_v(\chi)} d\Lambda(\chi) \right) d\lambda(u) d\lambda(v) \\ &= \int_G \int_G \langle f(u), g(v) \rangle \langle \delta_u, \delta_v \rangle_{\widehat{G}} d\lambda(u) d\lambda(v) \\ &= \int_G \int_G \langle f(u), g(v) \rangle \delta_{u,v} d\lambda(u) d\lambda(v) \\ &= \int_G \langle f(v), g(v) \rangle d\lambda(v) \\ &= \langle f, g \rangle_{L_2(G, \mathcal{M})}. \end{aligned}$$

$\blacksquare$

We have the following norms

$$\|\cdot\|_{L_2(G, \mathcal{M})} = \|\langle \cdot, \cdot \rangle_{L_2(G, \mathcal{M})}\|_{\mathcal{A}}^{\frac{1}{2}}$$

and

$$\|\cdot\|_{L_2(\widehat{G}, \mathcal{M})} = \|\langle \cdot, \cdot \rangle_{L_2(\widehat{G}, \mathcal{M})}\|_{\mathcal{A}}^{\frac{1}{2}}$$

on $L_2(G, \mathcal{M})$ and on $L_2(\widehat{G}, \mathcal{M})$ respectively, [8].

Proposition 3.10. Let G be a discret abelian group. Then the map $L_2(G, \mathcal{M}) \rightarrow L_2(\widehat{G}, \mathcal{M})$, $f \mapsto \widehat{f}$ is a linear isometry.

Proof. Let $f \in L_1(G, \mathcal{M}) \cap L_2(G, \mathcal{M})$. Then

$$\begin{aligned} \int_{\widehat{G}} \|\widehat{f}(\chi)\|_{\mathcal{M}}^2 d\Lambda(\chi) &= \int_{\widehat{G}} \|\langle \widehat{f}(\chi), \widehat{f}(\chi) \rangle\|_{\mathcal{A}} d\Lambda(\chi) \\ &= \int_{\widehat{G}} \|\langle \int_G f(t) \overline{\chi(t)} d\lambda(t), \int_G f(s) \overline{\chi(s)} d\lambda(s) \rangle\|_{\mathcal{A}} d\Lambda(\chi) \\ &= \int_{\widehat{G}} \|\int_G \int_G \langle f(t), f(s) \rangle \chi(t) \overline{\chi(s)} d\lambda(t) d\lambda(s)\|_{\mathcal{A}} d\Lambda(\chi) \\ &\leq \int_{\widehat{G}} \left(\int_G \int_G \|\langle f(t), f(s) \rangle\|_{\mathcal{A}} |\chi(t) \overline{\chi(s)}| d\lambda(t) d\lambda(s) \right) d\Lambda(\chi) \\ &= \int_{\widehat{G}} \left(\int_G \int_G \|\langle f(t), f(s) \rangle\|_{\mathcal{A}} |\chi(t) \overline{\chi(s)}| d\lambda(t) d\lambda(s) \right) d\Lambda(\chi) \\ &= \int_{\widehat{G}} \left(\int_G \int_G \|\langle f(t), f(s) \rangle\|_{\mathcal{A}} d\lambda(t) d\lambda(s) \right) d\Lambda(\chi) \\ &= \Lambda(\widehat{G}) \int_G \int_G \|\langle f(t), f(s) \rangle\|_{\mathcal{A}} d\lambda(t) d\lambda(s) \\ &\leq \Lambda(\widehat{G}) \int_G \int_G \|f(t)\|_{\mathcal{M}} \|f(s)\|_{\mathcal{M}} d\lambda(t) d\lambda(s) \\ &= \Lambda(\widehat{G}) \left(\int_G \|f(t)\|_{\mathcal{M}} d\lambda(t) \right) \left(\int_G \|f(s)\|_{\mathcal{M}} d\lambda(s) \right) \\ &= \Lambda(\widehat{G}) \left(\int_G \|f(t)\|_{\mathcal{M}} d\lambda(t) \right)^2 < +\infty. \end{aligned}$$

Hence $\widehat{f} \in L_2(\widehat{G}, \mathcal{M})$ and since $L_1(G, \mathcal{M}) \cap L_2(G, \mathcal{M})$ is dense in $L_2(G, \mathcal{M})$ it follows that the Fourier transform can be extended in one way to the map $L_2(G, \mathcal{M}) \rightarrow L_2(\widehat{G}, \mathcal{M})$, $f \mapsto \widehat{f}$.

