Super edge-magic total labelings of connected unicyclic (p,q) graphs with magic constant $p + q + 3$

K.Kayathri¹ and A.Shakila Jemima²

¹PG & Research Department of Mathematics, Thiagarajar College,
Madurai, Tamil Nadu, India² N.M.S.S.V. N.College, Madurai, TamilNadu, India

Abstract

A graph that admits a super edge-magic total labeling is called a SEMT graph. In this paper, we study the structure of connected unicyclic SEMT graphs with magic constant $\lambda = p + q + 3$. We also enumerate SEMT labelings with $\lambda = p + q + 3$, for connected unicyclic graphs that contain a star of maximum size m , with $p - 4 \leq m \leq p - 1$

Keywords: Super edge-magic, unicyclic graphs

AMS subject classification: 05C78

1. Introduction

In 1970, Kotzig and Rosa [3] defined a magic valuation of a graph. This notion was rediscovered by Ringel and Llado [4] in 1996 and they called it by the name edge-magic labeling. Enomoto et al. [1] introduced the concept of super edge-magic total labeling.

Let G be a simple, connected and undirected graph with p vertices and q edges. An edge-magic total labeling f is a bijection from $V(G) \cup E(G)$ to the set of integers $\{1, 2, \dots, p + q\}$ such that if xy is an edge of G , then $f(x) + f(y) + f(xy) = \lambda$ for some integer constant λ .

A super edge-magic total labeling is an edge-magic total labeling f with $f(V(G)) = \{1, 2, \dots, p\}$ and $f(E(G)) = \{p + 1, p + 2, \dots, p + q\}$.

Some of the super edge-magic graphs are C_n iff n is odd, $K_{m,n}$ iff $m = 1$ or $n = 1$,

K_n iff $n = 1, 2$ or 3 , kP_n if k is odd, $K_{1,m} \cup 2nK_{1,2}$ for all m and n , $C_m \cup P_n$ if $m \geq 6$ even and $n \geq \frac{m}{2} + 2$ and a connected 3-regular graph on p vertices iff $p \equiv 2 \pmod{4}$. For a detailed list of SEMT graphs, refer Gallian Survey of Graph Labeling [2].

In this paper, we study SEMT labelings with $\lambda = p + q + 3$ for connected unicyclic graphs that contain a star of maximum size m with $p - 4 \leq m \leq p - 1$.

2. Notations

1. Let $\mathcal{G}$ denote the class of all connected unicyclic SEMT graphs with $\lambda = p + q + 3$.
2. Let G be any graph in $\mathcal{G}$, with a SEMT labeling f . For our convenience, we fix $V(G)$ and $E(G)$ as follows: $V(G) = \{v_1, v_2, \dots, v_p\}$, $E(G) = \{e_1, e_2, \dots, e_q\}$, where $f(v_i) = i$ and $f(e_i) = p + q + 1 - i$.
3. Let $\mathcal{G}_m$ denote the class of all connected unicyclic SEMT graphs that contain a star of maximum size m , and magic constant $\lambda = p + q + 3$.

3. Graphs in $\mathcal{G}$

Let G be any graph in $\mathcal{G}$. We first make the following **Observations**:

1. If $S = \{f(u) + f(v) / uv \in E(G)\}$, then $S = \{3, 4, \dots, p + 1, p + 2\}$.
2. v_p is not adjacent with any v_i for $3 \leq i \leq p$, since $f(u) + f(v) \leq p + 2$ for all $uv \in E(G)$.
3. Sum of the labels of the vertices incident with the edge e_i is $i + 2$ for $1 \leq i \leq q$.
4. $e_1 = v_1 v_2$ and $e_2 = v_1 v_3$.
5. $e_q (= e_p)$ is not incident with v_1 .

Theorem 1 Let G be any graph in $\mathcal{G}$. Then G contains K_3 (and so $K_3 = C_3$ is the only cycle in G). Moreover, in any SEMT labeling with $\lambda = p + q + 3$, the vertex v_1 lies in K_3 .

Proof Let G be a given graph satisfying the given hypothesis with the super edge-magic labeling f . By Observation 4, $e_1 = v_1 v_2$ and $e_2 = v_1 v_3$. Also $e_3 = v_2 v_3$ or $v_1 v_4$.

If $e_3 = v_2 v_3$, then G contains K_3 and v_1 lies in K_3 . If $e_3 = v_1 v_4$, then the edges e_1, e_2, e_3 form $K_{1,3}$ with v_1 as the root.

Suppose that the edge set $\{e_1, e_2, \dots, e_r\}$ forms $K_{1,r}$ where $3 \leq r \leq q - 2$ with v_1 as the

root and the set $\{e_1, e_2, \dots, e_{r+1}\}$ does not form $K_{1,r+1}$.

