The Neumann Type Boundary Value Problem in the Theory of Thermoelasticity with Microtemperatures for a Plane with Circular Hole

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Abstract

This paper studies the linear theory of thermoelastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures. The present work considers the 2D equilibrium theory of thermoelasticity for solids with microtemperatures. This paper is devoted to the explicit solution of the Neumann type boundary value problem for an elastic plane, with microtemperatures having a circular hole. Special representations of the regular solutions of the considered equations are constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. Using the Fourier method, we presented the solution of the Neumann type boundary value problem for the plane with circular hole in the explicit form.

1. Introduction

The linear theory of thermoelasticity for materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was established by Grot [1]. Ieşan and Quintanilla in [2] considered the dynamical theory of homogeneous and isotropic thermoelastic bodies, have formulated the boundary value problems and presented an uniqueness theorems, also they established a counterpart of the Boussinesq-Somigliana-Galerkin solution in the classical elastostatics. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze in [3]. The representations of the Galerkin type and general solutions of the system of static of the above mentioned theory were obtained by Scalia, Svanadze, and Tracina [4]. The linear theory of micromorphic elastic solids with microtemperatures was considered by Ieşan [5].

For applications, it is especially important to construct the solutions boundary value problems (BVPs) in explicit form because such solutions enable us to effectively perform quantitative analysis of the investigated problems. Some works of the 2D and 3D theories of elasticity for materials with microstructures can be seen in [6-30], in which give the main results and bibliographical data and also the explicit solutions are constructed for some BVPs for the concrete domains.

The present paper considers the 2D equilibrium theory of thermoelasticity for solids with microtemperatures. The present article is arranged as follows: in Section 2, the basic equations are presented in terms of the displacement vector, temperature and the microtemperatures. The basic BVP is formulated for a plane with a circular hole. In Section 3, the regular solutions of the considered equations are given by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. In Section 4 the Neumann type BVP is solved explicitly for a plane with a circular hole. The obtained solutions are given in the explicit form.

2. Basic Equations. Boundary value problems

Let $X = (x_1, x_2)$ be a point in the Euclidean two-dimensional space E^2 . Let D^- be the whole plane with a circular hole, bounded by the circumference S with center at the origin and radius R . Let us assume that the domain D^- is filled with an isotropic thermoelastic material with micro-temperatures.

The governing homogeneous system of the theory of thermoelasticity with microtemperatures has the form [3]

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad} \text{div} \mathbf{u} - \text{grad}(\beta \theta) = 0, \quad (1)$$

$$k \Delta \theta + k_1 \text{div} \mathbf{w} = 0, \quad (2)$$

$$k_6 \Delta \mathbf{w} + (k_4 + k_5) \text{grad} \text{div} \mathbf{w} - k_3 \text{grad} \theta - k_2 \mathbf{w} = 0, \quad (3)$$

where $\mathbf{u}(\mathbf{x}) = (u_1, u_2)^T$ is the displacement vector in a solid,

$\mathbf{w} = (w_1, w_2)^T$ is called microtemperatures vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0 > 0$), of the natural state, $k_j, k, \lambda, \mu, \beta$ are the

constitutive coefficients, Δ is the 2D Laplace operator. superscript "T" stands for the transpose operation.

Definition. A vector-function $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)^T$ defined in the domain D^- is called regular if $\mathbf{U} \in C^2(D^-) \cap C^1(\overline{D^-})$ and

the vector $\mathbf{U}$ should additionally satisfy the following conditions at

$$\mathbf{U} \in O(|\mathbf{x}|^{-1}), \quad \frac{\partial \mathbf{U}}{\partial x_j} \in O(|\mathbf{x}|^{-2}), \quad j = 1, 2,$$

infinity:

$$|\mathbf{x}|^2 = x_1^2 + x_2^2.$$

For the equations (1)-(3) we consider the following basic BVP.

