Mathematical Analysis: How Would AI Tackle Math Olympiad Problems?

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ABSTRACT: Artificial Intelligence (AI) has increasingly gained attention as a novel approach to problem solving. Computer scientists are currently in a race to build an AI system that will win a gold medal at the world's premier math competition, the International Mathematical Olympiad (IMO). In order to succeed, it is critical to design and evaluate a way to assess how AI undertakes Olympiad questions. This paper examines the ways in which students are evaluated for their mathematical skills and suggests that such assessment be implemented for the AI in order for it to successfully solve math Olympiad questions. The challenge lies in determining a method of applying mathematical creativity and thinking ability to AI.

KEYWORDS: Mathematics; Other; Artificial Intelligence; Problem Solving; International Mathematical Olympiad.

■ Introduction

The fields of engineering and natural sciences have contributed significantly to the development of modern technology. Among many, mathematics has had the most crucial impacts on technology because it has helped humans think logically, critically, and innovatively. Therefore, to master mathematics well, the government establishes mathematics as one of the fields of science studied at every level of education from elementary school to college in most countries.¹

As mathematics is highly valued, there has been many attempts to assess one's understanding. However, it has remained a difficult task to find valid and standardized instruments that can identify those gifted in mathematics. After long years of research, mathematicians have come to a conclusion that mathematical creativity and thinking ability are the fundamental requirements to be a talented mathematician.² And now, computer scientists are taking this a step further and trying to build an Artificial Intelligence (AI) system that can win a gold medal at the world's premier math competition, the International Mathematical Olympiad (IMO).³ To achieve this, it is critical to design and evaluate an approach that will assess how AI undertakes Olympiad questions.

■ Results and Discussion

Values Required for Olympiad Problems:

According to Kim and Cho, mathematical creativity is like a plan and it contains four main aspects: fluency, flexibility, originality, and elaboration.² In their explanations, fluency addresses how well the person can generate and create multiple responses, while flexibility addresses how well the person can generate different categories of responses that are not fixed on just one idea. Originality describes how unique the responses are, while elaboration describes how extensive a person can develop a simple design into a more complex one.²

On the other hand, mathematical thinking ability can be more like a set of techniques that help you execute a plan. Thinking abilities are composed of several sub-abilities includ-

ing intuitive insight (picking up critical cues in the problem), organization of knowledge necessary for solving the problem, visualization of the given information, abstraction of the problem into concepts, symbols, formulas, and figures, reasoning through inductive and deductive thinking, generalization of the mathematical relationships, and reflection of own problem-solving instances in the past.²

Questions in mathematical competitions such as the IMO often require both mathematical creativity and thinking abilities at the same time. Thus, the biggest task is to educate AI so that it can apply both mathematical creativities and thinking abilities appropriately.

How to Perceive and Solve Olympiad Questions

A. Overview

Solving mathematical problems is similar to hunting for the hidden treasure; there is a final goal to reach and several clues are given upfront. For easy questions, the information is too obvious to miss, or the amount of information given is more than enough. However, the more challenging questions in the IMO will give you indirect clues and these will often be ill-defined so that the clues are not easily noticed. At the same time, many different types of questions will cover a wide and high level of mathematics and demand both creativity and thinking ability.

There is always a good place to start no matter how complicated a question is and how peculiar its topic is. Usually, problem-solving strategies such as heuristics, is very effective in making progress on unfamiliar and non-standard problems without evident clues. According to Cheung, one should first understand the problem and then devise, carry out, and revise the plan to tackle unfriendly problems successfully.⁴

In most cases, searching for a pattern is a good start because it could lead to a conjecture or even a proof. Sometimes, few modifications can be made to a question in order to find the pattern. Simply, the question can be restated in terms of simpler words or mathematical symbols if it contains too many

difficult terms that confuse you. Once the question is turned into a similar but easier one, a pattern can be more easily noted.