Let $f \in L_2(G, \mathcal{M})$. We have

$$\|\widehat{f}\|_{L_2(\widehat{G}, \mathcal{M})} = \|\langle \widehat{f}, \widehat{f} \rangle_{L_2(\widehat{G}, \mathcal{M})}\|_{\mathcal{A}}^{\frac{1}{2}} = \|\langle f, f \rangle_{L_2(G, \mathcal{M})}\|_{\mathcal{A}}^{\frac{1}{2}} = \|f\|_{L_2(G, \mathcal{M})}.$$

■

Let $g \in L_2(\widehat{G}, \mathcal{M})$, the inverse Fourier transform is defined for all $t \in G$ by

$$\check{g}(t) = \int_{\widehat{G}} \chi(t) g(\chi) d\Lambda(\chi).$$

Proposition 3.11. Let G be a discrete abelian group, then for $f \in L_2(G, \mathcal{M})$, $\check{f} = f$.

Proof. Let $f \in L_2(G, \mathcal{M})$, for all $t \in G$, we have

$$\begin{aligned}
 \check{f}(t) &= \int_{\widehat{G}} \chi(t) \widehat{f}(\chi) d\Lambda(\chi) \\
 &= \int_{\widehat{G}} \chi(t) \left(\int_G f(s) \overline{\chi(s)} d\lambda(s) \right) d\Lambda(\chi) \\
 &= \int_{\widehat{G}} \int_G \chi(t) \overline{\chi(s)} f(s) d\lambda(s) d\Lambda(\chi) \\
 &= \int_{\widehat{G}} \int_G \delta_t(\chi) \overline{\delta_s(\chi)} f(s) d\lambda(s) d\Lambda(\chi) \\
 &= \int_G f(s) \left(\int_{\widehat{G}} \delta_t(\chi) \overline{\delta_s(\chi)} d\Lambda(\chi) \right) d\lambda(s) \\
 &= \int_G f(s) \langle \delta_t, \delta_s \rangle_{\widehat{G}} d\lambda(s) \\
 &= \int_G f(s) \delta_{s,t} d\lambda(s) \\
 &= f(t).
 \end{aligned}$$

■

Proposition 3.12. Let G be a compact abelian group, then for $g \in \mathcal{C}_0(\widehat{G}, \mathcal{M}) \cap L_2(\widehat{G}, \mathcal{M})$, we have $\widehat{\check{g}} = g$.

Proof. Let $\chi \in \widehat{G}$. We have

$$\begin{aligned}
 \widehat{\check{g}}(\chi) &= \int_G \overline{\chi(t)} \check{g}(t) d\lambda(t) \\
 &= \int_G \overline{\chi(t)} \left(\int_{\widehat{G}} \eta(t) g(\eta) d\Lambda(\eta) \right) d\lambda(t) \\
 &= \int_G \left(\int_{\widehat{G}} \overline{\chi(t)} \eta(t) g(\eta) d\Lambda(\eta) \right) d\lambda(t) \\
 &= \int_{\widehat{G}} g(\eta) \left(\int_G \overline{\chi(t)} \eta(t) d\lambda(t) \right) d\Lambda(\eta) \\
 &= \int_{\widehat{G}} g(\eta) \langle \chi, \eta \rangle_{\widehat{G}} d\Lambda(\eta) \\
 &= g(\chi).
 \end{aligned}$$

■

Proposition 3.13. Let $f \in L_2(G, \mathcal{M})$ and $g \in L_2(\widehat{G}, \mathcal{M})$, then $\langle f, \check{g} \rangle_{L_2(G, \mathcal{M})} = \langle \widehat{f}, g \rangle_{L_2(\widehat{G}, \mathcal{M})}$.

Proof. Let $f \in L_2(G, \mathcal{M})$ and $g \in L_2(\widehat{G}, \mathcal{M})$. We have

$$\begin{aligned}
 \langle f, \check{g} \rangle_{L_2(G, \mathcal{M})} &= \int_G \langle f(t), \check{g}(t) \rangle d\lambda(t) \\
 &= \int_G \langle f(t), \int_{\widehat{G}} \chi(t) g(\chi) d\Lambda(\chi) \rangle d\lambda(t) \\
 &= \int_G \int_{\widehat{G}} \langle f(t), \chi(t) g(\chi) \rangle d\Lambda(\chi) d\lambda(t) \\
 &= \int_G \int_{\widehat{G}} \chi(t) \langle f(t), g(\chi) \rangle d\Lambda(\chi) d\lambda(t) \\
 &= \int_G \int_{\widehat{G}} \overline{\chi(t)} \langle f(t), g(\chi) \rangle d\Lambda(\chi) d\lambda(t) \\
 &= \int_{\widehat{G}} \langle \int_G \overline{\chi(t)} f(t) d\lambda(t), g(\chi) \rangle d\Lambda(\chi) \\
 &= \int_{\widehat{G}} \langle \widehat{f}(\chi), g(\chi) \rangle d\Lambda(\chi) \\
 &= \langle \widehat{f}, g \rangle_{L_2(\widehat{G}, \mathcal{M})}.
 \end{aligned}$$

■

In Hilbert C^* -module-valued functions settings we set the following definition for the convolution product.