Then e_{r+1} is not incident with the vertex v_1 . Since the sum of the labels of the end vertices of the edge e_{r+1} is $r + 3$, $v_i v_j \in E(G)$ for some $2 \leq i < j \leq r$.

Thus G contains K_3 and v_1 lies in K_3 .

Next, suppose that the edge set $\{e_1, e_2, \dots, e_{q-1}\}$ forms $K_{1,q-1}$ with v_1 as the root.

Then $G = K_{1,q-1} + e$ and the result follows. ■

4. Graphs in $\mathcal{G}_{p-1}$

Note that $K_{1,p-1} + e$ is the only graph in $\mathcal{G}_{p-1}$.

Theorem 2 The graph $K_{1,p-1} + e$ with $p \geq 4$ has $\left\lfloor \frac{p+1}{2} \right\rfloor$ SEMT labelings with $\lambda = p + q + 3$.

Proof Let G be a graph satisfying the given hypothesis with a SEMT labeling f .

Let v_i , $1 \leq i \leq p-1$ be the root of $K_{1,p-1}$.

By Observation 2, $v_i \neq v_j$ for $3 \leq j \leq p$.

Case 1 $v_i = v_1$

Now $e_j = v_1 v_{j+1}$ where $1 \leq j \leq q-1$ and $e = e_q = v_k v_{p+2-k}$ where $2 \leq k \leq \left\lfloor \frac{p+1}{2} \right\rfloor$.

Thus the number of SEMT labelings of G is $\left\lfloor \frac{p+1}{2} \right\rfloor - 1$.

Case 2 $v_i = v_2$

Now $e = e_2 = v_1 v_3$, $e_1 = v_1 v_2$ and $e_j = v_2 v_j$ for $3 \leq j \leq q$.

Thus **G has only one SEMT labeling.**

Thus the total number of SEMT labelings of G is $\left\lfloor \frac{p+1}{2} \right\rfloor - 1 + 1 = \left\lfloor \frac{p+1}{2} \right\rfloor$. ■

5. Graphs in $\mathcal{G}_{p-2}$

Theorem 3 Let G be a graph in $\mathcal{G}_{p-2}$ with $p \geq 6$. Then the number of SEMT labelings

$$\text{of } G \text{ is } \begin{cases} \frac{(p-3)(p-2)+6}{4} & \text{if } p \equiv 0 \pmod{4} \\ \frac{(p-3)(p-2)+4}{4} & \text{if } p \equiv 2 \pmod{4} \\ \frac{(p-3)(p-1)+4}{4} & \text{if } p \equiv 1 \text{ or } 3 \pmod{4}. \end{cases}$$

Proof Let i be the smallest index such that v_i is the root of $K_{1,p-2}$. Then $i \leq p-1$.

Let v_r be the vertex not in $K_{1,p-2}$ in G .

By Theorem 1, $K_3 = C_3$ is the only cycle in G_{p-2} and v_1 lies in this cycle.

Now $v_r v_i \notin E(G)$. Hence, if $r+i-2 \leq q$, e_{r+i-2} is an edge, and it is not incident with v_r and v_i . Hence the edge e_{r+i-2} and the vertex v_i form a triangle; and by Theorem 1, v_1 lies in this triangle... (1)

Since $q = p$, it follows that $\deg(v_r) = 1$.

By Observation 4, $\deg(v_1) \geq 2$; and so $v_r \neq v_1$.

Case 1 $v_i = v_1$

Since $v_1 v_2$ and $v_1 v_3 \in E(G)$, $r \geq 4$. By Observation 5, e_q is not incident with v_1 .

Now $e_j = v_1 v_{j+1} \in E(G)$ for $1 \leq j \leq q-1$ and $j \neq r-1$.

Using (1), $e_{r-1} = v_k v_{r+1-k}$, where $2 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor$.

Hence the number of chances for the edge e_{r-1} is $\left\lfloor \frac{r}{2} \right\rfloor - 1$.

Since $r \leq q$, $e_{r-1} \neq e_q$. Hence $e_q = e_p = v_r v_{p+2-r}$.

Then $r \neq p+2-r$; and so $r \neq \frac{p+2}{2}$.

Hence the number of SEMT labelings of G is

$$S_1 = \sum_{\substack{r=4 \\ r \neq \frac{p+2}{2}}}^p \left(\left\lfloor \frac{r}{2} \right\rfloor - 1 \right) = \begin{cases} \frac{(p-3)(p-2)+2}{4} & \text{if } p \equiv 0 \pmod{4} \\ \frac{(p-3)(p-2)}{4} & \text{if } p \equiv 2 \pmod{4} \\ \frac{(p-3)(p-1)}{4} & \text{if } p \equiv 1 \text{ or } 3 \pmod{4}. \end{cases}$$

Case 2 $v_i = v_2$

Using (1), $e_{r+i-2} = e_r$ is incident with v_1 and $e_r = v_1 v_{r+1}$. Also, $v_1 v_3 \in E(G)$; and so $r = 3$. Thus G has **only one SEMT labeling**.

