Interfacial Velocities, Slip Parameters and Other Theoretical Expressions Arising in the Beavers and Joseph Condition

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Abstract: - The slip hypothesis and the interfacial condition of Beavers and Joseph are discussed in this work, which reports on extensions of this condition to flow through a free-space channel over non-Darcy porous layers, flow through composite porous layers, and flow of pressure-dependent fluids through and over porous layers. Expressions for velocities at the interface and relationships between slip parameters are obtained.

Key-Words: - Slip hypothesis, Beavers and Joseph condition, Pressure-dependent viscosity.

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1 Introduction

Fluid flow through free-space channels underlain by porous sediments, and the flow through two or more porous layers, have received considerable attention over the past half century due to their importance in many practical applications in lubrication theory, groundwater flow, the design of cooling and heating systems, and in oil recovery, (*cf.* [1] and the references therein). Of importance are the interfacial conditions between the two flow domains due to the influence they have on heat, mass and momentum transfer across layers, [2].

Prior to 1967, it was customary in lubrication studies to use a no-slip condition at the interface between a fluid-layer in free-space and a fluid layer saturating a porous medium, [3]. The experiments of Beavers and Jospeh, [4], however, introduced a slip hypothesis and an interfacial condition to replace the no-slip condition in their attempt to explain the increase in mass flux in the channel associated with the presence of a permeable boundary over its value when the no-slip condition is applied. Their analysis resulted in the empirical slip-flow condition of Beavers and Joseph (referred to hereafter as the BJ condition), which ascertains that velocity shear at the interface between free-space and a porous layer is proportional to the difference between the interfacial velocity and the velocity in the porous medium. Their slip hypothesis agreed well with their experiment of flow through a Navier-Stokes channel terminated by a semi-infinite Darcy porous layer. The constant of proportionality in the BJ condition includes what is termed a slip parameter, α, which is an empirical, dimensionless slip coefficient that is independent of fluid viscosity and dependent on the porous medium properties, Reynolds number and flow direction at the interface.

In its original form, the BJ condition has been validated for flow through a channel over a Darcy porous layer. However, more recent work reports on modifications of the BJ condition to extend it to flow over a Forchheimer porous layer, [5], and flow through composite porous layers, [6]. Furthermore, flow due to gravity has also been considered, [7].

Mathematical modelling of flow over porous layers include the following variations:

a) Navier-Stokes flow in a channel underlain by an infinite porous layer. Flow through the porous layer is governed by Darcy's equation or Forchheimer's equation, [4,5].
b) Navier-Stokes flow in a channel underlain by a finite porous layer. Flow through the porous layer is governed by Brinkman's equation or by Forchheimer-Brinkman's equation, [8-10].
c) Flow through a finite porous layer underlain by an infinite porous layer. Flow in the finite porous layer is governed by Brinkman's equation or by Forchheimer-Brinkman's equation, and flow in the infinite layer is governed by Darcy's equation or Forchheimer's equation, [2,6].
d) Flow through a finite porous layer underlain by another finite porous layer. Flow in each of the finite porous layers is governed by Brinkman's equation or by Forchheimer-Brinkman's equation. When the flow is governed by the same equation, permeabilities are different. The flow domain can also be inclined at an angle, and the flow is gravity-driven, [2,7].
e) Navier-Stokes flow through a channel underlain by a semi-infinite porous medium with a transition layer. Flow in the semi-infinite layer is governed by Dary's equation or Forchheimer's equation with constant permeability, and flow in the transition layer is governed by Brinkman's equation with variable permeability, [11].
f) Navier-Stokes flow through an inclined channel underlain by a Darcy layer of constant or variable permeability, and the fluid is one with pressure-dependent viscosity. This problem is considered in this work.

Unidirectional forms of the governing equations in the above configurations are listed in what follows for convenience and ease of reference. Their ranges of validity have been discussed in the literature (*cf.* [1,11,12] and the references therein).

Navier-Stokes equations:

$$\frac{d^2v}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \tag{1}$$

Darcy's equation:

$$u_d = -\frac{k}{\mu}\frac{dp}{dx} \tag{2}$$

Forchheimer's equation:

$$\frac{\mu}{k}u_f + \frac{dp}{dx} + \frac{\rho C_f}{\sqrt{k}}{u_f}^2 = 0 \tag{3}$$

Brinkman's equation:

$$\mu_e\frac{d^2u}{dy^2} - \frac{\mu}{k}u = \frac{dp}{dx} \tag{4}$$

Brinkman-Forchheimer's equation:

$$\mu_e\frac{d^2u}{dy^2} - \frac{\mu}{k}u - \frac{\rho C_f}{\sqrt{k}}u^2 = \frac{dp}{dx} \tag{5}$$

In equations (1)-(5), k is the permeability, μ is the base-fluid viscosity, μ_e is the effective viscosity (that is, viscosity of the fluid saturating the porous medium), ρ is the fluid density, C_f is the Forchheimer drag coefficient, $\frac{dp}{dx} < 0$ is the driving pressure gradient, v is the Navier-Stokes velocity, u_d is the Darcy velocity, u_f is the Forchheimer velocity, and u is the ensemble average Brinkman velocity.

