Dynamic Behavior of the Solutions for a Two-layer Neural Network Model with Delays

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Abstract

In this paper, a two-layer neural network model with delays is investigated. By means of the generalized Chafee's criterion, some sufficient conditions to guarantee the existence of oscillatory solution for the model are obtained. Computer simulation indicates that the decay rate in the model affects the stability and instability.

Keywords: A two-layer neural network model, delay, stability, oscillation

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1. INTRODUCTION

In the last decades, bifurcation analysis in various practical models is a good research topic. Many excellent and interesting results have been obtained [1-12]. For example, Hajnova and Pribylova have studied bifurcation manifolds using the Grobner basis computation for the following predator-prey model [1]:

$$\begin{cases} \frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) - \frac{a_0(h+P)^2NP}{a_0hN+(h+P)^2}, \\ \frac{dP}{dt} = \frac{ea_0(h+P)^2NP}{a_0hN+(h+P)^2} - mP. \end{cases} \quad (1)$$

The authors provided formulas for bifurcation manifolds in commonly studied cases in applied research for the fold, transcritical, cusp, Hopf and Bogdanov-Takens bifurcations. Cao has considered the existence of bifurcations in an internet congestion control system with distributed delay as follows:

$$p'(t) = kp(t)[f(\int_{-\infty}^t (\frac{m}{T})^m \frac{(t-r)^{m-1} \exp(-\frac{m(t-r)}{T})}{(m-1)!} dr) - c]. \quad (2)$$

Some criteria of stability and bifurcation conditions for the positive equilibrium of the proposed system are precisely established with the help of the Routh-Hurwitz criterion [2]. Hou *et al.* have investigated the bifurcation and stability of a nonlinear rotor system subjected to constant excitation and rub-impact [3]:

$$\begin{cases} q_1'' + \omega_1^2 q_1 + a_1 q_4 + a_2 q_1' + a_3 q_4' + a_4(q_1^3 + q_1 q_2^2) = p_1 + E \cos \tau, \\ q_2'' + \omega_1^2 q_2 + a_1 q_3 + a_2 q_2' + a_3 q_3' + a_4(q_1^2 q_2 + q_2^3) = p_2 + E \sin \tau + G_1, \\ q_3'' + \omega_2^2 q_3 + b_1 q_2 + b_1 q_3' + b_3 q_4' + b_4 q_2' = 0 \\ q_4'' + \omega_2^2 q_4 + b_1 q_1 + b_2 q_4' - b_3 q_3' + b_4 q_1' = G_2. \end{cases} \quad (3)$$

The authors used harmonic balance method combined with an alternating frequency domain procedure to derive the approximate periodic solutions of the system. The bifurcation behavior caused by flight maneuvers in an aircraft rub-impact rotor system with Duffing type nonlinearity has been discussed.

Recently, Wang *et al.* provided a two-layer neural network model with delays [4]:

$$\begin{cases} x_1'(t) = -k_1 x_1(t) + c_{1,n+1} f_{1,n+1}(x_{n+1}(t - \tau_1)) + c_{1,n+2} f_{1,n+2}(x_{n+2}(t - \tau_1)), \\ x_2'(t) = -k_2 x_2(t) + c_{2,n+1} f_{2,n+1}(x_{n+1}(t - \tau_1)) + c_{2,n+2} f_{2,n+2}(x_{n+2}(t - \tau_1)) \\ \quad + c_{2,n+3} f_{2,n+3}(x_{n+3}(t - \tau_1)), \\ \dots, \\ x_n'(t) = -k_n x_n(t) + c_{n,2n-1} f_{n,2n-1}(x_{2n-1}(t - \tau_1)) + c_{n,2n} f_{n,2n}(x_{2n}(t - \tau_1)), \\ x_{n+1}'(t) = -k_{n+1} x_{n+1}(t) + c_{n+1,1} f_{n+1,1}(x_1(t - \tau_2)) + c_{n+1,2} f_{n+1,2}(x_2(t - \tau_2)), \\ x_{n+2}'(t) = -k_{n+2} x_{n+2}(t) + c_{n+2,1} f_{n+2,1}(x_1(t - \tau_2)) + c_{n+2,2} f_{n+2,2}(x_2(t - \tau_2)) \\ \quad + c_{n+2,3} f_{n+2,3}(x_3(t - \tau_2)), \\ \dots, \\ x_{2n}'(t) = -k_{2n} x_{2n}(t) + c_{2n,n-1} f_{2n,n-1}(x_{n-1}(t - \tau_2)) + c_{2n,n} f_{2n,n}(x_n(t - \tau_2)). \end{cases} \quad (4)$$