Note that $\deg(v_1) = 3 < p - 2$; and so this labeling is not counted in Case 1.

Case 3 $v_i = v_j$ where $3 \leq j \leq p - 1$

By Observation 2, $v_r = v_p$.

If $v_i = v_3$, then $v_3v_1v_2v_3$ is a cycle. On the otherhand, $e_4 = v_1v_5$ or v_2v_4 would form another cycle $v_3v_1v_5v_3$ or $v_3v_2v_4v_3$, a contradiction.

If $v_i \neq v_3$, then we have two cycles $v_iv_1v_2v_i$ and $v_iv_1v_3v_i$, a contradiction.

Considering all the three cases, the number of SEMT labelings of G is $S_1 + 1$; and we get the result. ■

6. Graphs in $\mathcal{G}_{p-3}$

Theorem 4 Let G be a graph in $\mathcal{G}_{p-3}$ with $p \geq 7$. Then the number of SEMT labelings of G is

$$S = \begin{cases} \frac{2p^3 - 18p^2 + 85p - 168}{24} & \text{if } p \equiv 0 \pmod{8} \\ \frac{4p^3 - 33p^2 + 134p - 201}{48} & \text{if } p \equiv 1 \text{ or } 5 \pmod{8} \\ \frac{2p^3 - 18p^2 + 82p - 132}{24} & \text{if } p \equiv 2 \text{ or } 6 \pmod{8} \\ \frac{4p^3 - 33p^2 + 134p - 213}{48} & \text{if } p \equiv 3 \text{ or } 7 \pmod{8} \\ \frac{2p^3 - 18p^2 + 85p - 156}{24} & \text{if } p \equiv 4 \pmod{8}. \end{cases}$$

Proof Let i be the smallest index such that v_i is the root of $K_{1,p-3}$.

Let v_r, v_s be two vertices not in $K_{1,p-3}$.

Without loss of generality, let $r < s$. Then $r \leq p - 1$.

Case 1 $v_i = v_1$

Now $e_j = v_1v_{j+1}$ where $j \neq r - 1, s - 1$ and $1 \leq j \leq q - 1$.

Moreover, if $e_{r-1} = v_s v_{r+1-s}$, then $r + 1 > s$ which implies that $s \leq r$, a contradiction.

Hence $e_{r-1} = v_k v_{r+1-k}$, where

$$2 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor \dots \dots \dots (1).$$

Hence the number of chances for the edge e_{r-1} is $\left\lfloor \frac{r-2}{2} \right\rfloor$.

Now $\langle V(G) - \{v_r, v_s\} \rangle$ has $p-3+1 = p-2$ edges. Hence, it is easy to note that the edges e_{s-1}, e_q are incident with v_r and v_s .

Also $e_{s-1} \neq v_s v_1$. Hence $e_{s-1} = v_r v_{s+1-r}$ and $e_q = v_s v_{p+2-s}$, where $r \neq s+1-r$ and $s \neq p+2-s$. Hence $s \neq 2r-1$ and $s \neq \frac{p+2}{2}$. Note that $2r-1 > r$.

Next, we count the number of chances of s , by considering the following possibilities:

Case 1.1 $2r-1 \leq p$ and $2r-1 \neq \frac{p+2}{2}$

Case 1.2 $2r-1 \leq p$ and $2r-1 = \frac{p+2}{2}$

Case 1.3 $2r-1 > p$

Case 1.1 $2r-1 \leq p$ and $2r-1 \neq \frac{p+2}{2}$

Now $s \neq 1, 2, 3, \dots, r, 2r-1, \frac{p+2}{2}$; and so the number of chances of s is **p-r-2** if p is even;

And **p-r-1** if p is odd.

Also $2r-1 \leq p$ implies that $r \leq \frac{p+1}{2}$; and $2r-1 \neq \frac{p+2}{2}$ implies that $r \neq \frac{p+4}{4}$.

Hence Case 1.1 arises only when $r \neq \frac{p+4}{4}$ and $r \leq \frac{p+1}{2}$.

Case 1.2 $2r-1 \leq p$ and $2r-1 = \frac{p+2}{2}$

Now $s \neq 1, 2, 3, \dots, r, 2r-1$. Thus the number of chances of s is **p-r-1**.

Also $2r-1 = \frac{p+2}{2}$ implies that $r = \frac{p+4}{4}$.