Although heuristics are handy in understanding and analyzing problems, they often end up providing tedious and possibly inefficient solutions. In other words, the most efficient and interesting strategies are topic-oriented and interdisciplinary. Therefore, a skilled student should be able to apply different approaches for different types of problems. The below examples are some of the popular types of approaches that are categorized and analyzed by Cheung.⁴

B. Methods of Proof:

Proof problems are easily found in the IMO, but most students often find them tricky. The reason why students find the proof problems difficult is because it takes more knowledge, discipline, and effort to prove a general statement than to prove only a single case. Unlike what is expected, general methods of proof usually do not require multiple proofs to prove a proposition to be true. Instead, they involve two different proofs with connecting logic.

There are two major approaches to proof problems: induction and deduction. Proof by induction is very common in Olympiad problems. As mathematical induction is a technique for proving a statement for natural numbers only, if one can easily prove a statement; the statement should then be proven to be true for the first iteration and its $(n+1)^{\text{th}}$ iteration is then proven to be true when the statement still holds for the n^{th} case. Contrarily, proof by deduction establishes a true proposition by providing well-known mathematical principles that have already been found true. It often seems to be easier to employ proof by deduction, but it requires more caution as one needs to make sure certain principles are appropriate and applicable to specific cases.

C. Strategies on Counting:

Problems on counting are common in mathematical competitions and there are a couple of strategies to tackle such questions.

The pigeonhole principle is one of the most well-known and commonly used principles. The principle states that if n items are to be put into m containers and n is greater than m , then at least one container contains more than one item. This seemingly obvious principle is extremely useful when a question involves matching items or minimum number of trials.

The next helpful technique is the principle of inclusion and exclusion. The basic idea of the principle is that adding all the number of elements that satisfy at least one of several conditions and subtracting all the overlaps, then, any element would not be accounted more than once. The simplest case would be the following.

D. Geometry:

Unlike algebraic problems, there is almost no limit to the variety in geometric problems. Simply, there are infinitely many dots, lines, faces, and three-dimensional figures in three-dimensional space. There is always a general approach

to geometric problems, but it is often topic-oriented and unfortunately inefficient most of the time.

As mentioned above, breaking a complex polygon down to simpler pieces such as triangles and circles would be a decent start. The properties of a triangle are fairly simple but useful and so are those of a circle. Therefore, using their properties might lead to another essential equation and so on. For some cases, the properties of different types of quadrilaterals could be helpful but those of quadrilaterals are generally not as useful or not applicable to most of the cases.

E. Synthesis versus Analysis:

Cheung not only provided useful strategies to various types of Olympiad problems, but also analyzed different general approaches.⁴ In proving equations or inequalities, synthesis and analysis are the two fundamental techniques to solve such problems.

If the equation or inequality is solved synthetically, the given condition is modified multiple times using basic mathematical principles to deduce the final equation or inequality. Mostly, the basic principles are applicable unconditionally but there are often cases where some restrictions should be made in order to employ modifications based on the basic principles.

While a synthetic approach starts from given conditions, building up adequate relations, and eventually deduce a final solution, the analytic approach is exactly the opposite. In an analytic approach, the final equation or inequality is altered to obtain the initial conditions. In other words, synthesis is to figure out necessary conditions for the given condition whereas analysis looks for a sufficient condition for the final equation or inequality. Most importantly, every deduced step in both processes should be reversible.

F. Partial Attack versus Global Views:

Just like the case of synthesis and analysis above, partial attack and global view are simply different perspectives.

Partial attack is useful when a general relation should be concluded, and few pieces of information are given initially. This approach is naturally used in finding a general formula, for example, finding the general n^{th} term of a geometric sequence. The common ratio could easily be calculated if any two terms of the sequence are known. Accordingly, the value of the first term can also be found since the ratio is known. Then, the general formula for n^{th} term of the sequence, a_n , would be $a_n = a_1 r^{n-1}$ where a_1 is the first term and r is the common ratio.