For a special case $n = 3$, and $k_i = k(1 \leq i \leq 6)$, $a_1 = a_3$, set $\tau = \tau_1 + \tau_2$, $u_i(t) = x_i(t - \tau_2)(1 \leq i \leq 3)$, $u_j(t) = x_j(t)$, $4 \leq j \leq 6$, the authors discussed the following

linearized system:

$$\begin{cases} u'_1(t) = -ku_1(t) + a_1u_4(t - \tau) + b_1u_5(t - \tau), \\ u'_2(t) = -ku_2(t) + c_1u_4(t - \tau) + a_2u_5(t - \tau) + b_2u_6(t - \tau), \\ u'_3(t) = -ku_3(t) + c_2u_5(t - \tau) + a_3u_6(t - \tau), \\ u'_4(t) = -ku_4(t) + a_1u_1(t) + b_1u_2(t), \\ u'_5(t) = -ku_5(t) + c_1u_1(t) + a_2u_2(t) + b_2u_3(t), \\ u'_6(t) = -ku_6(t) + c_2u_2(t) + a_3u_3(t). \end{cases} \quad (5)$$

By means of matrix decomposition method, the Hopf bifurcation analysis has been investigated. Based on the normal form method and the center manifold theorem, the explicit formulas about the stability of the bifurcating periodic solution and the direction of the Hopf bifurcation are established. However, even if $n = 3$, if $a_1 \neq a_3$, k_i are different positive numbers, bifurcating method is still hard to deal with model (5). Because the characteristic equation according to authors' method will be very complex like the following (see [4], equations (6) and (7)):

$$R_1 \times R_2 = 0, \quad (6)$$

where $R_1 = (\lambda + k_1 + a_1e^{-\frac{1}{2}\lambda\tau})(\lambda + k_2 + a_2e^{-\frac{1}{2}\lambda\tau})(\lambda + k_3 + a_3e^{-\frac{1}{2}\lambda\tau}) - (\lambda + k_4 + a_3e^{-\frac{1}{2}\lambda\tau})b_1c_1e^{-\lambda\tau} - (\lambda + k_5 + a_1e^{-\frac{1}{2}\lambda\tau})b_2c_2e^{-\lambda\tau}$, $R_2 = (\lambda + k_1 - a_1e^{-\frac{1}{2}\lambda\tau})(\lambda + k_2 - a_2e^{-\frac{1}{2}\lambda\tau})(\lambda + k_3 - a_3e^{-\frac{1}{2}\lambda\tau}) - (\lambda + k_4 - a_3e^{-\frac{1}{2}\lambda\tau})b_1c_1e^{-\lambda\tau} - (\lambda + k_5 - a_1e^{-\frac{1}{2}\lambda\tau})b_2c_2e^{-\lambda\tau}$. One is hard to simplify $R_1 \times R_2 = 0$, since $a_1 \neq a_3$ and k_i are different positive numbers. In this paper, we shall study the periodic oscillation and the stability of the solutions for model (4) by mathematical analysis method. We assume that the activation functions f_{ij} are continuous bounded differentiable functions, satisfying:

$$f_{ij}(0) = 0, u f_{ij}(u) > 0 (u \neq 0). \quad (7)$$

The general activation functions such as $\tanh(x)$, $\arctan(x)$ satisfy condition (7). The linearized system of (4) at origin is the following:

$$\begin{cases} x'_1(t) = -k_1x_1(t) + b_{1,n+1}x_{n+1}(t - \tau_1) + b_{1,n+2}x_{n+2}(t - \tau_1), \\ x'_2(t) = -k_2x_2(t) + b_{2,n+1}x_{n+1}(t - \tau_1) + b_{2,n+2}x_{n+2}(t - \tau_1) + b_{2,n+3}x_{n+3}(t - \tau_1), \\ \dots\dots\dots, \\ x'_n(t) = -k_nx_n(t) + b_{n,2n-1}x_{2n-1}(t - \tau_1) + b_{n,2n}x_{2n}(t - \tau_1), \\ x'_{n+1}(t) = -k_{n+1}x_{n+1}(t) + b_{n+1,1}x_1(t - \tau_2) + b_{n+1,2}x_2(t - \tau_2), \\ x'_{n+2}(t) = -k_{n+2}x_{n+2}(t) + b_{n+2,1}x_1(t - \tau_2) + b_{n+2,2}x_2(t - \tau_2) + b_{n+2,3}x_3(t - \tau_2), \\ \dots\dots\dots, \\ x'_{2n}(t) = -k_{2n}x_{2n}(t) + b_{2n,n-1}x_{n-1}(t - \tau_2) + b_{2n,n}x_n(t - \tau_2). \end{cases} \quad (8)$$