It should be noted that Brinkman's and Brinkman-Forchheimer's equations are compatible in differential order with Navier-Stokes equations, while Darcy's and Forchheimer's equations are of a lower differential order. This fact influences conditions at the interface between layers, which in turn influence momentum, heat and mass transfer between layers.

For a complete account of the available interfacial conditions and their applications and validity, one is referred to the elegant work of Ehrhardt, [13], where the following three popular conditions are discussed:

i) Beavers and Joseph (BJ) condition: This condition is the main focus of this work and is used when the porous layers are semi-infinite and differential orders of the governing equations are not compatible.

ii) Velocity and shear stress continuity at the interface: This condition is used when the porous layers are finite and the differential orders of the governing equations are compatible.

iii) The jump-stress condition of Ochoa-Tapia and Whitaker, [10]: This is the most popular, and most widely used interfacial condition in handling discontinuities at the interface.

The focus of the current work is the BJ condition. While this condition has been applied originally to the case of Navier-Stokes flow over a semi-infinite Darcy porous layer (variation (a), above), recent work shows that it can be modified for variations (b), (c), and (e), above. An overview of its applicability and modification for variations (b) and (c) are provided in this work, together with its modification for Navier-Stokes flow through a channel underlain by an inclined, semi-infinite Darcy layer. The case of flow of pressure-dependent viscosity fluids is also considered in this work where the flow in an inclined Navier-Stokes channel underlain by an infinite Darcy layer is analyzed. Expressions for velocity in the channel and velocity at the interface are obtained, and the relationship between slip parameters in this variable viscosity flow and in the case of constant viscosity, is derived.

2 Beavers and Joseph's Slip Hypothesis

The BJ condition states that the velocity shear at the interface between free-space and a porous layer is proportional to the difference between the interfacial velocity and the velocity in the porous medium. The constant of proportionality is known as the slip parameter, α, and, mathematically, the BJ condition is expressed in the following form, valid at the assumingly sharp interface at $y = 0$ (in **Fig. 1**):

$$\frac{dv}{dy} = \frac{\alpha}{\sqrt{k}}(u_i - u_d) \tag{6}$$

where u_i is the velocity at the interface.

Beavers and Joseph, [4], used condition (1) to obtain the following expression for the interfacial velocity, u_i:

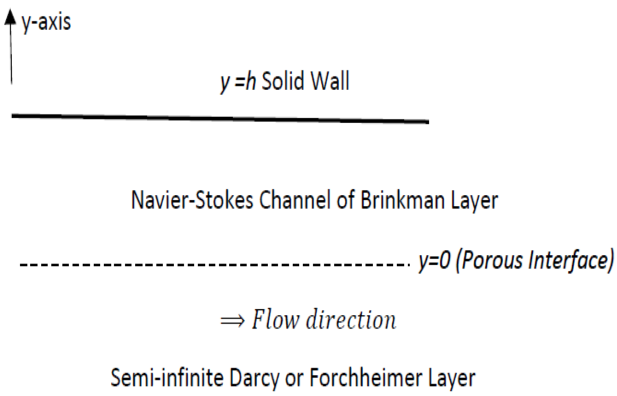

Fig. 1. Representative Sketch

$$u_i = -\frac{k}{2\mu}\left[\frac{\sigma^2+2\alpha\sigma}{1+\alpha\sigma}\right]\frac{dp}{dx} \tag{7}$$

wherein h is the channel depth, $\sigma = \frac{h}{\sqrt{k}}$, and $\frac{dp}{dx} < 0$ is the common driving pressure gradient.

Velocity distribution in the channel is given by:

$$v(y) = \left\{\left[\frac{y^2}{2\mu} + \frac{\alpha\sqrt{k}}{\mu}y\right] - \frac{k}{2\mu}\left(1 + \frac{\alpha}{\sqrt{k}}y\right)\left[\frac{\sigma^2+2\alpha\sigma}{1+\alpha\sigma}\right]\right\}\frac{dp}{dx} \tag{8}$$

The BJ condition has received considerable attention in the literature. For a more complete review of the available literature, one is referred to the work of Nield and Bejan, [1].

The slip parameter, α, in the BJ condition has been discussed extensively by many authors, including Nield, [14], who concluded that α is an empirical, dimensionless coefficient that is independent of fluid viscosity and dependent on the porous medium properties and flow direction at the interface. Nield, [14], provided the range of 0.01 to 5 for the value of α, and reported that in the experiments of Beavers and Joseph, the values α used were 0.78, 1.45, and 4.0 for Foametal having average pore sizes of 0.016, 0.034, and 0.045 inches, respectively, and 0.1 for Aloxite with average pore size of 0.013 or 0.027 inches.