Hence Case 1.2 arises only when $r = \frac{p+4}{4}$ (and $p \equiv 0(\text{mod } 4)$).

Case 1.3 $2r-1 > p$

Now $\frac{p+1}{2} < r$; and so $\frac{p+2}{2} \leq r$.

Thus $s \neq 1, 2, 3, \dots, r$ and the number of chances of s is $p-r$.

Combining Cases 1.1, 1.2 and 1.3 and using (1), the number of SEMT labelings of G in Case 1 is

$$S_1 = \begin{cases} \sum_{r=4}^{\frac{p}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r-2) + \left\lfloor \frac{\frac{p+4}{4}-2}{2} \right\rfloor (p-\frac{p+4}{4}-1) + \sum_{r=\frac{p+2}{2}}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) & \text{if } p \equiv 0 \pmod{4} \\ \sum_{r=4}^{\frac{p}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r-2) + \sum_{r=\frac{p+2}{2}}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) & \text{if } p \equiv 2 \pmod{4} \\ \sum_{r=4}^{\frac{p+1}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r-1) + \sum_{r=\frac{p+3}{2}}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) & \text{if } p \text{ is odd} \end{cases}$$

$$= \begin{cases} \sum_{r=4}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) - 2 \sum_{r=4}^{\frac{p}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor + \left\lfloor \frac{\frac{p+4}{4}-2}{2} \right\rfloor & \text{if } p \equiv 0 \pmod{4} \\ \sum_{r=4}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) - 2 \sum_{r=4}^{\frac{p}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor & \text{if } p \equiv 2 \pmod{4} \\ \sum_{r=4}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-r) - \sum_{r=4}^{\frac{p+1}{2}} \left\lfloor \frac{r-2}{2} \right\rfloor & \text{if } p \text{ is odd.} \end{cases}$$

On simplification, we get

$$S_1 = \begin{cases} \frac{2p^3 - 18p^2 + 61p - 96}{24} & \text{if } p \equiv 0 \pmod{8} \\ \frac{4p^3 - 33p^2 + 86p - 57}{48} & \text{if } p \equiv 1 \text{ or } 5 \pmod{8} \\ \frac{2p^3 - 18p^2 + 58p - 60}{24} & \text{if } p \equiv 2 \text{ or } 6 \pmod{8} \\ \frac{4p^3 - 33p^2 + 86p - 69}{48} & \text{if } p \equiv 3 \text{ or } 7 \pmod{8} \\ \frac{2p^3 - 18p^2 + 61p - 84}{24} & \text{if } p \equiv 4 \pmod{8}. \end{cases}$$

Case 2 $v_i = v_2$

As $v_2 v_1 \in E(G)$, we have $3 \leq r < s \leq p$.

We consider two cases: $r = 3$ and $r > 3$.

Case 2.1 $r = 3$

Then $e_3 = v_1 v_4$. Also $e_1 = v_1 v_2$ and $e_2 = v_1 v_3$.

Again two subcases arise: (i) $s = 4$ (ii) $s > 4$.

Case 2.1.1 $r = 3$ and $s = 4$

Now $e_4 \neq v_2v_4$; and so $e_4 = v_1v_5$. Hence $e_j = v_2v_j$, for $5 \leq j \leq p$; and G has **exactly one SEMT labeling** in this case.

Case 2.1.2 $r = 3$ and $s > 4$

Now $e_3 = v_1v_4$ and $v_1v_2v_4v_1$ is a cycle. Also $e_j = v_2v_j$ for $4 \leq j \leq p$ and $j \neq s$. Now $e_s \neq v_kv_l$ for $4 \leq k < l \leq p$ and $k, l \neq s$; otherwise $v_2v_kv_lv_2$ would form another cycle, a contradiction. Then $e_s = v_1v_{s+1}$ or v_3v_{s-2} ; and so $v_1v_2v_{s+1}v_1$ or $v_2v_1v_3v_{s-2}v_2$ is another cycle, a contradiction.

Case 2.2 $r > 3$

Then $4 \leq r < s \leq p$. Now $v_2v_1v_3v_2$ is a cycle. Hence $\langle V(G) - \{v_r, v_s\} \rangle$ has $p - 3 + 1 = p - 2$ edges. Also, $e_r \neq v_2v_r$ and $e_s \neq v_2v_s$.

Hence the edges e_r and e_s must be incident with v_s and v_r respectively.

So $e_r = v_sv_{r+2-s}$ and $e_s = v_rv_{s+2-r}$. Thus $r < s < r + 2$ implies that $s = r + 1$.

Hence the number of chances of r (and s) is $p - 4$; and so there are **$p - 4$ SEMT labelings**

in Case 2.2.

Hence, the number of SEMT labelings of G in Case 2 is **$S_2 = 1 + p - 4 = p - 3$** .