Proof An equilibrium point $x^* = [x_1^*, x_2^*, \dots, x_{2n}^*]^T$ of system (4) is a constant solution of the following algebraic equation

$$\left\{ \begin{array}{l} -k_1 x_1^* + c_{1,n+1} f_{1,n+1}(x_{n+1}^*) + c_{1,n+2} f_{1,n+2}(x_{n+2}^*) = 0, \\ -k_2 x_2^* + c_{2,n+1} f_{2,n+1}(x_{n+1}^*) + c_{2,n+2} f_{2,n+2}(x_{n+2}^*) + c_{2,n+3} f_{2,n+3}(x_{n+3}^*) = 0, \\ \dots\dots\dots, \\ -k_n x_n^* + c_{n,2n-1} f_{n,2n-1}(x_{2n-1}^*) + c_{n,2n} f_{n,2n}(x_{2n}^*) = 0, \\ -k_{n+1} x_{n+1}^* + c_{n+1,1} f_{n+1,1}(x_1^*) + c_{n+1,2} f_{n+1,2}(x_2^*) = 0, \\ -k_{n+2} x_{n+2}^* + c_{n+2,1} f_{n+2,1}(x_1^*) + c_{n+2,2} f_{n+2,2}(x_2^*) + c_{n+2,3} f_{n+2,3}(x_3^*) = 0, \\ \dots\dots\dots, \\ -k_{2n} x_{2n}^* + c_{2n,n-1} f_{2n,n-1}(x_{n-1}^*) + c_{2n,n} f_{2n,n}(x_n^*) = 0. \end{array} \right. \quad (11)$$

From condition (7) we have $f_{ij}(0) = 0$. This means that zero is an equilibrium point of system (11). System (11) can be rewritten as the follows:

$$\left\{ \begin{array}{l} c_{1,n+1} f_{1,n+1}(x_{n+1}^*) + c_{1,n+2} f_{1,n+2}(x_{n+2}^*) = k_1 x_1^*, \\ c_{2,n+1} f_{2,n+1}(x_{n+1}^*) + c_{2,n+2} f_{2,n+2}(x_{n+2}^*) + c_{2,n+3} f_{2,n+3}(x_{n+3}^*) = k_2 x_2^*, \\ \dots\dots\dots, \\ c_{n,2n-1} f_{n,2n-1}(x_{2n-1}^*) + c_{n,2n} f_{n,2n}(x_{2n}^*) = k_n x_n^*, \\ c_{n+1,1} f_{n+1,1}(x_1^*) + c_{n+1,2} f_{n+1,2}(x_2^*) = k_{n+1} x_{n+1}^*, \\ c_{n+2,1} f_{n+2,1}(x_1^*) + c_{n+2,2} f_{n+2,2}(x_2^*) + c_{n+2,3} f_{n+2,3}(x_3^*) = k_{n+2} x_{n+2}^*, \\ \dots\dots\dots, \\ c_{2n,n-1} f_{2n,n-1}(x_{n-1}^*) + c_{2n,n} f_{2n,n}(x_n^*) = k_{2n} x_{2n}^*. \end{array} \right. \quad (12)$$

Noting that condition (7) implies that $f_{ij}(x_j^*) > 0$ when $x_j^* > 0$, and $f_{ij}(x_j^*) < 0$ when $x_j^* < 0$. Since $k_i > 0 (i = 1, 2, \dots, 2n)$, when we select $x_j^* > 0$, then the right hand of system (12) is a positive number, however one can not guarantee that the left hand of system (12) is a positive number since B is not a positive definite matrix. The same as when we select $x_j^* < 0$, the right hand of system (12) is a negative number, but one can not ensure that the left hand of system (12) is a negative number. Therefore, system (11), namely system (4) has a unique equilibrium point, it is exactly the zero point.

Lemma 2 Assume that $k_i > 0 (i = 1, 2, \dots, 2n)$ and the condition (7) holds, then all solutions of system (4) are bounded.

Proof From condition (7), the activation functions f_{ij} are bounded. Let $|f_{ij}| \leq N$, then

[illegible][illegible]

Lemma 3 Assume that the conditions of Lemma 1 and Lemma 2 hold. If the unique equilibrium point of system (4) is unstable, then there exists a limit cycle, namely, a periodic oscillatory solution of system (4).

In this paper we adopt the following norms of vectors and matrices [15]: $\|x(t)\| = \sum_{i=1}^{2n} |x_i(t)|$, the measure $\mu(D)$ of a matrix D is defined by $\mu(D) = \lim_{\theta \rightarrow 0^+} \frac{\|I + \theta D\| - 1}{\theta}$, which for the chosen norms reduces to $\mu(D) = \max_{1 \leq j \leq 2n} [d_{jj} + \sum_{i=1, i \neq j}^{2n} |d_{ij}|]$. $D > 0$ (respectively, $D < 0$) which indicates that D is a positive (negative) definite matrix.