Beavers and Joseph, [4], based their analysis on flow in a channel over a Darcy porous layer. Some authors, however, favoured the use of Brinkman's equation due to compatibility of differential orders or Brinkman's and the Navier-Stokes equations. Neale and Nader, [8], suggested the use of velocity and shear stress continuity at the interface and recovered Beavers and Joseph's results by defining the slip parameter in terms of the base fluid viscosity coefficient, μ, and the effective viscosity coefficient of the fluid saturating the porous medium, μ_e, as

$$\alpha^2 = \frac{\mu_e}{\mu} \tag{9}$$

Many other investigations point to a general agreement that conditions at the interface must emphasize velocity continuity and shear stress continuity in order to facilitate the matching of flow in the channel with the flow through the porous medium. Rudraiah, [9], concluded that Brinkman's equation is a more appropriate model when the porous layer is of finite depth, while the BJ condition is suitable for semi-infinite porous layers. The case of flow through a channel bounded by a Brinkman's layer of constant permeability, however, results in permeability discontinuity at the interface. This has been remedied by Nield and Kuznetsov, [11], with the introduction of a transition layer between a constant permeability layer and the free-space channel. This is variation (e), discussed in the introduction.

3 Modified BJ Condition for a Forchheimer Layer

Abu Zaytoon and Hamdan, [5], provided an answer to the following question: If the structure of the porous medium in Beavers and Joseph's work is changed to one where the Forchheimer equation is valid, could the above values of the slip parameter be used?

To furnish an answer, Abu Zaytoon and Hamdan, [5], derived a relationship between the slip velocity in the flow over a Darcy layer and the slip velocity in the flow over a Forchheimer porous layer. Since the slip parameter α is empirical and its determination may not be easy (*cf.* Mierzwiczak *et.al.,* [15]), existing values of α can be modified to find values of the slip parameter to be used in the flow over a Forchheimer layer. When the flow is governed by equation (1) in the free-space channel, and equation (3) in the semi-infinite Forchheimer layer, subject to the no-slip condition at $y = h$ and a BJ condition at the interface of the form

$$\frac{dv}{dy} = \frac{\beta}{\sqrt{k}}\left(u_i - u_f\right) \tag{10}$$

where β is the slip parameter associated with the Forchheimer case, solutions are as follows.

Velocity distribution in the free-space channel is given by:

$$v(y) = \frac{(y^2-h^2)}{2\mu}\frac{dp}{dx} + \beta\left\{\frac{u_i}{\sqrt{k}} + \frac{\mu}{2\rho k C_f} - \frac{1}{2\rho k C_f}\sqrt{\mu^2 - 4\rho\sqrt{k}\,C_f\,\frac{dp}{dx}}\right\}(y-h) \tag{11}$$

Velocity distribution in the Forchheimer porous layer is given by:

$$u_f = -\frac{\omega_1}{\mu}\frac{dp}{dx};\;\; y < 0 \tag{12}$$

where

$$\omega_1 = \frac{\mu}{2\rho C_f\sqrt{k}\,\frac{dp}{dx}}\left[\mu - \sqrt{\mu^2 - 4\rho k\sqrt{k}\,C_f\,\frac{dp}{dx}}\right] \tag{13}$$

Velocity at the interface between the channel and porous layer has been calculated as

$$u_i = -\frac{k}{2\mu}\left[\frac{\sigma^2+2\beta\sigma\omega_1}{1+\beta\sigma}\right]\frac{dp}{dx} \tag{14}$$

The following relationship between slip parameters α and β has been derived,

$$\beta = \frac{\alpha[\sigma^2-2]}{[\sigma^2+2\alpha\sigma(1-\omega_1)-2\omega_1]} \tag{15}$$

4 Modified BJ Condition for Inclined Layers

In the experiment of Beavers and Joseph, [4], the flow configuration included a horizontal free-space channel underlain by a Darcy porous layer wherein the flow was driven by a common pressure gradient.

The case of flow due to gravity in a configuration inclined at an angle ϑ, was considered by Abu Zaytoon *et.al*, [7], who provided theoretical analysis of the Navier-Stokes flow in a channel bounded at $y = h$ by a solid, impermeable wall, and underlain by a semi-infinite porous layer. The configuration is shown in **Fig. 2**. Both cases of Darcy and Forchheimer layers were considered.

Flow in the channel is governed by the following equations, wherein $p = p(y)$ is the pressure and g is the gravitational acceleration:

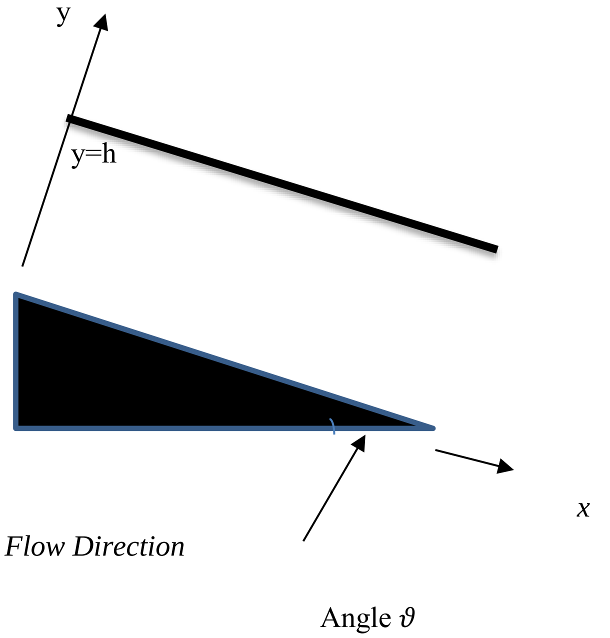

Fig. 2. Representative Sketch for Flow Down an Incline

$$\frac{d^2v}{dy^2} = -\frac{\rho g}{\mu} sin\vartheta \tag{16}$$

$$\frac{dp}{dy} + \rho g cos\vartheta = 0 \tag{17}$$

Flow in the Darcy porous layer is governed by the following equations, wherein $p_d = p_d(y)$ is the interstitial pressure:

$$\rho g sin\vartheta - \frac{\mu}{k} u_d = 0 \tag{18}$$

$$\frac{dp_d}{dy} + \rho g cos\vartheta = 0 \tag{19}$$

Flow in the Forchheimer layer is governed by the following equations, wherein $p_f = p_f(y)$ is the pressure distribution in the Forchheimer layer:

$$-\frac{\mu}{k} u_f - \frac{\rho C_f}{\sqrt{k}} (u_f)^2 + \rho g sin\vartheta = 0 \tag{20}$$

$$\frac{dp_f}{dy} + \rho g cos\vartheta = 0 \tag{21}$$

Boundary conditions associated with this configuration are:

$$v(h) = 0 \tag{22}$$

$$p(h) = p_0 \tag{23}$$

Interfacial conditions in the case of the Darcy porous layer are:

$$\frac{dv}{dy}(0) = \frac{\alpha_1}{\sqrt{k}}(u_{i1} - u_d) \tag{24}$$

$$p_d(0) = p(0) \tag{25}$$

where u_{i1} is the velocity at the interface between the Navier-Stokes fluid layer and Darcy's layer, and α_1 is the associated slip parameter.

Interfacial conditions in the case of Forchheimer's porous layer are:

$$\frac{dv}{dy}(0) = \frac{\alpha_2}{\sqrt{k}}(u_{i2} - u_f) \tag{26}$$

$$p_f(0) = p(0) \tag{27}$$

where u_{i2} is the velocity at the interface between the Navier-Stokes fluid layer and Forchheimer's layer, and α_2 is the associated slip parameter.

Solutions to the above two problems are as follows.

4.1 Darcy's Case

Velocity and pressure profiles in the Navier-Stokes channel are given by:

$$v(y) = \frac{\rho g sin\vartheta}{2\mu}\left[(h^2 - y^2) + \frac{\alpha_1(\sigma^2-2)}{(1+\alpha_1\sigma)}(y-h)\right] \tag{28}$$

$$p(y) = p_0 + \rho g cos\vartheta\,(h-y);\ \ 0 \le y \le h \tag{29}$$

Darcy's velocity and pressure profiles are given by:

$$u_d = \frac{\rho g k}{\mu} sin\vartheta \tag{30}$$

$$p_d(y) = p_0 + \rho g cos\vartheta\,(h-y);\ \ y \le 0 \tag{31}$$

Velocity and pressure at the interface are given by:

$$u_{i1} = \rho g sin\vartheta \frac{k}{2\mu}\left[\frac{\sigma^2+2\alpha_1\sigma}{(1+\alpha_1\sigma)}\right] \tag{32}$$

$$p_d(0) = p(0) = p_0 + \rho g h cos\vartheta \tag{33}$$

4.2 Forchheimer's Case

Velocity and pressure profiles in the Navier-Stokes channel are given by:

$$v(y) = \frac{\rho g}{2\mu} sin\vartheta (h^2 - y^2) + \frac{\alpha_2}{\sqrt{k}} \frac{\rho g sin\vartheta}{2\mu} \left[\frac{h^2 - 2k\omega_2}{(1+\alpha_2\sigma)}\right](y-h) \quad (34)$$

$$p(y) = p_0 + \rho g cos\vartheta\,(h-y);\ \ 0 \le y \le h \quad (35)$$

Forchheimer's velocity and pressure profiles are given by:

$$u_f = -\frac{1}{2\rho C_f\sqrt{k}}\left[\mu - \sqrt{\mu^2 + 4C_f k\sqrt{k}\rho^2 g sin\vartheta}\right];$$

$$y < 0 \quad (36)$$

$$p_f(y) = p_0 + \rho g cos\vartheta\,(h-y);\ \ y \le 0 \quad (37)$$

Velocity and pressure at the interface are given by:

$$u_{i2} = \frac{\rho g sin\vartheta}{2\mu}\left[\frac{h^2}{(1+\alpha_2\sigma)} + \frac{2k\alpha_2\sigma\omega_2}{1+\alpha_2\sigma}\right] \quad (38)$$

where

$$\omega_2 = \frac{\mu}{\rho g k sin\vartheta} u_f \quad (39)$$

$$p_f(0) = p(0) = p_0 + \rho g h cos\vartheta \quad (40)$$

4.3 Relationship between α_1 and α_2

Equating the interfacial velocities u_{i1} and u_{i2} gives the following relationship between the slip parameter α_1 that was used in Beavers and Joseph experiment, and slip parameter α_2 that should be used when a Darcy inclined porous layer is replaced by a Forchheimer porous layer:

$$\alpha_2 = \frac{\alpha_1[\sigma^2 - 2]}{\sigma^2 + 2\alpha_1\sigma[1-\omega_2] - 2\omega_2} \quad (41)$$

5 Modified BJ Condition for Composite Porous Layers

In the mathematical modelling of flow over porous layers, variation (d) in the introduction involves replacing the Navier-Stokes channel with a Brinkman porous layer. This layer is underlain by either a semi-infinite Darcy layer or a Forchheimer layer. Both of these cases were considered by Silva-Zea *et.al.*, [6].