Case 3 $v_i = v_3$

As v_p is not adjacent with v_j for $3 \leq j \leq p - 1$, $v_s = v_p$.

Also v_p is adjacent with v_1 or v_2 .

Case 3.1 $v_1v_p \in E(G)$

Now $e_{p-1} = v_1v_p$; and so $e_{p-1} \neq v_3v_{p-2}$. Hence $v_r = v_{p-2}$, so that $r \geq 5$.

Then $v_3v_1, v_3v_2, \dots, v_3v_{p-3}$ and $v_3v_{p-1} \in E(G)$ and $v_3v_1v_2v_3$ is a cycle.

Then $e_4 \neq v_2v_4$ (otherwise $v_3v_2v_4v_3$ is another cycle, a contradiction); and so $e_4 = v_1v_5$.

If $r \neq 5$, then $v_3v_1v_5v_3$ is another cycle, a contradiction. Hence $r = 5$ and $p = 7$.

Thus $\deg(v_1) = 4 = p - 3$ and v_1 is also a 3333root of $K_{1,p-3}$, contradicting the choice of the root v_i .

Case 3.2 $v_2v_p \in E(G)$

Now $e_p = v_2v_p$; and so $e_p \neq v_3v_{p-1}$. Hence $v_r = v_{p-1}$; and so $r \geq 6$.

Then $v_3v_1, v_3v_2, \dots, v_3v_{p-2} \in E(G)$. Hence $v_3v_1v_2v_3$ is a cycle.

But $e_4 = v_2v_4$ or v_1v_5 would form another cycle, a contradiction.

Case 4 $v_i = v_j$ for $4 \leq j \leq p-1$

Now $V_1 = \{v_{p-j+3}, v_{p-j+4}, \dots, v_p\} \subseteq \{v_r, v_s\}$; and so $j \leq 4$. Hence $v_i = v_4$. Thus $v_r = v_{p-1}$ and $v_s = v_p$. As $p \geq 7$, $r \geq 6$. But $v_4v_1v_2v_4$ and $v_4v_1v_3v_4$ are two cycles, a contradiction.

Considering all the four cases, the number of SEMT labelings of G is $S_1 + S_2$.

On simplification, we get the result. ■

7. Graphs in $\mathcal{G}_{p-4}$

Theorem 5 Let G be a graph in $\mathcal{G}_{p-4}$ with $p \geq 9$. Then the number of SEMT labelings of G is

$$S = \begin{cases} \frac{4p^4 - 56p^3 + 317p^2 - 544p - 576}{192} & \text{if } p \equiv 0 \pmod{16} \text{ and when } p < 24 \text{ (i.e. when } p = 16) \\ \frac{4p^4 - 56p^3 + 317p^2 - 556p - 384}{192} & \text{if } p \equiv 0 \pmod{16} \text{ and when } p \geq 24 \\ \frac{4p^4 - 56p^3 + 317p^2 - 556p - 480}{192} & \text{if } p \equiv 8 \pmod{16} \\ \frac{4p^4 - 56p^3 + 302p^2 - 340p - 1446}{192} & \text{if } p \equiv 1 \pmod{8} \\ \frac{4p^4 - 56p^3 + 314p^2 - 472p - 1080}{192} & \text{if } p \equiv 2 \pmod{8} \\ \frac{4p^4 - 56p^3 + 302p^2 - 364p - 1206}{192} & \text{if } p \equiv 3 \pmod{8} \\ \frac{4p^4 - 56p^3 + 317p^2 - 544p - 720}{192} & \text{if } p \equiv 4 \pmod{8} \\ \frac{4p^4 - 56p^3 + 302p^2 - 340p - 1350}{192} & \text{if } p \equiv 5 \pmod{8} \\ \frac{4p^4 - 56p^3 + 314p^2 - 472p - 984}{192} & \text{if } p \equiv 6 \pmod{8} \\ \frac{4p^4 - 56p^3 + 302p^2 - 364p - 1110}{192} & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Proof Let i be the smallest index such that v_i is the root of $K_{1,p-4}$. Let v_r, v_s, v_t be the three vertices not in K_{p-4} . Note that $r \leq p-2$.

Case 1 $v_i = v_1$

We have $e_j = v_1v_{j+1}$ where $j \neq r-1, s-1, t-1$ and $1 \leq j \leq q-1$.

Now e_{r-1} is not incident with v_1 and v_r . Moreover if $e_{r-1} = v_s v_{r+1-s}$, then $s < r+1$ which implies that $s \leq r$, a contradiction. Similarly $e_{r-1} \neq v_t v_{r+1-t}$.

Hence $e_{r-1} = v_k v_{r+1-k}$ where $2 \leq k \leq \left\lfloor \frac{r}{2} \right\rfloor$.