Theorem 1 Let $k_* = \min\{k_1, k_2, \dots, k_{2n}\}$. Assume that the conditions of Lemma 1 and Lemma 2 hold. Let $\beta_1, \beta_2, \dots, \beta_{2n}$ represent the eigenvalues of matrix B . If all eigenvalues of matrix B are negative numbers or have negative real parts, also there exists a positive number r_* ($r_* < k_*$) such that

$$\frac{(\|B\| + \|D\|)e^{r_*\tau}}{k_* - r_*} \leq 1 \quad (15)$$

where $D = (d_{ij})_{2n \times 2n}$, $d_{ij} = \frac{1}{2}c_{ij} \max_{\{\xi_{ij} \in [0, x]\}} |f''_{ij}|$. Then the trivial solution of system (4) is stable.

Proof According to the basic theory of differential equation, since all eigenvalues of matrix A are negative numbers, and all eigenvalues of matrix B are negative numbers or have negative real parts, then all characteristic values of system (9) are negative numbers or have negative real parts. The trivial solution of system (9) is stable. Now we need to prove that the trivial solution of system (4) is stable. From condition (7), by the Taylor's Theorem, we have $f(x(t - \tau)) = f'(0)x(t - \tau) + \frac{1}{2}f''(\xi)x^2(t - \tau)$, where ξ lies between 0 and x . Therefore, system (4) can be written as a matrix form:

$$X'(t) = AX(t) + BX(t - \tau) + \tilde{C}X^2(t - \tau) \quad (16)$$

where matrix $\tilde{C} = (\tilde{c}_{ij})_{2n \times 2n}$, $\tilde{c}_{ij} = \frac{1}{2}c_{ij}f''_{ij}(\xi_{ij})$. Using the variations of constants formula, from (16) we have

$$X(t) = e^{A(t-\tau)}X(\tau) + \int_{\tau}^t [e^{A(t-s)}BX(s - \tau) + \tilde{C}X^2(s - \tau)]ds, t > \tau. \quad (17)$$

where $D = (d_{ij})_{2n \times 2n}$, and $d_{ij} = \frac{1}{2}c_{ij} \max_{\{\xi_{ij} \in [0, x]\}} |f''_{ij}|$. Noting that A is a diagonal matrix and $-k_i < 0$, from (17) we have

$$\begin{aligned} \|X(t)\| &\leq e^{-k_*(t-\tau)}\|X(\tau)\| + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|\|X(s - \tau)\| \\ &\quad + \|D\|\|X^2(s - \tau)\|)\}ds, t > \tau. \end{aligned} \quad (18)$$

Let $L = \sup_{t \in [-\tau, \tau]} \|X(t)\|$, then

$$\begin{aligned} \|X(t)\| &\leq Le^{-k_*(t-\tau)} + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|\|X(s - \tau)\| \\ &\quad + \|D\|\|X^2(s - \tau)\|)\}ds, t > \tau. \end{aligned} \quad (19)$$

Define

$$\begin{aligned} \|Y(t)\| &= Le^{-k_*(t-\tau)} + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|\|Y(s - \tau)\| \\ &\quad + \|D\|\|Y^2(s - \tau)\|)\}ds, t > \tau. \end{aligned} \quad (20)$$

According to the comparison theorem of differential theorem, we have $\|X(t)\| \leq \|Y(t)\|$. In the following we prove that there exist positive number $r_*(r_* < k_*)$ such that $Y(t)$ can be represented as the form:

$$\|Y(t)\| \leq Le^{-r_*(t-\tau)}, t \geq \tau. \quad (21)$$

Noting that $e^{-2r_*(t-\tau)} < e^{-r_*(t-\tau)}$, using condition (15) we have

$$\begin{aligned}
& Le^{-k_*(t-\tau)} + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|\|Y(s-\tau)\| + \|D\|\|Y^2(s-\tau)\|)\}ds, \\
&= Le^{-k_*(t-\tau)} + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|Le^{-r_*(s-2\tau)} + \|D\|Le^{-2r_*(s-2\tau)})\}ds \\
&< Le^{-k_*(t-\tau)} + \int_{\tau}^t \{e^{-k_*(t-s)}(\|B\|Le^{-r_*(s-2\tau)} + \|D\|Le^{-r_*(s-2\tau)})\}ds \\
&= Le^{-k_*(t-\tau)} + L(\|B\| + \|D\|)e^{-k_*t}e^{2r_*\tau} \int_{\tau}^t e^{(k_*-r_*)s}ds \\
&= Le^{-k_*(t-\tau)} + \frac{L(\|B\| + \|D\|)e^{-k_*t}e^{2r_*\tau}}{k_* - r_*}(e^{(k_*-r_*)t} - e^{(k_*-r_*)\tau}) \\
&= Le^{-k_*(t-\tau)} + \frac{L(\|B\| + \|D\|)e^{r_*\tau}}{k_* - r_*}(e^{-r_*(t-\tau)} - e^{-k_*(t-\tau)}) \\
&\leq Le^{-r_*(t-\tau)}
\end{aligned} \tag{22}$$

Therefore we have $\|X(t)\| \leq Le^{-r_*(t-\tau)}$, implying that the trivial solution of system (4) is stable.

3. THE EXISTENCE OF THE OSCILLATORY SOLUTIONS

Theorem 2 Assume that the conditions of Lemma 1 and Lemma 2 hold. Let $\beta_1, \beta_2, \dots, \beta_{2n}$ represent the eigenvalues of matrix B . If there exists one eigenvalue, say β_1 , which is a positive real number or is a positive real part of a complex number, satisfying $k_* < \beta_1$ or $k_* < \operatorname{Re}(\beta_1)$. Then system (4) generates a limit cycle, namely, a periodic solution.

Proof It is known that the trivial solution of the linearized system (8) is unstable, then the trivial solution of original system (4) is unstable. Since (9) is an equivalent system of (8), therefore, for proving the instability of the trivial solution of system (4) we only need to prove the instability of the trivial solution of system (9). Thus in the following we discuss the instability of the trivial solution of system (9). Since A is a diagonal matrix, the eigenvalues of matrix A are $-k_1, -k_2, \dots, -k_{2n}$, and the eigenvalues of matrix B are $\beta_1, \beta_2, \dots, \beta_{2n}$, system (9) has the following characteristic equation:

$$\det(\lambda I - A - Be^{-\lambda\tau}) = 0 \tag{23}$$

equation (23) is equivalent to

$$\prod_{i=1}^{2n} (\lambda + k_i - \beta_i e^{-\lambda\tau}) = 0 \tag{24}$$

Without loss of generality, we assume that $k_1 = k_*$, so from (24) we have

$$\lambda + k_1 - \beta_1 e^{-\lambda\tau} = 0 \quad (25)$$

We will show that equation (25) has a positive eigenvalue or a positive real part of a complex number eigenvalue. Let $h(\lambda) = \lambda + k_1 - \beta_1 e^{-\lambda\tau}$, then $h(\lambda)$ is a continuous function of λ . If β_1 is a positive number, $h(0) = k_1 - \beta_1 < 0$. Noting that $\lim_{\lambda \rightarrow +\infty} e^{-\lambda\tau} = 0$. Therefore, there exists a large $\tilde{\lambda} > 0$ such that $h(\tilde{\lambda}) = \tilde{\lambda} + k_1 - \beta_1 e^{-\tilde{\lambda}\tau} > 0$. By the Intermediate Value Theorem of continuous function, there exists a λ , say $\lambda_1 \in (0, \tilde{\lambda})$ such that $h(\lambda_1) = \lambda_1 + k_1 - \beta_1 e^{-\lambda_1\tau} = 0$. This means that there exists a positive eigenvalue of equation (25). If $Re(\beta_1) > k_1$, let $\lambda = \rho + i\omega$, where ρ is a real part of λ , and ω is an imaginary part of λ . From (25) we get

$$\rho + k_1 - e^{-\rho\tau}[Re(\beta_1) \cos(\omega\tau) - Im(\beta_1) \sin(\omega\tau)] = 0 \quad (26)$$

Let $g(\rho) = \rho + k_1 - e^{-\rho\tau}[Re(\beta_1) \cos(\omega\tau) - Im(\beta_1) \sin(\omega\tau)]$, then $g(\rho)$ is a continuous function of ρ . Obviously, there is a suitably large $\tilde{\rho}(> 0)$ such that $g(\tilde{\rho}) > 0$. On the other hand, if $\rho = 0$, τ is small ($0 < \tau \ll 1$) then $\cos(\omega\tau) \sim 1$, $\sin(\omega\tau) \sim 0$ and $g(0) \sim k_1 - Re(\beta_1)$. Since $k_1 < Re(\beta_1)$, thus $g(0) < 0$. Again from the Intermediate Value Theorem of continuous function, there exists a ρ , say $\rho_1 \in (0, \tilde{\rho})$ such that $g(\rho_1) = \rho_1 + k_1 - e^{-\rho_1\tau}[Re(\beta_1) \cos(\omega\tau) - Im(\beta_1) \sin(\omega\tau)] = 0$. This proves that there is a positive real part of characteristic equation (25). Therefore, the trivial solution is unstable of system (9) when time delay τ is small ($0 < \tau \ll 1$). According to the theory of functional differential equation, if for small delay the trivial solution of the system is unstable, then the instability of the trivial solution is still maintained when delay increases [16]. Therefore, for any delay τ the trivial solution of system (9) is unstable, implying that the trivial solution of system (4) is unstable. Based on Lemma 3, since all solutions of system (4) are bounded and the unique equilibrium point is unstable, it forced system (4) to generate a limit cycle, namely, a periodic solution. The proof is completed.