The global view is technically the opposite of the partial attack. Appropriate equations and relations can be applied to calculate a solution for specific conditions. This can be as simple as using the quadratic formula to calculate the roots for a specific quadratic equation. Indeed, Olympiad problems are much more complicated than this.

How to Evaluate and Improve One's Competence for Solving Olympiad Questions

A. Overview:

Using the strategies suggested above, a student can possibly tackle many Olympiad problems successfully. However, there can be a number of steps that could go wrong in the middle of the so-called strategies. For instance, the student might not understand the problem at all, not know the purpose of

the problem, lack planning skills, not know the concepts and formulae that must be used to solve the question, and simply make mistakes. According to Hidatati's research, applying appropriate but incorrect formulae and applying random formulae were the most common errors.¹

B. Cognitive Conflict Strategy:

In order to minimize such errors, cognitive conflict strategy was applied to improve students' mathematical critical thinking ability.⁵ If a student makes a mistake, he or she is given relevant counter-information, an analogy, or relevant guided questions that challenge his or her solution. Given the information, he or she is encouraged to go over his/her own answer and hopefully they figure out what went wrong. This way of learning, cognitive conflict strategy, has helped many students think critically and realistically. Furthermore, it has made the students more positive and enthusiastic.⁵

C. Analogy in Scientific Thinking :

Clement has analyzed a problem-solving protocol and found the role of analogy in scientific thinking.⁶ Given the initial conception A, Clement emphasized that a wide range of knowledge is required to come up with an analogous conception B, and the conception B and the analogous relation must be confirmed.⁶ Once the analogy is established, it can be used to solve unfamiliar problems; they can be related to several analogous ones that are more familiar. As long as the modified problems are taken care of, the original problems can be revised in a similar manner.

D. Scientific and Problem-Solving Approach :

For the process of solving a question in general, two basic methods exist: the scientific approach and problem-solving approach. The scientific approach refers to an empirical method to construct a formula or a principle. To learn scientifically, one should observe a problem, build a hypothesis, test if the hypothesis holds, and conclude the relation. Thus, the scientific approach is useful in developing a formula in general while the problem-solving approach focuses more on individual cases. In other words, one should identify the problem, define the problem, explore the solution, act on a strategy, come up with a conclusion and review the problem.⁷ The problems that use the scientific approach are characterized by the number of trials are required until hypothesis is proven to be true. Contrarily, a number of true cases should be obtained to establish a general solution for the problem-solving approach. According to Chairuddin, students that practiced the scientific approach scored about 6 % higher mean scores than those practiced the problem-solving approach.⁷

E. Development of Metacognitive Level :

The higher the metacognitive level, the greater the creative thinking abilities. As students' metacognitive abilities develop, they are not only aware of their use but will progressively use these abilities strategically and will reflect back and evaluate these abilities.⁸

Student's metacognitive skill develops with age. With repetition, maturity, biological development, and good Olympic training, they will be able to improve on their creative thinking abilities and behavior.⁸

Students must not fixate on the concepts taught in school but must be continually exposed to multiple forms of questions to promote their creative thinking skills.⁸

Progress of AI and its Current Location

A. Overview:

While we have been flooded with a tremendous amount of decent AI applications in recent years, machines are still not good enough to understand problems, provide appropriate answers, and have serious discussions with others. To secure the next level of AI applications, it is important to understand the history, realize the current issues, know the principles and apply them properly to come up with advanced solutions. Only then, AI that can solve Olympiad problems can be invented.⁹

B. History and Application of AI:

In 1936, Alan Turing, a British mathematician, invented a Turing machine that manipulates symbols on a strip of tape based on pre-made principles. Consequently, he published an article about how to test if machines can think, if machines can learn, and how intelligent machines can be developed in a paper called *Computing Machinery and Intelligence* in 1950. Later, his work fostered the standard for modern computer technology and is seen as the beginning of the history of artificial intelligence.¹⁰