Flow through the finite Brinkman layer is governed by equation (4), where η is the permeability of the Brinkman layer. Flow in the Darcy layer by equation (2) and that through the Forchheimer layer by equation (3). In both the Darcy and Forchheimer layers permeability is denoted by k.

The no-slip velocity condition is used on the upper bounding wall, $y = h$. At the interface between layers, $y = 0$, the BJ condition (6) is valid for the case of the Darcy layer, and is assumed to be of the following form for the Forchheimer layer:

$$\frac{du}{dy} = \frac{\beta}{\sqrt{k}}(u_i - u_f) \quad (42)$$

Silva-Zea *et.al.*, [6], obtained the following results.

5.1 Flow through a finite porous layer underlain by a semi-infinite Darcy porous layer

Velocity in the Darcy layer is given by equation (2) and the velocity distribution in the Brinkman layer is given by:

$$u(y) = \left[\frac{\eta p_x}{\mu} sech\frac{\sigma}{\sqrt{\lambda}} - \alpha\sqrt{\lambda\eta}\left(\frac{u_i}{\sqrt{k}} + \frac{\sqrt{k}p_x}{\mu}\right) tanh\frac{\sigma}{\sqrt{\lambda}}\right] cosh\frac{y}{\sqrt{\lambda\eta}} + \left(\alpha\sqrt{\frac{\lambda\eta}{k}}u_i + \alpha\sqrt{\lambda\eta k}\frac{p_x}{\mu}\right) sinh\frac{y}{\sqrt{\lambda\eta}} - \frac{\eta p_x}{\mu} \quad (43)$$

where

$$\lambda = \frac{\mu_e}{\mu} \quad (44)$$

and the interfacial velocity is given by:

$$u_i = \frac{\eta p_x}{\mu}\frac{\left(1 - cosh\frac{\sigma}{\sqrt{\lambda}}\right) - \alpha\sqrt{\frac{\lambda k}{\eta}} sinh\frac{\sigma}{\sqrt{\lambda}}}{cosh\frac{\sigma}{\sqrt{\lambda}} + (\alpha\sqrt{\lambda\eta k}) sinh\frac{\sigma}{\sqrt{\lambda}}} \quad (45)$$

If $\eta = k$ then

$$u_i = \left[\frac{1}{cosh\frac{\sigma}{\sqrt{\lambda}} + \alpha\sqrt{\lambda} sinh\frac{\sigma}{\sqrt{\lambda}}} - 1\right]\frac{k p_x}{\mu} \quad (46)$$

Since $\lambda = \frac{\mu_e}{\mu}$ and equation (9) gives $\alpha^2 = \frac{\mu_e}{\mu}$, it follows that $\alpha^2 = \lambda$ or $\alpha = \sqrt{\lambda}$, and u_i can be written as:

$$u_i = \left[\frac{1}{\cosh\frac{\sigma}{\alpha}+\alpha^2\sinh\frac{\sigma}{\alpha}} - 1\right]\frac{kp_x}{\mu} \qquad (47)$$

5.2 Flow through a finite porous layer underlain by a semi-infinite Forchheimer porous layer

Velocity in the Forchheimer layer is given by:

$$u_f = -\frac{\mu}{2\rho C_f\sqrt{k}} + \sqrt{\frac{\mu^2}{4\rho^2 C_f^2 k} - \frac{\sqrt{k}\,p_x}{\rho C_f}} \qquad (48)$$

and the velocity distribution in the Brinkman layer is given by:

$$u = \left\{\frac{\eta p_x}{\mu}\operatorname{sech}\frac{h}{\sqrt{\lambda\eta}} - \beta\sqrt{\lambda\eta/k}\,\tanh\frac{h}{\sqrt{\lambda\eta}}\,u_i - \beta\sqrt{\lambda\eta/k}\left[\frac{\mu}{2\rho C_f\sqrt{k}} - \sqrt{\left(\frac{\mu}{2\rho C_f\sqrt{k}}\right)^2 - \frac{\sqrt{k}p_x}{\rho C_f}}\right]\tanh\frac{h}{\sqrt{\lambda\eta}}\right\}\cosh\frac{y}{\sqrt{\lambda\eta}} + \beta\sqrt{\lambda\eta/k}\left[u_i + \frac{\mu}{2\rho C_f\sqrt{k}} - \sqrt{\left(\frac{\mu}{2\rho C_f\sqrt{k}}\right)^2 - \frac{\sqrt{k}p_x}{\rho C_f}}\right]\sinh\frac{y}{\sqrt{\lambda\eta}} - \frac{\eta p_x}{\mu} \qquad (49)$$