Theorem 3 Assume that the conditions of Lemma 1 and Lemma 2 hold. For matrix B , let $\sigma = \max_{1 \leq j \leq 2n} \sum_{i=1}^{2n} |b_{ij}|$. If

$$k_* < \sigma \quad (27)$$

then the trivial solution of system (9) is unstable, implying that the trivial solution of system (4) is unstable. Then system (4) generates a limit cycle, namely, a periodic solution.

Proof For equation $\frac{du(t)}{dt} = -k_* u(t)$, if $u(t) < 0$, then $\frac{d|u(t)|}{dt} = -\frac{du(t)}{dt} = k_* u(t) = -k_* (-u(t)) = -k_* |u(t)|$. While when $u(t) > 0$, then $\frac{d|u(t)|}{dt} = \frac{du(t)}{dt} = -k_* u(t) =$

$-k_*|u(t)|$. Thus, from system (6) we have

$$\left\{ \begin{array}{l} \frac{d|u_1(t)|}{dt} \leq -k_1|u_1(t)| + |b_{1,n+1}||u_{n+1}(t-\tau)| + |b_{1,n+2}||u_{n+2}(t-\tau)|, \\ \frac{d|u_2(t)|}{dt} \leq -k_2|u_2(t)| + |b_{2,n+1}||u_{n+1}(t-\tau)| + |b_{2,n+2}||u_{n+2}(t-\tau)| + |b_{2,n+3}||u_{n+3}(t-\tau)|, \\ \dots\dots\dots, \\ \frac{d|u_n(t)|}{dt} \leq -k_n|u_n(t)| + |b_{n,2n-1}||u_{2n-1}(t-\tau)| + |b_{n,2n}||u_{2n}(t-\tau)|, \\ \frac{d|u_{n+1}(t)|}{dt} \leq -k_{n+1}|u_{n+1}(t)| + |b_{n+1,1}||u_1(t-\tau)| + |b_{n+1,2}||u_2(t-\tau)|, \\ \frac{d|u_{n+2}(t)|}{dt} \leq -k_{n+2}|u_{n+2}(t)| + |b_{n+2,1}||u_1(t-\tau)| + |b_{n+2,2}||u_2(t-\tau)| + |b_{n+2,3}||u_3(t-\tau)|, \\ \dots\dots\dots, \\ \frac{d|u_{2n}(t)|}{dt} \leq -k_{2n}|u_{2n}(t)| + |b_{2n,n-1}||u_{n-1}(t-\tau)| + |b_{2n,n}||u_n(t-\tau)|. \end{array} \right. \quad (28)$$

Let $z(t) = \sum_{i=1}^{2n} |u_i(t)|$, then we make the sum of equations and get:

$$\frac{dz(t)}{dt} \leq -k_*z(t) + \sigma z(t-\tau) \quad (29)$$

Consider equation

$$\frac{dv(t)}{dt} = -k_*v(t) + \sigma v(t-\tau) \quad (30)$$

The characteristic equation of (30) is

$$\lambda = -k_* + \sigma e^{\lambda\tau} \quad (31)$$

Condition $k_* < \sigma$ means that there has a positive eigenvalue for equation (30), implying that the trivial solution of (30) is unstable. In other words, there exists a sequence $\{t_k\}_{t_0 > \tau}$ such that for an arbitrary $\varepsilon > 0$ we have $|v(t_k)| > \varepsilon$, where $v(t)$ represents the trivial solution of (30). Now we prove that the trivial solution of (29) is unstable. Noting that $\sum_{i=1}^{2n} |u_i(t)| \leq v(t)$. This means that there exists at least one component of $z(t)$, say $|u_1(t)|$ and a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ such that $|u_1(t_{k_j})| = \frac{1}{2n}|v(t_{k_j})| > \frac{\varepsilon}{2n}$. Since ε is an arbitrary sufficiently small positive number, then $\frac{\varepsilon}{2n}$ is still an arbitrary sufficiently small positive number. Therefore, the component $u_1(t)$ is unstable. According to definition, the trivial solution of system (9) is unstable, implying that the trivial solution of system (4) is unstable. Based on Lemma 3, system (4) generates a limit cycle, namely, a periodic solution. The proof is completed.