Problem 1. Find a regular solution $\mathbf{U}$ to the equations (1)-(3) in the domain D^- , satisfying the following boundary conditions on S :

$$(\mathbf{PU})^- = \mathbf{f}^-(\mathbf{z}), \quad \mathbf{w}^- = \mathbf{F}^-(\mathbf{z}), \quad \theta^- = f_3^-(\mathbf{z}),$$

$$z \in S,$$

where $(\cdot)^-$ denotes the limits of vector $\mathbf{V} = (\mathbf{Pu}, \mathbf{w}, \theta)^T$ on $z \in S$ from D^-

$$\mathbf{V}^-(\mathbf{z}) = \lim_{D^- \ni \mathbf{x} \rightarrow \mathbf{z} \in S} \mathbf{V}(\mathbf{x}),$$

$\mathbf{f}(f_1, f_2)$, $\mathbf{F}(F_1, F_2)$ and f_3 are the given functions. Moreover we assume that the boundary values are absolute integrable functions. The vector $\mathbf{P}(\partial_x, \mathbf{n})\mathbf{U}$ is defined in the following form

$$\mathbf{P}(\partial_x, \mathbf{n})\mathbf{U} = \mathbf{T}(\partial_x, \mathbf{n})\mathbf{u} - \mathbf{n}(\beta\theta), \quad (4)$$

$\mathbf{T}(\partial_x, \mathbf{n})\mathbf{u}$ is the stress vector in the classical theory of elasticity

$$\mathbf{T}(\partial_x, \mathbf{n})\mathbf{u} = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda \mathbf{n} \operatorname{div} \mathbf{u} + \mu \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right),$$

$\mathbf{n}(n_1, n_2)$ is a normal vector, and $\frac{\partial}{\partial n}$ is the derivative along the vector $\mathbf{n}(n_1, n_2)$.

3. Representation of regular solutions of the system (1)-(3)

In [10] it is proved that the regular solution of the system (2),(3) can be represented in the form

$$\mathbf{w} = -\operatorname{grad} \left[\frac{k_3}{k_2} \mathcal{G} + \frac{k}{k_1} \mathcal{G}_1 \right] + \overset{2}{\mathbf{w}},$$

$$\overset{2}{\mathbf{w}} = -\frac{k_6}{k_2} \operatorname{rotrot} \mathbf{w}, \quad \theta = \mathcal{G} + \mathcal{G}_1,$$

where $\Delta \mathcal{G} = 0$, $\operatorname{div} \overset{2}{\mathbf{w}} = 0$,

$$(\Delta - s_1^2) \mathcal{G}_1 = 0, \quad (\Delta - s_2^2) \overset{2}{\mathbf{w}} = 0, \quad s_2^2 = \frac{k_2}{k_6} > 0,$$

$$\operatorname{div} \mathbf{w} = -\frac{k}{k_1} s_1^2 \mathcal{G}_1, \quad s_1^2 = \frac{kk_2 - k_1 k_3}{k^* k_7} > 0.$$

Let us assume that θ is known functions when $\mathbf{x} \in D^-$. Then from (1), after obvious transformations, for $\mathbf{u}(\mathbf{x})$ and $\operatorname{div} \mathbf{u}$ we obtain the following non-homogeneous equations

$$\Delta \operatorname{div} \mathbf{u} = \frac{\beta s_1^2}{\mu_0} \mathcal{G}_1, \quad \mu_0 = \lambda + 2\mu,$$

$$\Delta \mathbf{u} = -\operatorname{grad} \left[\frac{\lambda + \mu}{\mu} \psi - \frac{\beta}{\mu} \mathcal{G} - \frac{\beta}{\mu_-} \mathcal{G}_1 \right], \quad (5)$$

solutions of which can be written as

$$\operatorname{div} \mathbf{u} = \psi + \frac{\beta}{\mu_0} \mathcal{G}_1,$$

$$\mathbf{u} = \Psi - \operatorname{grad} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right],$$

where Ψ and ψ are an arbitrary harmonic functions $\Delta \Psi = 0$, $\Delta \psi = 0$, the functions $\mathcal{G}$, ψ , $\mathcal{G}_0$, ψ_0 and Ψ are interrelated by the following relations:

$$\Delta \mathcal{G}_0 = \mathcal{G}, \quad \Delta \psi_0 = \psi, \quad \operatorname{div} \Psi = \frac{\mu_0}{\mu} \psi - \frac{\beta}{\mu} \mathcal{G}.$$

From the above reasoning we have proved the following theorem:

Theorem 3 The regular solution $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)^T$ of equations (1)-(3) admits in the domain D a representation

$$\mathbf{u} = \Psi - \operatorname{grad} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right]. \quad (6)$$

$$\mathbf{w} = -\operatorname{grad} \left[\frac{k_3}{k_2} \mathcal{G} + \frac{k}{k_1} \mathcal{G}_1 \right] + \overset{2}{\mathbf{w}}, \quad \overset{2}{\mathbf{w}} = -\frac{k_6}{k_2} \operatorname{rotrot} \mathbf{w},$$

$$\theta = \mathcal{G} + \mathcal{G}_1, \quad \operatorname{div} \Psi = \frac{\mu_0}{\mu} \psi - \frac{\beta}{\mu} \mathcal{G},$$

$$\operatorname{div} \mathbf{u} = \psi + \frac{\beta}{\mu_0} \mathcal{G}_1, \quad \operatorname{div} \mathbf{w} = -\frac{k}{k_1} s_1^2 \mathcal{G}_1.$$