Velocity at the interface is given by:

$$u_i = \frac{\left\{\frac{\eta p_x}{\mu}\right\}}{\left[\cosh\frac{h}{\sqrt{\lambda\eta}} + \beta\sqrt{\lambda\eta/k}\sinh\frac{h}{\sqrt{\lambda\eta}}\right]} - \frac{\beta\sqrt{\lambda\eta/k}\sinh\frac{h}{\sqrt{\lambda\eta}}\frac{\mu}{2\rho C_f\sqrt{k}}}{\left[\cosh\frac{h}{\sqrt{\lambda\eta}} + \beta\sqrt{\lambda\eta/k}\sinh\frac{h}{\sqrt{\lambda\eta}}\right]} + \frac{\beta\sqrt{\lambda\eta/k}\sinh\frac{h}{\sqrt{\lambda\eta}}\sqrt{\left(\frac{\mu}{2\rho C_f\sqrt{k}}\right)^2 - \frac{\sqrt{k}p_x}{\rho C_f}}}{\left[\cosh\frac{h}{\sqrt{\lambda\eta}} + \beta\sqrt{\lambda\eta/k}\sinh\frac{h}{\sqrt{\lambda\eta}}\right]} \qquad (50)$$

If $\eta = k$ then (50) can be written in the form:

$$u_i = \frac{\left\{\frac{kp_x}{\mu} - \beta\sqrt{\lambda}\xi\sinh\frac{\sigma}{\sqrt{\lambda}}\right\}}{\left[\cosh\frac{\sigma}{\sqrt{\lambda}} + \beta\sqrt{\lambda}\sinh\frac{\sigma}{\sqrt{\lambda}}\right]} \qquad (51)$$

where

$$\xi = \frac{\mu}{2\rho C_f\sqrt{k}} - \sqrt{\left(\frac{\mu}{2\rho C_f\sqrt{k}}\right)^2 - \frac{\sqrt{k}p_x}{\rho C_f}} \qquad (52)$$

Equating the interfacial velocities in (47) and (51), the following expression is obtained for β in terms of α:

$$\beta = \frac{\sinh\frac{\sigma}{\sqrt{\lambda}}\cosh\frac{\sigma}{\sqrt{\lambda}}\left[1 - \left(1 - \frac{\mu}{kp_x}\xi\right)\right]}{\frac{1}{\sqrt{\lambda}}\cosh^2\frac{\sigma}{\sqrt{\lambda}} + \alpha\sinh\frac{\sigma}{\sqrt{\lambda}}\left(1 + \cosh\frac{\sigma}{\sqrt{\lambda}}\right)} - \frac{\alpha\sinh^2\frac{\sigma}{\sqrt{\lambda}}\left[\sqrt{\lambda} - \frac{\sigma\mu}{kp_x}\xi\right]}{\frac{1}{\sqrt{\lambda}}\cosh^2\frac{\sigma}{\sqrt{\lambda}} + \alpha\sinh\frac{\sigma}{\sqrt{\lambda}}\left(1 + \cosh\frac{\sigma}{\sqrt{\lambda}}\right)} \qquad (53)$$

Upon using $\alpha = \sqrt{\lambda}$ in (53), the following relationship between β and α is obtained:

$$\beta = \frac{\sinh\frac{\sigma}{\alpha}\left\{\left[1 - \left(1 - \frac{\mu}{kp_x}\xi\right)\cosh\frac{\sigma}{\alpha}\right] - \left[\alpha^2 - \frac{\sigma\mu\sqrt{\vartheta}}{kp_x}\xi\right]\sinh\frac{\sigma}{\alpha}\right\}}{\frac{1}{\alpha}\cosh^2\frac{\sigma}{\alpha} + \alpha\sinh\frac{\sigma}{\alpha}\left(1 + \cosh\frac{\sigma}{\alpha}\right)} \qquad (54)$$

6 Modified BJ Condition for Inclined Flow of Pressure-dependent Viscosity Fluids

The preceding models and interfacial conditions have recently reviewed by Roach and Hamdan, [16]. In what follows, the problem of flow of a pressure-dependent viscosity fluid through a channel over a semi-infinite Darcy layer. Expressions for the interfacial velocity are obtained and relationships between the slip parameters in this type of flow and the slip parameter in case of flow of fluids with constant viscosities are derived.

Consider the steady flow of an incompressible fluid with pressure-dependent viscosity through a channel of depth h underlain by a semi-infinite porous layer inclined at an angle ϑ to the horizontal, as shown in **Fig. 2.** The channel is bounded at $y = h$ by a solid, impermeable wall, on which the no-slip condition is imposed. Fluid in the channel and the fluid saturating the porous layer are assumed to be of the same densities.

The following governing equations are based on models of pressure-dependent viscosity fluids, reported in the works of Fusi *et.al.*, [17], and Housiadas *et.al.*, [18]. Boundary and interfacial conditions are based rooted in the works of Abu Zaytoon and Hamdan, [19].