4. SIMULATION RESULT

Example 1 Firstly consider the following $n = 3$ system:

$$\left\{ \begin{array}{l} x'_1(t) = -k_1x_1(t) + c_{1,4}f_{1,4}(x_4(t-\tau_1)) + c_{1,5}f_{1,5}(x_5(t-\tau_1)), \\ x'_2(t) = -k_2x_2(t) + c_{2,4}f_{2,4}(x_4(t-\tau_1)) + c_{2,5}f_{2,5}(x_5(t-\tau_1)) + c_{2,6}f_{2,6}(x_6(t-\tau_1)), \\ x'_3(t) = -k_3x_3(t) + c_{3,5}f_{3,5}(x_5(t-\tau_1)) + c_{3,6}f_{3,6}(x_6(t-\tau_1)), \\ x'_4(t) = -k_4x_4(t) + c_{4,1}f_{4,1}(x_1(t-\tau_2)) + c_{4,2}f_{4,2}(x_2(t-\tau_2)), \\ x'_5(t) = -k_5x_5(t) + c_{5,1}f_{5,1}(x_1(t-\tau_2)) + c_{5,2}f_{5,2}(x_2(t-\tau_2)) + c_{5,3}f_{5,3}(x_3(t-\tau_2)), \\ x'_6(t) = -k_6x_6(t) + c_{6,2}f_{6,2}(x_2(t-\tau_2)) + c_{6,3}f_{6,3}(x_3(t-\tau_2)). \end{array} \right. \quad (32)$$

We selected the activation function as $f(x) = \tanh(x)$ and then $f'(x) = 1 - \tanh^2(x)$, $f''(x) = 2(\tanh^3(x) - \tanh(x))$, so $f'(0) = 1$, and $\max f''(x) = \frac{8\sqrt{3}}{9}$. The parameters $k_1 = 3.5, k_2 = 3.55, k_3 = 3.65, k_4 = 3.6, k_5 = 3.64, k_6 = 3.62$; $c_{14} = 0.65, c_{15} = 0.35, c_{24} = -0.6, c_{25} = 0.58, c_{26} = 0.56, c_{35} = 0.45, c_{36} = -0.75, c_{41} = 0.15, c_{42} = 0.46, c_{51} = -0.24, c_{52} = 0.36, c_{53} = 0.25, c_{62} = 0.28, c_{63} = -0.42, \tau_1 = 0.8, \tau_2 = 0.75$. The characteristic values of matrix B are $-0.4324 \pm 0.3778i, 0.4324 \pm 0.3778i, \pm 0.6649$. Therefore, B is not a positive definite matrix. From Lemma 1, this system has a unique equilibrium point, namely, the zero point. Thus, $k_* = 3.5, \|B\| = 1.7, \|D\| = 2.6150$. We select $r_* = 0.0001$, so $\frac{(\|B\| + \|D\|)e^{r_*\tau}}{k_* - r_*} = 0.9475 < 1$. Based on Theorem 1, the trivial solution is convergent (see Fig 1).

Then we select parameters $k_1 = 0.05, k_2 = 0.15, k_3 = 0.16, k_4 = 0.18, k_5 = 0.12, k_6 = 0.2$; $c_{14} = -0.75, c_{15} = 0.85, c_{24} = -0.16, c_{25} = 0.18, c_{26} = 0.45, c_{35} = -0.65, c_{36} = 0.35, c_{41} = 0.96, c_{42} = 0.24, c_{51} = -0.86, c_{52} = 0.35, c_{53} = 0.92, c_{62} = -0.78, c_{63} = 0.72, \tau_1 = 0.5, \tau_2 = 0.4$. The characteristic values of matrix B are $\pm 1.2537i, -0.2906 \pm 0.5447i, 0.2906 \pm 0.5447i$. Since there is a characteristic value $0.2906 \pm 0.5447i$ which the real part is $0.2906 > 0.12 = k_*$. Based on Theorem 2, there is an oscillatory solution (see Fig 2). When we increase time delays as

$\tau_1 = 1.5, \tau_2 = 1.4$, the other parameters are the same as in Fig 2, we see that the oscillatory behavior is still maintained, but the oscillatory frequency is changed (Fig 3).

Remark 1 Since the values k_i are different numbers, the bifurcating method in [4] is invalid for example 1.