4. Explicit solution of the Problem 1

Let us introduce the polar coordinates:

$$x_1 = \rho \cos \eta, \quad x_2 = \rho \sin \eta,$$

$$\rho = \sqrt{x_1^2 + x_2^2}, \quad 0 \leq \eta \leq 2\pi.$$

Note that in 2D space "rot" is defined as a scalar

$$\operatorname{rot} \Phi = \frac{\partial \Phi_2}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_2}$$

for a vector $\Phi = (\Phi_1, \Phi_2)$ and as a vector

$$\operatorname{rot} \psi = \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right) \quad \text{for a scalar } \psi.$$

The following identities are holds:

$$\frac{\partial}{\partial n} \operatorname{grad} g = \frac{1}{\rho} \operatorname{grad} \left(\rho \frac{\partial}{\partial \rho} - 1 \right) g,$$

$$\operatorname{div} \mathbf{n} g = \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) g,$$

$$\operatorname{div} \frac{\partial \mathbf{u}}{\partial n} = \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) \operatorname{div} \mathbf{u} - \frac{1}{\rho} \left(\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial n} \right) =$$

$$\left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) \operatorname{div} \mathbf{u} - \frac{1}{\rho^2} \left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n} \right)$$

$$+ \frac{1}{\rho} \frac{\partial^2}{\partial \rho^2} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right],$$

$$\operatorname{div} \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} \left(\frac{\partial \Psi_2}{\partial x_1} - \frac{\partial \Psi_1}{\partial x_2} \right) = -\frac{\partial}{\partial \rho} \operatorname{div} \Psi,$$

$$\text{if } \Delta \Psi = 0,$$

$$\left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n} \right) = \left(\frac{\partial}{\partial n} - \frac{1}{\rho} \right) (\mathbf{x} \cdot \Psi),$$

$$\lambda \operatorname{div} \mathbf{u} - \beta \theta = \lambda \psi - \beta \mathcal{G} - \frac{2\mu\beta}{\mu_0} \mathcal{G}_1.$$

Using these identities, the stress vector takes the form

$$\begin{aligned} \mathbf{P}(\partial_x, \mathbf{n}) \mathbf{U} &= 2\mu \frac{\partial \Psi}{\partial n} - \\ &\frac{2\mu}{\rho} \operatorname{grad} \left(\rho \frac{\partial}{\partial \rho} - 1 \right) \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right] \quad (7) \\ &+ \mathbf{n} [\lambda \operatorname{div} \mathbf{u} - \beta \theta] + \mu \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} \left(\frac{\partial \Psi_2}{\partial x_1} - \frac{\partial \Psi_1}{\partial x_2} \right). \end{aligned}$$

From (4), (6) and (7), by direct calculation, we readily get:

$$\begin{aligned} (\mathbf{x} \cdot \mathbf{P} \mathbf{u}) &= 2\mu \left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n} \right) \\ &- 2\mu \rho \frac{\partial^2}{\partial \rho^2} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right] \\ &+ \rho [\lambda \operatorname{div} \mathbf{u} - \beta \theta], \end{aligned}$$

$$\begin{aligned} \operatorname{div} \mathbf{P} \mathbf{u} &= -\frac{2\mu}{\rho^2} \left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n} \right) \\ &+ \frac{2\mu}{\rho} \frac{\partial^2}{\partial \rho^2} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right] \quad (8) \\ &+ \frac{\mu_0 \psi - \beta \mathcal{G}}{\rho}, \end{aligned}$$

$$\begin{aligned} \frac{1}{\rho^2} (\mathbf{x} \cdot \mathbf{P} \mathbf{u}) + \operatorname{div} \mathbf{P} \mathbf{u} &= \\ \frac{2}{\rho} \left((\lambda + \mu) \psi - \frac{\mu\beta}{\mu_0} \mathcal{G}_1 - \beta \mathcal{G} \right), \end{aligned}$$

$$(\mathbf{x} \cdot \mathbf{w}) = -\rho \frac{\partial}{\partial \rho} \left[\frac{k_3}{k_2} \mathcal{G} + \frac{k^*}{k_1^*} \mathcal{G}_1 \right] + \left(\mathbf{x} \cdot \mathbf{w}^2 \right),$$

$$\theta = \mathcal{G} + \mathcal{G}_1, \quad \operatorname{div} \mathbf{w} = -\frac{k}{k_1} s_1^2 \mathcal{G}_1,$$

$$(\Delta - s_2^2) \left(\mathbf{x} \cdot \mathbf{w}^2 \right) = 0,$$

where $(\mathbf{x} \cdot \mathbf{u})$ denotes the scalar product of two vectors.