Hence the number of chances for the edge e_{r-1} is $\left\lfloor \frac{r-2}{2} \right\rfloor \dots \dots \dots (1)$

As v_r, v_s, v_t are not isolated vertices, these vertices are incident with one of the edges e_{s-1}, e_{t-1}, e_q . Note that $t \geq s+1$. Hence $e_{s-1} \neq v_t v_{s+1-t}$.

Also $e_{s-1} \neq v_s v_1$. Hence $e_{s-1} = v_r v_{s+1-r}$. Then $v_{s+1-r} \neq v_r$; and so $s \neq 2r-1$. Also $v_{s+1-r} \neq v_s$ as $r \neq 1$. Moreover $v_{s+1-r} \neq v_t$ as $t \geq s+1$.

Thus e_{s-1} is incident with v_r , but not incident with v_s and v_t .

Also $e_{t-1} \neq v_t v_1$. Hence $e_q = v_t v_{p+2-t}$ and $e_{t-1} = v_s v_{t+1-s}$ where $p+2-t \neq t$; $t+1-s \neq s$;

Hence $s \neq 2r-1$; $t \neq 2s-1$; $t \neq \frac{p+2}{2} \dots \dots \dots (2)$

Note that $r < s \leq p-1$.

Now fixing r , we count the number of chances of t , by considering the following possibilities:

Case 1.1 $2s-1 \leq p$ and $2s-1 = \frac{p+2}{2}$

Case 1.2 $2s-1 \leq p$ and $2s-1 \neq \frac{p+2}{2}$

Case 1.3 $2r-1 \leq p$ and $2s-1 > p$

Case 1.4 $2r-1 > p$

Note that $2s-1 \leq p$ implies that $2r-1 \leq p$.

Case 1.1 $2s-1 \leq p$ and $2s-1 = \frac{p+2}{2}$

Now $2s-1 = \frac{p+2}{2}$ implies that $s = \frac{p+4}{4}$.

Hence this case arises only when $p \equiv 0 \pmod{4}$.

Since $t > s$, using (2), we have $t \neq 1, 2, \dots, s, 2s-1 (= \frac{p+2}{2})$.

Thus the number of chances of t is $p-s-1 = p - \frac{p+4}{4} - 1 = \frac{3p-8}{4}$.

Since $r < s$, $r < \frac{p+4}{4}$ and so $r \leq \frac{p}{4}$.

Also $s \neq 2r-1$. Then $\frac{p+4}{4} \neq 2r-1$; and so $r \neq \frac{p+8}{8}$.

Note that $4 \leq \frac{p+8}{8} \leq \frac{p}{4}$ only if $p \geq 24$.

Hence the number of SEMT labeling of G in this case is

$$S_1 = \begin{cases} \sum_{r=4}^{\lfloor \frac{p}{2} \rfloor} \left\lfloor \frac{r-2}{2} \right\rfloor \left\lceil \frac{3p-8}{4} \right\rceil & \text{if } p \geq 24 \text{ and } p \equiv 0 \pmod{8} \\ \sum_{r=4}^{\lfloor \frac{p}{2} \rfloor} \left\lfloor \frac{r-2}{2} \right\rfloor \left\lfloor \frac{3p-8}{4} \right\rfloor & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{3p^3 - 56p^2 + 320p - 512}{256} & \text{if } p \equiv 0 \pmod{8} \text{ and } 8 \leq p < 24 \\ \frac{3p^3 - 56p^2 + 272p - 384}{256} & \text{if } p \equiv 4 \pmod{8} \text{ and } 8 \leq p < 24 \\ \frac{3p^3 - 68p^2 + 544p - 1024}{256} & \text{if } p \equiv 0 \pmod{16} \text{ and } p \geq 24 \\ \frac{3p^3 - 68p^2 + 448p - 768}{256} & \text{if } p \equiv 8 \pmod{16} \text{ and } p \geq 24. \end{cases}$$

Case 1.2 $2s-1 \leq p$ and $2s-1 \neq \frac{p+2}{2}$

Then $r < s \leq \frac{p+1}{2}$; and so $4 \leq r \leq \frac{p-1}{2}$.

Now $2s-1 \neq \frac{p+2}{2}$ implies that $s \neq \frac{p+4}{4}$. Also $s \neq 2r-1$.

Hence $t \neq 1, 2, \dots, s, 2s-1, \frac{p+2}{2}$ and the number of chances of t is $p-s-2$.