Despite Alan's early contribution to artificial intelligence, the term artificial intelligence first appeared a few years later, in a meeting known as the Dartmouth Conference in 1956. John McCarthy, a mathematician and scientist at Dartmouth College, planned a summer research project on artificial intelligence and invited his fellow researchers. In the invitation letter, he wrote "We propose that a 2 month, 10-man study of artificial intelligence be carried out during the summer of 1956 at Dartmouth College in Hanover, New Hampshire."¹¹ At this meeting, the participants discussed how to embody Turing's "thinking machine" and transform it into a system of logic and form.

Since then, early AI scholars have shown a high level of trust and confidence in the potential of artificial intelligence. Allen Newell and Herbert Alexander Simon said that computers would be the champions of chess and the computers would be able to compose music of aesthetic values. Furthermore, Herbert even stressed that machines would be able to do whatever humans can do in the next couple of decades.¹³

Although it fell short of his prophecy, remarkable progress was made over the next 20 years. Artificial intelligence research centers were established at MIT and Carnegie Mellon University and a number of possibilities were studied. In 1959, Simon and Alan created a general problem-solving algorithm that was able to solve the "Top of Hanoi" puzzle.¹³

Later, research on artificial intelligence had been split into smaller topics and started to focus on practical purposes. The 'experts system' has been introduced and it refers to a system that organizes, expresses, and stores expert knowledge possessed by humans in a specific field, so that the general public can use the knowledge. A few good examples are the following: Logic Theorist of Carnegie-Mellon University in 1956, DENDRAL, a molecular structure identification pro

gram created in the 1960s, and Mycin, a program designed to diagnose patients with infectious blood diseases and prescribe antibiotics.¹⁴

During the 2000s, based on Professor Hinton's Deep Belief Network, deep neural network (deep learning) technology began to show potential for practical use. In particular, Deep-CNN (Convolution Neural Network: specialized model for image recognition/classification) showed a 26 % recognition error rate.¹⁵

As soon as computers could perceive problems more accurately and effectively, their performance has soared abruptly. Google's AlphaGo 1.0 beat a Go world champion, Sedol Lee, in 2016 and AlphaGo 2.0, the master version of AlphaGo 1.0, beat Ke Jie, the number one ranked player at the time, at the 2017 Future of Go Summit. Even at this moment, scientists and engineers have computers collect and study large amounts of data and research what fields and applications artificial intelligence can be used to enrich human life.

■ Conclusion

In order for AI to successfully solve math Olympiad questions, one must learn how to educate AI so that it is capable of utilizing mathematical creativity and thinking abilities. This paper examines the ways in which students can be assessed for their mathematical skills and suggests that such assessment be made for AI to possibly develop AI capable of solving math Olympiad questions. In the past, the four-color theorem was finally approved by mathematicians once they developed a computer model to assist the proof. At first, this theorem, which states that only four colors are needed to color the regions of a map such that no two neighboring regions sharing a common boundary have the same color, was extremely difficult to prove, and many mathematicians held opposing views. However, the conflict was resolved once researchers developed a computer model to prove the theorem. Similar to this, the seemingly challenging math Olympiad questions may also be tackled by AI. In order to achieve this, this paper suggests that it is critical to understand the foundations behind students' learning and problem-solving skills and to devise a way to implement such skills in AI. Previously, researchers at Cornell University successfully devised open Internet collaboration on mathematical research problems.⁹

As seen in the examples of AlphaZero and AlphaGo, AI has outperformed humans as soon as they fully understood the rules of Go or Chess. Likewise, this paper has sorted some common types of mathematics Olympiad problems and provided basic approaches to them in order to help AI understand and eventually solve the problems by itself. Hopefully, this paper will lay the foundation for application of AI for solving mathematical problems, and consequently, AI systems that solve mathematics Olympiad questions themselves accurately will soon follow.

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