A solution of Problem 1 is sought in the domain D^- in the form (6), where the harmonic and the meta-harmonic functions

$\mathcal{G}$, ψ , $\mathcal{G}_1$, and $\left(\mathbf{x} \cdot \mathbf{w}^2 \right)$ can be represented as follows

$$\mathcal{G}(\mathbf{x}) = \frac{E_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{\rho} \right)^n (E_n \cos n\eta + M_n \sin n\eta),$$

$$\psi(\mathbf{x}) = \frac{A'_0}{2} + \sum_{n=1}^{\infty} \left(\frac{R}{\rho} \right)^n (A'_n \cos n\eta + B'_n \sin n\eta),$$

$$\mathcal{G}_1 = \frac{C_{10}}{2} \frac{H_0^{(1)}(is_1 \rho)}{H_0^{(1)}(is_1 R)}$$

$$+ \sum_{n=1}^{\infty} \frac{H_n^{(1)}(is_1 \rho)}{H_n^{(1)}(is_1 R)} (C_{1n} \cos n\eta + D_{1n} \sin n\eta),$$

$$\left(\mathbf{x} \cdot \mathbf{w}\right)^2 = \frac{C_{10}''}{2} \frac{H_0^{(1)}(is_2 \rho)}{H_0^{(1)}(is_2 R)} + \sum_{n=1}^{\infty} \frac{H_n^{(1)}(is_2 \rho)}{H_n^{(1)}(is_2 R)} (C_{1n}'' \cos n\eta + D_{1n}'' \sin n\eta), \quad (9)$$

where $E_n, M_n, \dots$ are the sought coefficients, $H_n^{(1)}(z)$ is the Hankel's function with the index n . Taking into account (9), we can write the particular solutions of equations $\Delta \mathcal{G}_0 = \mathcal{G}$, $\Delta \psi_0 = \psi$ in the form

$$\begin{aligned} \mathcal{G}_0(\mathbf{x}) &= \frac{E_0 \rho^2}{8} - \frac{\rho^2}{4} \sum_{n=1}^{\infty} \frac{1}{n-1} \left(\frac{R}{\rho}\right)^n (E_n \cos n\eta + M_n \sin n\eta), \\ \psi_0(\mathbf{x}) &= \frac{A_0' \rho^2}{8} - \frac{\rho^2}{4} \sum_{n=1}^{\infty} \frac{1}{n-1} \left(\frac{R}{\rho}\right)^n (A_n' \cos n\eta + B_n' \sin n\eta). \end{aligned} \quad (10)$$

We introduce the following notations:

$$\begin{aligned} (\mathbf{x} \cdot \mathbf{f})^- &= h_1^-, \quad (\operatorname{div} \mathbf{f})^- = h_2^-, \quad (\mathbf{x} \cdot \mathbf{F})^- = h_3^-, \\ (\operatorname{div} \mathbf{F})^- &= h_4^-, \quad \theta = h_5^-. \end{aligned}$$

From (8), passing to the limit as $\rho \rightarrow R$, to determine unknown functions, we obtain the following conditions

$$\begin{aligned} \left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n}\right) &= R \frac{\partial^2}{\partial \rho^2} \left[\frac{\lambda + \mu}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 - \frac{\beta}{\mu_0 s_1^2} \mathcal{G}_1 \right] \\ &+ \left[\frac{\mu_0 \psi - \beta \mathcal{G}}{R} - h_2 \right] \frac{R^2}{2\mu} = G_4, \\ (\lambda + \mu) \psi - \frac{\mu \beta}{\mu_0} \mathcal{G}_1 - \beta \mathcal{G} &= \frac{R}{2} \left(h_2 + \frac{h_1}{R^2} \right), \\ \mathcal{G} + \mathcal{G}_1 &= h_5, \quad -\frac{k}{k_1} s_1^2 \mathcal{G}_1 = h_4, \end{aligned} \quad (11)$$

$$\left(\mathbf{x} \cdot \mathbf{w}\right)^- = R \frac{\partial}{\partial \rho} \left[\frac{k_3}{k_2} \mathcal{G} + \frac{k^*}{k_1^*} \mathcal{G}_1 \right] + h_3 = G_3.$$