Thus the number of SEMT labelings of G in this case is

$$S_2 = \sum_{r=4}^{\lfloor \frac{p-1}{2} \rfloor} \sum_{\substack{s=r+1 \\ s \neq 2r-1 \\ s \neq \frac{p+4}{4}}}^{\lfloor \frac{p+1}{2} \rfloor} \left\lfloor \frac{r-2}{2} \right\rfloor (p-s-2)$$

Case 1.3 $2r-1 \leq p$ and $2s-1 > p$

Then $r \leq \frac{p+1}{2}$ and $s \geq \frac{p+2}{2}$. Also $s \neq 2r-1$; and so $s \geq \frac{p+2}{2}$.

Then using (2), we have $t \neq 1, 2, \dots, s$ and the number of chances of t is $p-s$.

Thus the number of SEMT labelings of G in this case is

$$S_3 = \sum_{r=4}^{\lfloor \frac{p+1}{2} \rfloor} \sum_{\substack{s=\lfloor \frac{p+2}{2} \rfloor \\ s \neq 2r-1}}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-s)$$

$$= \begin{cases} \frac{3p^4 - 34p^3 + 120p^2 - 80p}{384} & \text{if } p \equiv 0 \pmod{8} \\ \frac{3p^4 - 34p^3 + 138p^2 - 266p + 159}{384} & \text{if } p \equiv 1 \pmod{8} \\ \frac{3p^4 - 34p^3 + 120p^2 - 200p + 144}{384} & \text{if } p \equiv 2 \pmod{8} \\ \frac{3p^4 - 34p^3 + 138p^2 - 218p + 87}{384} & \text{if } p \equiv 3 \pmod{8} \\ \frac{3p^4 - 34p^3 + 120p^2 - 128p}{384} & \text{if } p \equiv 4 \pmod{8} \\ \frac{3p^4 - 34p^3 + 138p^2 - 218p + 15}{384} & \text{if } p \equiv 5 \pmod{8} \\ \frac{3p^4 - 34p^3 + 120p^2 - 152p + 48}{384} & \text{if } p \equiv 6 \pmod{8} \\ \frac{3p^4 - 34p^3 + 138p^2 - 170p + 39}{384} & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Case 1.4 $2r - 1 > p$

Then $r \geq \frac{p+2}{2}$; and so $s \geq \frac{p+4}{2}$.

Hence using (2), $t \neq 1, 2, \dots, s$ and the number of chances of t is $p - s$.

Thus the number of SEMT labelings in this case is

$$S_4 = \sum_{r=\left\lceil \frac{p+2}{2} \right\rceil}^{p-1} \left\lfloor \frac{r-2}{2} \right\rfloor (p-s)$$

$$= \begin{cases} \frac{5p^4 - 48p^3 + 136p^2 - 96p}{768} & \text{if } p \equiv 0 \pmod{4} \\ \frac{5p^4 - 60p^3 + 262p^2 - 492p + 285}{768} & \text{if } p \equiv 1 \pmod{4} \\ \frac{5p^4 - 48p^3 + 160p^2 - 192p + 48}{768} & \text{if } p \equiv 2 \pmod{4} \\ \frac{5p^4 - 60p^3 + 238p^2 - 348p + 117}{768} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Combining the above four cases, **the number of SEMT labelings of G in Case 1 is $S_1 + S_2 + S_3 + S_4$.**

Case 2 $v_i = v_2$

Then $e_j = v_2 v_j$ for $1 \leq j \leq p$, except for $j = r, s, t$.

We have two cases: $r > 3$ and $r = 3$.

Case 2.1 $r > 3$

Then $\langle \{v_1, v_2, v_3\} \rangle$ is K_3 . Since v_r, v_s, v_t are not isolated vertices, these vertices are incident with one of the edges e_r, e_s, e_t .

Also note that $t \geq r+2$. Hence $e_r \neq v_t v_{r+2-t}$. Also $e_r \neq v_r v_2$; and so $e_r = v_s v_{r+2-s}$. Then $s < r+2$; and so $\mathbf{s=r+1}$.

Moreover $e_s = v_t v_{s+2-t}$, $e_t = v_r v_{t+2-r}$; and so $t = s+1 = r+2$. Then $e_t = v_r v_4$; and so $r \neq 4$. Hence $5 \leq r \leq p-2$.

Thus we have a SEMT labeling for every r with $5 \leq r \leq p-2$; and so **G has $p-6$ SEMT labelings in this case.**

Case 2.2 $r=3$

We have two cases: $s=4$ and $s>4$.

Case 2.2.1 $r=3$ and $s=4$

Now $e_1 = v_1 v_2$, $e_2 = v_1 v_3 = v_1 v_r$, $e_r = e_3 = v_1 v_4 = v_1 v_s$. Since v_t is not an isolated vertex, it follows that $e_s = v_t v_{s+2-t}$. Hence $t = s+1 = 5$. Thus we have **only one SEMT labeling**

Case 2.2.2 $r=3$ and $s>4$

Now $e_r = e_3 = v_1 v_4$ and so $\langle \{v_1, v_2, v_4\} \rangle$ is a cycle. Also $v_r v_1 = v_3 v_1 \in E(G)$.