Governing equations for flow in the channel, $0 < y < h$, are given by:

$$\mu\frac{d^2v}{dy^2}+\frac{d\mu}{dy}\frac{dv}{dy}+\rho gsin\vartheta=0 \quad (55)$$

$$\frac{dp}{dy}+\rho gcos\vartheta=0 \quad (56)$$

and for flow in the porous layer by:

$$\rho gsin\vartheta-\gamma u_d=0 \quad (57)$$

$$\frac{dp_d}{dy}+\rho gcos\vartheta=0 \quad (58)$$

where γ is a function of velocity, pressure and position.

Pressure conditions at the boundary and at the interface, and the pressure distributions across the channel and the porous layer as the same as given by equations (29) and (31).

It is assumed that the viscosities are functions of pressure, so that the governing equations (55)-(58) represent a determinate system of four scalar equations in four unknowns.

Fluid viscosity variations with pressure are assumed to be according to the following relationships:

In the porous layer:

$$\gamma=\frac{\mu_d(p_d)}{k}=\frac{\mu_1e^{\alpha_1p_d}}{k} \quad (59)$$

In the channel:

$$\mu(p)=\mu_2e^{\alpha_2p} \quad (60)$$

where $\mu_1>0$ and $\mu_2>0$ are reference constant viscosities, $a_1>0$ and $a_2>0$ are parameters.

From (57) and (59), the Darcy tangential velocity in the porous layer takes the form:

$$u_d=\delta\frac{k}{\mu_1}e^{-a_1p_d} \quad (61)$$

where

$$\delta=\rho gsin\vartheta \quad (62)$$

Viscosities must be equal at the interface. Equations (59) and (60) give the following relationships when viscosities are set to be equal at $y=0$:

$$ln(\mu_2/\mu_1)=(a_1-a_2)[p_0+\beta h] \quad (63)$$

Equation (63) gives $\mu_2=\mu_1$ then $a_1=a_2$. If $\mu_2<\mu_1$ then $a_1<a_2$ and if $\mu_2>\mu_1$ then $a_1>a_2$.

For convenience, the independent variable is changed from y to p, and derivatives with respect to y and with respect to p are related by:

$$\frac{d}{dy}=-A\frac{d}{dp} \quad \text{and} \quad \frac{d^2}{dy^2}=A^2\frac{d^2}{dp^2} \quad (64)$$

wherein

$$A=\rho gcos\vartheta \quad (65)$$

Equation (55) thus takes the following form:

$$\mu A^2\frac{d^2v}{dp}+A^2\frac{d\mu}{dp}\frac{dv}{dp}+\delta=0 \quad (66)$$

Using (60) in (66), and subsequent division by $\mu_2e^{a_2p}A^2$, equation (66) takes the form

$$\frac{d^2v}{dp^2}+a_2\frac{dv}{dp}=-\frac{\delta}{\mu_2A^2}e^{-a_2p} \quad (67)$$

Equation (67) admits the general solution

$$v(p)=c_1+c_2e^{-a_2p}+\frac{\delta}{a_2\mu_2A^2}pe^{-a_2p} \quad (68)$$

or equivalently,

$$v(y)=c_1+c_2e^{-a_2[p_0+A(h-y)]}+\frac{\delta}{a_2\mu_2A^2}[p_0+A(h-y)]e^{-a_2[p_0+A(h-y)]} \quad (69)$$

where c_1 and c_2 are arbitrary constants.

At the upper bounding wall, the no-slip condition $v(y=1)=0$, takes the following form in terms of pressure:

$$v(p_0)=0 \quad (70)$$

Equation (68) then gives:

$$\frac{dv}{dp} = -a_2 c_2 e^{-a_2 p} + \frac{\delta}{a_2 \mu_2 A^2}[1 - a_2 p]e^{-a_2 p} \quad (71)$$

The Beavers and Joseph condition is given by the following derivative form at $y = 0$

$$\frac{dv}{dy} = \frac{\varepsilon}{\sqrt{k}}(u_i - u_d(0)) \quad (72)$$

where ε is a slip parameter (for the flow at hand) and $u_i = v(0^+)$ is the interfacial velocity.

In terms of pressure, this condition is replaced by the following form, valid at $p(0) = p_0 + Ah$

$$\frac{dv}{dp} = -\frac{\varepsilon}{A\sqrt{k}}(u_i - u_d(p_0 + Ah)) \quad (73)$$

where

$$u_d(y = 0) = \delta \frac{k}{\mu_1} e^{-a_1 p_d(0)} = \delta \frac{k}{\mu_1} e^{-a_1[p_0 + Ah]} \quad (74)$$