Definition 3.14. Let $f, g \in L_1(G, \mathcal{M})$, $f \boxtimes g$ is defined by

$$f \boxtimes g(t) = \int_G \langle f(xt^{-1}), g(x) \rangle_{\mathcal{M}} d\lambda(x) \text{ for all } t \in G.$$

Let $L_1(G, \mathcal{A})$ denote the space of Bochner-integrable $\mathcal{A}$ -valued functions on G . It is a Banach space with respect to the norm

$$\|f\|_{\mathcal{A}}^1 = \int_G \|f(t)\|_{\mathcal{A}} d\lambda(t) \text{ for all } f \in L_1(G, \mathcal{A}).$$

Proposition 3.15. If $f, g \in L_1(G, \mathcal{M})$ then $f \boxtimes g \in L_1(G, \mathcal{A})$ and $\|f \boxtimes g\|_{\mathcal{A}}^1 \leq \|f\|_1 \|g\|_1$.

Proof. Let $f, g \in L_1(G, \mathcal{M})$, we have

$$\begin{aligned}
 \|f \boxtimes g\|_{\mathcal{A}}^1 &= \int_G \|f * g(t)\|_{\mathcal{A}} d\lambda(t) \\
 &= \int_G \left\| \int_G \langle f(xt^{-1}), g(x) \rangle_{\mathcal{M}} d\lambda(x) \right\|_{\mathcal{A}} d\lambda(t) \\
 &\leq \int_G \int_G \|\langle f(xt^{-1}), g(x) \rangle_{\mathcal{M}}\|_{\mathcal{A}} d\lambda(x) d\lambda(t) \\
 &\leq \int_G \int_G \|f(xt^{-1})\|_{\mathcal{M}} \|g(x)\|_{\mathcal{M}} d\lambda(x) d\lambda(t) \\
 &= \int_G \|g(x)\|_{\mathcal{M}} \left(\int_G \|f(xt^{-1})\|_{\mathcal{M}} d\lambda(t) \right) d\lambda(x) \\
 &= \int_G \|g(x)\|_{\mathcal{M}} \left(\int_G \|f(y)\|_{\mathcal{M}} d\lambda(y) \right) d\lambda(x) \\
 &= \int_G \|f(y)\|_{\mathcal{M}} d\lambda(y) \int_G \|g(x)\|_{\mathcal{M}} d\lambda(x) \\
 &= \|f\|_1 \|g\|_1.
 \end{aligned}$$

Hence follows the statement. ■

We obtain the following analogue of the convolution theorem.

Proposition 3.16. Let $f, g \in L_1(G, \mathcal{M})$, then

$$\widehat{f \boxtimes g}(\chi) = \langle \widehat{f}(\chi), \widehat{g}(\chi) \rangle_{\mathcal{M}} \text{ for all } \chi \in \widehat{G}.$$

Proof. Let $f, g \in L_1(G, \mathcal{M})$ and $\chi \in \widehat{G}$, we have

$$\begin{aligned}
 \widehat{f \boxtimes g}(\chi) &= \int_G \bar{\chi}(t) f \boxtimes g(t) d\lambda(t) \\
 &= \int_G \bar{\chi}(t) \left(\int_G \langle f(xt^{-1}), g(x) \rangle_{\mathcal{M}} d\lambda(x) \right) d\lambda(t) \\
 &= \int_G \bar{\chi}(y^{-1}x) \int_G \langle f(y), g(x) \rangle_{\mathcal{M}} d\lambda(x) d\lambda(y) \\
 &= \int_G \int_G \chi(y) \overline{\chi(x)} \langle f(y), g(x) \rangle_{\mathcal{M}} d\lambda(x) d\lambda(y) \\
 &= \left\langle \int_G \overline{\chi(y)} f(y) d\lambda(y), \int_G \overline{\chi(x)} g(x) d\lambda(x) \right\rangle_{\mathcal{M}} \\
 &= \langle \widehat{f}(\chi), \widehat{g}(\chi) \rangle_{\mathcal{M}}.
 \end{aligned}$$

Therefore we have $\widehat{f \boxtimes g}(\cdot) = \langle \widehat{f}(\cdot), \widehat{g}(\cdot) \rangle_{\mathcal{M}}$. ■

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