At first, from system (11), we find

$$\begin{aligned} \mathcal{G}_1^- &= -\frac{k_1}{k s_1^2} h_4^- = G_1, \quad \mathcal{G}^- = h_5^- - G_1 = G, \\ \psi^- &= \frac{1}{\lambda + \mu} \left[\beta G - \frac{\mu \beta}{\mu_0} G_1 + \frac{R}{2} \left(\frac{1}{R^2} h_1^- + h_2^- \right) \right] = G_2. \end{aligned}$$

On the other hand, for determining the unknown coefficients, from (9) it follows

$$\begin{aligned} \frac{E_0}{2} + \sum_{n=1}^{\infty} (E_n \cos n\eta + M_n \sin n\eta) &= G, \\ \frac{A_0'}{2} + \sum_{n=1}^{\infty} (A_n' \cos n\eta + B_n' \sin n\eta) &= G_2, \\ \frac{C_{10}}{2} + \sum_{n=1}^{\infty} (C_{1n} \cos n\eta + D_{1n} \sin n\eta) &= G_1, \\ \frac{C_{10}'}{2} + \sum_{n=1}^{\infty} (C_{1n}' \cos n\eta + D_{1n}' \sin n\eta) &= G_3, \end{aligned}$$

From here we get

$$\begin{aligned} E_n &= \frac{1}{\pi} \int_0^{2\pi} G^-(\eta) \cos n\eta d\eta, \\ M_n &= \frac{1}{\pi} \int_0^{2\pi} G^-(\eta) \sin n\eta d\eta, \\ A_n' &= \frac{1}{\pi} \int_0^{2\pi} G_2(\eta) \cos n\eta d\eta, \\ B_n' &= \frac{1}{\pi} \int_0^{2\pi} G_2(\eta) \sin n\eta d\eta, \\ C_{1n} &= \frac{1}{\pi} \int_0^{2\pi} G_1^-(\eta) \cos n\eta d\eta, \\ D_{1n} &= \frac{1}{\pi} \int_0^{2\pi} G_1^-(\eta) \sin n\eta d\eta, \\ C_{1n}' &= \frac{1}{\pi} \int_0^{2\pi} G_3^-(\eta) \cos n\eta d\eta, \\ D_{1n}' &= \frac{1}{\pi} \int_0^{2\pi} G_3^-(\eta) \sin n\eta d\eta, \end{aligned} \quad (12)$$

Substituting (12) into (9) and (10), we determine the functions

$$\begin{aligned} \mathcal{G}(\mathbf{x}) &= \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{R}{\rho}\right)^n \cos n(\eta - \eta_0) \right] G^-(\eta) d\eta \\ &= \frac{-1}{2\pi} \int_0^{2\pi} \frac{(R^2 - \rho^2) G^-(\eta) d\eta}{R^2 - 2R\rho \cos(\eta - \eta_0) + \rho^2}, \end{aligned}$$

$$\begin{aligned}\psi(\mathbf{x}) &= \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{R}{\rho} \right)^n \cos n(\eta - \eta_0) \right] G_2(\eta) d\eta \\ &= \frac{-1}{2\pi} \int_0^{2\pi} \frac{(R^2 - \rho^2) G_2(\eta) d\eta}{R^2 - 2R\rho \cos(\eta - \eta_0) + \rho^2}\end{aligned}$$

$$\begin{aligned}\left(\mathbf{x} \cdot \mathbf{w} \right) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{H_0^{(1)}(is_2\rho)}{H_0^{(1)}(is_2R)} G_3^-(\eta) d\eta \\ &+ \frac{1}{\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{H_n^{(1)}(is_2\rho)}{H_n^{(1)}(is_2R)} \cos n(\eta - \eta_0) G_3^-(\eta) d\eta, \\ \mathcal{G}_1(\mathbf{x}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{H_0^{(1)}(is_1\rho)}{H_0^{(1)}(is_1R)} G_1^-(\eta) d\eta, \\ &+ \frac{1}{\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{H_n^{(1)}(is_1\rho)}{H_n^{(1)}(is_1R)} \cos n(\eta - \eta_0) G_1^-(\eta) d\eta, \\ \mathbf{x} &= (\rho, \eta_0) \in D,\end{aligned}$$

$$\mathcal{G}_0(\mathbf{x}) = \frac{\rho^2}{8\pi} \int_0^{2\pi} G(\eta) d\eta,$$

$$\frac{\rho^2}{4\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n-1} \left(\frac{R}{\rho} \right)^n \cos n(\eta - \eta_0) G(\eta) d\eta,$$

$$\psi_0(\mathbf{x}) = \frac{\rho^2}{8\pi} \int_0^{2\pi} G_2(\eta) d\eta,$$

$$\frac{\rho^2}{4\pi} \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n-1} \left(\frac{R}{\rho} \right)^n \cos n(\eta - \eta_0) G_2(\eta) d\eta,$$