Since v_s, v_t are not isolated vertices, we have $e_s = v_t v_{s+2-t}$, $e_t = v_s v_{t+2-s}$ and $t = s+1$. Then $e_s = v_t v_1$ and $e_t = v_s v_3$. Thus we have a SEMT labeling for every s with $5 \leq s \leq p-1$ and **G has $p-5$ SEMT labelings in this case.**

Thus, **the number of SEMT labelings in Case 2 is $p-6 + 1 + p-5 = 2p-10$.**

Case 3 $v_i = v_3$

As v_p is not adjacent with v_j for $3 \leq j \leq p$, $v_t = v_p$.

Case 3.1 $v_1 v_p \in E(G)$

Then $v_3 v_{p-2} \notin E(G)$. Then $s = p-2$ or $r = p-2$.

Case 3.1.1 $s=p-2$

Then $e_{s+1} = e_{p-1} = v_1 v_p$. Now $e_{r+1} \neq v_r v_3$. Moreover v_s is not an isolated vertex. Hence $e_{r+1} = v_s v_{r+3-s}$ which implies that $r \geq s-2$; and so $r \geq p-4 \geq 5$. Thus $v_3 v_1, v_3 v_2, v_3 v_4 \in E(G)$ and $v_3 v_1 v_2 v_3$ is a cycle. Now if $e_4 = v_2 v_4$, it would form another cycle, a contradiction.

Hence $e_4 = v_1 v_5$ and $r=5$, $s=7$ and $p=9$.

Now $\langle v_1 v_2, v_1 v_3, v_1 v_5, v_1 v_7, v_1 v_9 \rangle = K_{1,5} = K_{1,p-4}$, contradicting the choice of the root v_i

Case 3.1.2 $r=p-2$

Then $r \geq 7$ and $v_3 v_1, v_3 v_2, \dots, v_3 v_6 \in E(G)$.

Now $v_1v_2v_3v_1$ is a cycle. Then $e_4 = v_1v_5$ or v_2v_4 would form another cycle, a contradiction.

Case 3.2 $v_1v_p \notin E(G)$

Then $v_2v_p \in E(G)$; and so $v_3v_{p-1} \notin E(G)$. Hence $s = p-1$.

Now $e_{s+1} = e_p = v_2v_p$. As $e_{r+1} \neq v_rv_3$, and v_s is not an isolated vertex, $e_{r+1} = v_sv_{r+3-s}$. Hence $r \geq s-2 = p-3 \geq 6$. Then as in Case 3.1.2, we would have two cycles, a contradiction.

Thus in Case 3, **G has no SEMT labeling.**

Case 4 $v_i = v_j, 4 \leq j \leq p-1$

Since $v_{p-j+k}v_j \notin E(G)$ for $k \geq 3$, we have $\{v_{p-j+3}, v_{p-j+4}, \dots, v_p\} \subseteq \{v_r, v_s, v_t\}$; and so $j \leq 5$. Hence $j = 4$ or 5 . Then v_{p-1}, v_p are not adjacent with v_j ; and so $v_s = v_{p-1}$ and $v_t = v_p$.

Now $v_t = v_p$ is adjacent with v_1 or v_2 .

Now $e_{r+j-2} \neq v_rv_j$. Moreover v_s is not an isolated vertex. Hence $e_{r+j-2} = v_sv_{r+j-s}$.

Then $r+j-s \geq 1$ which implies that $r \geq s-j+1 = p-j \geq p-5 \geq 4$.

Then $v_jv_1, v_jv_2, v_jv_3 \in E(G)$. Now $v_jv_1v_2v_j$ and $v_jv_1v_3v_j$ are two cycles, a contradiction.

Thus in Case 4, **G has no SEMT labeling.**

Considering all the four cases, the number of SEMT labelings of G is $S_1 + S_2 + S_3 + S_4 + 2p - 10$. On simplification, we get the result. ■

8. References

- [1] Enomota, H., Llado, A.S., Nakamigawa T., and Ringel. G., 1998, "Super edge-magic graphs", SUT J. Math., 34, pp. 105-109.
- [2] Joseph Gallian, A., 2014, "A Dynamic Survey of Graph Labeling", The Electronic J. Combinatorics, 17, pp. 98-142.
- [3] Kotzig, A., and Rosa, A., 1970, "Magic valuations of finite graphs", Canad. Math. Bull., 13, pp. 451-461.
- [4] Ringel, G., and Llado, A., 1996, "Another tree conjecture", Bull. Inst. Combin. Appl., 18, pp. 83-85.