Example 2 Consider the following $n = 4$ system:

$$\begin{cases} x'_1(t) = -k_1x_1(t) + c_{1,5}f_{1,5}(x_5(t - \tau_1) + c_{1,6}f_{1,6}(x_6(t - \tau_1), \\ x'_2(t) = -k_2x_2(t) + c_{2,5}f_{2,5}(x_5(t - \tau_1) + c_{2,6}f_{2,6}(x_6(t - \tau_1) + c_{2,7}f_{2,7}(x_7(t - \tau_1), \\ x'_3(t) = -k_3x_3(t) + c_{3,5}f_{3,5}(x_5(t - \tau_1) + c_{3,6}f_{3,6}(x_6(t - \tau_1), \\ x'_4(t) = -k_4x_4(t) + c_{4,7}f_{4,7}(x_7(t - \tau_1) + c_{4,8}f_{4,8}(x_8(t - \tau_1), \\ x'_5(t) = -k_5x_5(t) + c_{5,1}f_{5,1}(x_1(t - \tau_2) + c_{5,2}f_{5,2}(x_2(t - \tau_2), \\ x'_6(t) = -k_6x_6(t) + c_{6,1}f_{6,1}(x_1(t - \tau_2) + c_{6,2}f_{6,2}(x_2(t - \tau_2) + c_{6,3}f_{6,3}(x_3(t - \tau_2), \\ x'_7(t) = -k_7x_7(t) + c_{7,1}f_{7,1}(x_1(t - \tau_2) + c_{7,2}f_{7,2}(x_2(t - \tau_2), \\ x'_8(t) = -k_8x_8(t) + c_{8,3}f_{8,3}(x_3(t - \tau_2) + c_{8,4}f_{8,4}(x_4(t - \tau_2). \end{cases} \quad (33)$$

We selected the activation function as $f(x) = \arctan(x)$ and then $f'(x) = \frac{1}{1+x^2}$, $f''(x) = \frac{-2x}{(1+x^2)^2}$, so $f'(0) = 1$, and $\max f''(x) = \frac{3\sqrt{3}}{8}$. We select parameters $k_1 = 0.95, k_2 = 0.85, k_3 = 0.96, k_4 = 0.98, k_5 = 0.92, k_6 = 0.86, k_7 = 0.95, k_8 = 0.88$; $c_{15} = -0.35, c_{16} = 0.45, c_{25} = -0.36, c_{26} = 0.48, c_{27} = 0.65, c_{35} = -0.25, c_{36} = 0.55, c_{47} = 0.76, c_{48} = 0.25, c_{51} = -0.26, c_{52} = 0.35, c_{61} = -0.46, c_{62} = -0.72, c_{63} = 0.42, c_{71} = 0.35, c_{72} = 0.42, c_{83} = -0.68, c_{84} = 0.32$; $\tau_1 = 1.6, \tau_2 = 1.8$. The characteristic values of matrix B are $-0.3925 \pm 0.4440i, 0.3925 \pm 0.4440i, -0.2828, 0.2828, -0.2025, 0.2025$. B is not a positive definite matrix. The trivial solution of system (33) is convergent (see Fig 4).

However, we only change $k_1 = 0.15, k_2 = 0.13, k_3 = 0.24, k_4 = 0.18, k_5 = 0.12, k_6 = 0.16, k_7 = 0.15, k_8 = 0.18$, and $k_1 = 0.25, k_2 = 0.23, k_3 = 0.24, k_4 = 0.28, k_5 = 0.22, k_6 = 0.26, k_7 = 0.25, k_8 = 0.28$, respectively, the other parameters are the same as Fig 4. In this case $\sigma = \max_{1 \leq j \leq 8} \sum_{i=1}^8 |b_{ij}| = 1.48$, and $k_* = 0.13$ or $k_* = 0.22$,

in both cases we have $k_* < \sigma$. The conditions of Theorem 3 are satisfied and the oscillatory solutions appeared (see Fig 5 and Fig 6). All cases indicate that the values $k_i (i = 1, \dots, 8)$ affect the stability and oscillation.

Remark 2 Obviously, both in two examples, matrix B in the above two examples has a positive eigenvalue. Also condition (15) is very stronger restrictive condition. Therefore, Theorem 1 is only a sufficient condition.

5. CONCLUSION

This paper considers a two-layer network model with delays. Some sufficient conditions to guarantee the stability and instability of the solutions were obtained. A specific selection of parameters was used to demonstrate the proposed results. Computer simulation indicates that time delays affect oscillatory frequency. The values k_i is very important, it affects the stability and oscillation. The present method is easy to check, as compared to predicting the regions of bifurcation.

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