Thus, we have defined the functions

$$\mathcal{G}, \mathcal{G}_0, \psi, \psi_0, \mathcal{G}_1, \quad \text{in the domain } D^-.$$

Now let us consider the following equation

$$\left(\mathbf{x} \cdot \frac{\partial \Psi}{\partial n} \right) = \left(\frac{\partial}{\partial n} - \frac{1}{\rho} \right) (\mathbf{x} \cdot \Psi) = G_4.$$

It is easily to verified, that the function $(\mathbf{x} \cdot \Psi)$, is the solutions of the following equation

$$\Delta(\mathbf{x} \cdot \Psi) = 2 \operatorname{div} \Psi = 2 \left(\frac{\mu_0}{\mu} \psi - \frac{\beta}{\mu} \mathcal{G} \right).$$

From here we find

$$(\mathbf{x} \cdot \Psi) = \Omega + 2 \left(\frac{\mu_0}{\mu} \psi_0 - \frac{\beta}{\mu} \mathcal{G}_0 \right),$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0$ and the functions $\mathcal{G}_0$, and ψ_0 are chosen such that

$$\Delta\mathcal{G}_0 = \mathcal{G}, \quad \Delta\psi_0 = \psi.$$

Finally, for determining the function Ω we obtain the following Robin boundary value problem.

Find the harmonic function Ω in the domain D^- ,

$$\Delta\Omega = 0, \quad x \in D^-, \quad \text{satisfying the following boundary}$$

conditions on S :

$$\left(\frac{\partial \Omega}{\partial n} - \frac{\Omega}{R} \right)^- = G_4 - \frac{2}{\mu} \left(\frac{\partial}{\partial n} - \frac{1}{\rho} \right) (\mu_0 \psi_0 - \beta \mathcal{G}_0) = G_5^-,$$

solution of which has the form

$$\Omega(\mathbf{x}) = \frac{1}{\pi} \int_S \ln \frac{r(\mathbf{x}, \mathbf{y})}{r(\mathbf{x})} g(y) ds + \frac{1}{\pi} \int_S g(y) ds, \quad (13)$$

$$r^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

where $g(y)$ is a solution of the following Fredholm integral equation of second kind

$$\begin{aligned}g(x) &+ \frac{1}{\pi} \int_S \frac{\partial}{\partial n_x} \ln \frac{r(x, y)}{r(x)} g(y) ds \\ &- \frac{1}{R\pi} \int_S \ln \frac{r(x, y)}{r(x)} g(y) ds - \frac{1}{R\pi} \int_S g(y) ds = G_5^-. \quad (14)\end{aligned}$$

It is well known that integral equation (14) is always solvable.

Thus, we have defined all unknown functions. Through inserting the obtained values into (6), we get the final form for solution of the considered Problem 1.

We assume that the functions f_j , $j = 3, 4$, $\mathbf{f}$, $\mathbf{F}^+$, satisfy the following conditions on S

$$f_j \in C^3(S) \quad j = 3, 4, \quad \mathbf{f}, \quad \mathbf{F}^+ \in C^3(S).$$

Under these conditions the resulting series are absolutely and uniformly convergent.

Thus, the problem 1 is completely solved.

5. Conclusions

The main results of this work can be formulated as follows:

- The regular solution $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta)^T$ of equations (1)-(3) is obtained.
- Explicit solution of problem 1 for a plane with circular hole is presented. The obtained solutions are given by means of the

harmonic, bi-harmonic and meta-harmonic functions. For the harmonic functions $\mathcal{G}(\mathbf{x})$ and $\psi(\mathbf{x})$ the Poisson type formulas are obtained. The bi-harmonic and meta-harmonic functions are presented as absolutely and uniformly convergent series. The harmonic function $\Omega(\mathbf{x})$ is given by (13), where the function $g(y)$ is a solution of the Fredholm integral equation of second kind.

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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