Using (70), (71), (73) and (74), the following values for the arbitrary constants c_1 and c_2 are obtained:

$$c_1 = \left[h - \frac{1}{a_2 A}\right]\frac{\delta e^{-a_2 p_0}}{\mu_2 a_2 A} + \frac{\varepsilon e^{a_2 Ah}}{A\sqrt{k}a_2}\left[\frac{\delta k e^{-a_1(p_0+Ah)}}{\mu_1} - u_i\right] \quad (75)$$

$$c_2 = \frac{\delta}{\mu_2 (a_2 A)^2}[1 - a_2(p_0 + Ah)] + \frac{\varepsilon e^{a_2(p_0+Ah)}}{a_2 A\sqrt{k}}u_i - \frac{\varepsilon\sqrt{k}\,\delta e^{(a_2 - a_1)[p_0+Ah]}}{a_2 A\mu_1} \quad (76)$$

In order to evaluate the velocity at the interface, u_i, equation (69) is evaluated at $p(0) = p_0 + Ah$ and solved for u_i to get

$$u_i = \frac{\frac{\delta h\sqrt{k}e^{-a_2 p_0}}{\mu_2} + \left\{\frac{\delta\sqrt{k}e^{-a_2 p_0}}{\mu_2 a_2 A} - \frac{\varepsilon k\,\delta e^{-a_1[p_0+Ah]}}{\mu_1}\right\}[e^{-a_2 Ah} - 1]}{[\varepsilon(e^{a_2 Ah} - 1) + a_2 A\sqrt{k}]} \quad (77)$$

If $a_1 = a_2 = a$ and $\mu_1 = \mu_2 = \mu_0$ then:

$$u_i = \frac{\frac{\delta h\sqrt{k}e^{-ap_0}}{\mu_0} + \frac{\delta\sqrt{k}}{\mu_0}\left\{\frac{e^{-ap_0}}{aA} - \varepsilon\sqrt{k}\,e^{-a[p_0+Ah]}\right\}[e^{-aAh} - 1]}{[\varepsilon(e^{aAh} - 1) + aA\sqrt{k}]} \quad (78)$$

and velocity distribution in the channel is given by

$$v(y) = \left[\frac{\varepsilon u_i}{aA\sqrt{k}} + \frac{\delta e^{-ap_0}}{\mu_0 (aA)^2} - \frac{\varepsilon\sqrt{k}}{\mu_0}\frac{\delta}{aA}e^{-a(p_0+Ah)}\right][e^{-a[A(h-y)]} - 1] + \frac{\delta e^{-ap_0}}{a\mu_0 A}[(h-y)]e^{-a[A(h-y)]} \quad (79)$$

6.1 Relationship between slip parameters when $a_1 \neq a_2$ and $\mu_1 \neq \mu_2$

Let α_1 be the slip parameter for Navier-Stokes flow over an inclined Darcy layer without viscosity variations, with interfacial velocity is given by equation (32). The slip parameter for Navier-Stokes flow over an inclined Darcy layer with viscosity variations is denoted by ε, with interfacial velocity given by equation (77).

Equating the interfacial velocities in (32) and (77), and solving for ε in terms of α_1, yields

$$\varepsilon = -\frac{a_2 A\sqrt{k}}{1 - e^{-a_2 Ah}} + \frac{2\mu(\sqrt{k} + h\alpha_1)}{h^2 + 2\alpha_1 h\sqrt{k}}\left[\frac{\mu_1 e^{-a_2(p_0+Ah)} + \mu_2 a_2 A\sqrt{k}\,e^{-a_1(p_0+Ah)}}{\mu_1\mu_2 a_2 A}\right] + \frac{2\mu h(\sqrt{k} + h\alpha_1)}{\mu_2(h^2 + 2\alpha_1 h\sqrt{k})}\frac{e^{-a_2(p_0+Ah)}}{(1 - e^{-a_2 Ah})} \quad (80)$$

If $a_1 = a_2$ and $\mu_1 = \mu_2 = \mu_0$, where μ_0 is the constant viscosity is a Darcy layer, then equation (80) takes the form

$$\varepsilon = \frac{-\sqrt{k}aA}{(1 - e^{-aAh})} + \frac{2(\sqrt{k} + h\alpha_1)}{(h^2 + 2\alpha_1 h\sqrt{k})}e^{-a(p_0+Ah)}\left\{\frac{h}{1 - e^{-aAh}} + \frac{1}{aA} + \sqrt{k}\right\} \quad (81)$$

Equations (80) and (81) shows that the value of slip parameter ε is a function of material parameters k, h, p_0, parameters a_1, a_2, and a, reference viscosity coefficients, and $A = \rho g cos\vartheta$.

7 Conclusion

In this work, an overview of the extensions of the Beaver's and Joseph slip hypothesis has been provided and problems of interest in this field have been grouped into five variations. For each of the variations discussed in the literature to date, this work provided a summary of its findings, expressions for interfacial velocities, and relationships between their respective slip parameters and the Beavers and Joseph slip parameter. Solution to the problem of flow of pressure-dependent viscosity fluids through and over an inclined porous layer was obtained. The effects of viscosity variations on the interfacial velocity and the BJ slip parameter were discussed.

Future work includes analysis and simulation of the slip velocities, shear stresses at the interface, and mass flow rates across the finite layer in the flow configuration, and the effects of flow and medium parameters on interfacial quantities and flow characteristics.

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Contribution of individual authors

Authors worked independently and contributed equally to the literature survey, problem formulation and solution, and the writing of manuscript.

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