**CONCENTRATION RISK IN BANK LOAN PORTFOLIO'S:
MEASUREMENT, SINGLE OBLIGOR LIMITS, AND
CAPITAL ADEQUACY.**

Javier Márquez Díez-Canedo y Calixto López Castañón

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Abstract

Formal work on credit concentration risk has focused mainly on applying portfolio theory to portfolios of traded fixed income assets. No comparable counterpart has emerged however, for dealing with portfolios of everyday bank loans for which information compatible with portfolio theory is difficult or too costly to obtain. Based on the default behavior of the portfolio, as represented by default probabilities of the loans and their covariance matrix, a model is developed which relates a measure of concentration of the loan portfolio with value at risk in order to guarantee capital adequacy within a specified confidence level. It is seen that the “*Herfindahl-Hirshman*” index emerges naturally as a measure of concentration, and that there is a direct relation between this index and the “*single obligor limit*”, which is explored in detail. The results show how individual limits can be set on loans, along different dimensions of concentration, so as to ensure capital adequacy for the risk structure of the portfolio. Throughout the paper, the implications for risk management and regulation are discussed.

Keywords: Loan concentration risk, Herfindahl-Hirshman concentration index, single obligor limit, value at risk, capitalization ratio, default probabilities.

1. Introduction.

Loan concentration has long been identified as an important source of risk for banks and loan portfolios. Judging from current technical literature on credit risk¹, as far as concentration goes, the establishment of a generally accepted paradigm has remained elusive in spite of the importance of the problem. The more formal approaches which look to portfolio theory², have only been successful in dealing with portfolios of traded fixed income assets where information compatible with traditional Markowitz (1959) type models can be obtained in a cost effective manner. An interesting approach, which is oriented to avoiding the major data and analytical pitfalls while exploiting the virtues of modern portfolio theory, is presented in Altman and Saunders (1998). It must be pointed out however, that traditional portfolio theory approaches deal with the concentration issue indirectly, since the preoccupation is the allocation of assets through the well known mean-variance tradeoff, but a clear measure of concentration and its relation to risk has never been made explicit.

¹ A good reference for a review of the different approaches is Caouette et. al. 1998

² See for example Bennet 1984.

When dealing with portfolios of traditional bank loans, no comparable formal methodology seems to have emerged. As pointed out by Altman and Saunders (1998), the concentration measurement issue has been dealt with through subjective analysis. Typically, banks and other agents apply a scoring technique based on the opinion of a group of experts, about the degree of concentration observed along and across different segments of a portfolio, as regards to some classification criterion, in order to obtain an indicator of loan concentration. Generally, the number obtained is of more value in cardinal or hierarchical terms, than it is as a direct measure of risk that can quickly be translated into potential losses or value at risk³. Nonetheless, the results of the exercise provide the key elements for establishing limits on loans as a proportion of capital that can be allocated to the different areas where concentration can occur.

Obtaining a good measure of concentration is difficult for several reasons. First, one must identify or decide the classes or types of concentration that are relevant to a particular situation (e.g. by geographical region, industry, markets, product, etc.). Even if all loans along one of these *dimensions of possible concentration* are relatively small on an individual basis, excessive concentration along any dimension can be risky under adverse economic circumstances if default probabilities are highly correlated. Next, one must decide the adequate hierarchy within the particular classification. For example, what is more important: that the loan is a small mortgage, or that the debtor lives in a particular town, or both?

Another stumbling block is the traditional culture of credit analysts, which almost by definition are highly specialized. This makes it difficult for these analysts to perceive concentration risk, because each one is dedicated to a single dimension of concentration. Furthermore, the analyst that decides on home loans is definitely not the same one that looks at credit for a large metallurgical plant, nor do they employ the same analytical tools, or use the same jargon. In any case, the language employed is generally dissociated with currently accepted concepts and terms of risk management. Whereas credit analysts infer risks from such things as job stability and family income in the case of personal loans, or financial ratios, discounted cash flows and collateral, in the case of industrial loans, risk managers discuss in terms of volatilities, correlations, Greeks, risk profiles and value at risk. Thus, although there are no doubt credit analysts and risk managers who understand each other's terms, there is a conceptual and communications gap that needs to be filled, to establish a common set of risk measures, that provide the integral assessment that is necessary for addressing the concentration issue.

The approach adopted in the following analysis does not solve all the aforementioned problems, but it provides a theoretical framework that might eventually lead to a definition of concentration, from which formal and widely accepted measures of concentration can be obtained. In turn, these measures should be directly related to concentration risk using concepts akin to risk managers, and should also incorporate traditional credit risk analysis into the framework. We begin with the simplest possible case that assumes that all loans belong to a single dimension of concentration, and work our way up to a fairly general case where, at least theoretically, any number of dimensions of concentration can be dealt with.

³ See for example Moody's Investor services 1991, and the 1993 Coopers and Lybrand report.

2. Value at risk, Concentration and the “single obligor” limit: The simplest case.

Traditionally, banks deal with concentration risk by placing a limit on the maximum amount that can be loaned to a single debtor, along the different dimensions where concentration can occur; that is: By industry, geographical region, product, country etc. In what follows, it is assumed that *concentration occurs along only one dimension, and that the default probability “ p ” of any loan is identical for all the loans and each one is independent of the rest*. Normally, the “single obligor limit” is expressed as a proportion “ δ ” of the capital “ K ” of the bank. However, when discussing concentration, one normally addresses the issue of “how much of the pie is concentrated in an individual or group”. Thus, whatever the virtues of setting limits as a percentage of capital, this does not give much information as to the actual concentration of loans in the portfolio. In fact one can have highly concentrated portfolios respecting the limits as well as very diversified portfolios that also respect the constraint in terms of capital.⁴ We will therefore part from tradition, since for the purpose at hand it is better to think of concentration in terms of proportions of the total value of the loan portfolio, and fix limits accordingly. Throughout this paper, individual limits on loans will be expressed as proportions “ q ” of the total value of the loan portfolio “ V ”. Furthermore, no generality is lost since *d and q are linearly related*, so the results are not altered. To see this, let “ f_k ” denote the value of the k^{th} of “ N ” loans, and analyze the *single obligor limit* as represented by the following constraint:

$$f_k \leq dK = d \frac{K}{V} \bullet V = dyV = qV ; k = 0,1,2,3,\dots,N \dots(2.1)$$

where $y = \frac{K}{V}$ is the *capitalization ratio*. Thus, $q = dy$, and the single obligor limit will be expressed as:

$$f_k \leq qV ; k = 0,1,2,3,\dots,N \dots(2.2)$$

If concentration has to do with the number of debtors that have more credit, under the single obligor limit, maximum concentration occurs when all credit is concentrated in “ n ” individual debtors, each of which have loans in the limit θV . Symbolically, the following *pattern* would characterize the maximum concentration:

$$(2.3) f_k = \begin{cases} qV ; k = 0,1,2,3,\dots,n \\ 0 ; k = n+1, n+2,\dots,N \end{cases}$$

⁴ For example, if loans are constrained not to exceed 12% of capital, this can be done with only one loan in the portfolio in which case concentration is maximum. On the otherhand, if the portfolio has a thousand loans all representing 12% of capital, it would be a highly diversified portfolio. The problem in this last case would be one of capital adequacy, but that’s a different issue.

In the following argument we assume that “ n ” is always integer and satisfies⁵:

$$nqV = V \mathbf{P} nq = 1 \text{ i.e. } n = 1/q$$

Under these assumptions, the probability that “ m ” of the “ n ” loans where all credit is concentrated will default, is given by the binomial distribution; that is:

$$\Pr\{m; n\} = \binom{n}{m} p^m (1-p)^{n-m}$$

It is well known that for large “ N ”, the binomial can be approximated by the normal distribution with:

$$\text{Mean} = \mathbf{m} = np \text{ and Standard Deviaton } \mathbf{s} = \sqrt{np(1-p)}$$

Now, let “ $\mathbf{a}$ ” be the confidence level adopted and define:

$$n_{\mathbf{a}} = np + z_{\mathbf{a}} \sqrt{np(1-p)} \dots\dots(2.4)$$

Thus, the probability that more than “ $n_{\mathbf{a}}$ ” loans default, is “ α ”. In the preceding expression, “ z_{α} ” is the standard normal variable that corresponds to the chosen confidence level. Now, if “ qV ” is the value of every loan”, the previous means that the “*Value at risk*” due to concentration, with confidence level “ $\mathbf{a}$ ” is:

$$VAR_{\mathbf{a}} = n_{\mathbf{a}}qV$$

If the value at risk should not exceed the banks’ capital “ K ”, then:

$$VAR_{\mathbf{a}} = n_{\mathbf{a}}qV \leq K$$

That is:

$$\left[np + z_{\mathbf{a}} \sqrt{np(1-p)} \right] qV \leq K$$

Recalling that $nq = 1$, solving for q one arrives at:

⁵ Not much generality is lost under this assumption since the number of loans is usually large, and θ is always under the control of the risk manager.

$$\mathbf{q} \leq \frac{\left(\frac{K - pV}{z_a V} \right)^2}{p(1-p)} = \frac{(\mathbf{y} - p)^2}{z_a^2 p(1-p)} = \Theta(p, \mathbf{y}, \mathbf{a}) \quad \text{.....(2.5)}$$

As before, $\mathbf{y} = \frac{K}{V}$ is the banks' capitalization ratio, and the first implication in terms of capital adequacy is that this ratio cannot be less than the ratio of value at risk to the value of the total loan portfolio; that is:

$$\mathbf{y} \geq \frac{VAR_a}{V} \quad \text{.....(2.6)}$$

Expression (2.5) is very attractive, because it puts a bound on the single obligor limit “ $\mathbf{q}$ ”, which depends on the capitalization ratio of the bank “ $\mathbf{y}$ ”, the default probability “ p ”, and the value at risk confidence level through “ z_α ”. To the extent that the single obligor limit is associated to the minimum number of debtors where all credit can be held through $n = \frac{1}{\mathbf{q}}$, it is also an indicator of concentration, so the bound obtained can also be taken as a limit to the concentration of the portfolio. As shall be seen throughout this paper, this bound is very robust under much more general conditions.

3. A first generalization.

Clearly, the concentration pattern assumed in the preceding analysis, where a portfolio has exactly “ n ” loans in the limit, is not very likely to occur in practice. It is therefore desirable to see if an indicator for concentration can be obtained, which is consistent with the value at risk criterion, and at the same time can account for any type of loan distribution that a bank can hold. So now suppose that $F = (f_i) \in E^N$ is an arbitrary vector representing the loan portfolio ⁶, so that:

$$f_i = \text{amount of the } i^{\text{th}} \text{ loan in the portfolio; } i = 1, 2, \dots, N$$

If the default probability is “ p ”, and assuming independence, one can define “ N ” binary random variables “ x_i ” such that:

$$x_i = \begin{cases} f_i & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

⁶ E^N denotes Euclidian N -space.

Clearly $E(x_i) = pf_i$ and $\text{Variance}(x_i) = p(1-p)f_i^2$. Since the variables are independent:

$$\text{a) } \mathbf{m} = E\left(\sum_{i=1}^N x_i\right) = \sum_{i=1}^N pf_i = pV \text{ ; where } V = \sum_{i=1}^N f_i$$

$$\text{b) } \mathbf{s}^2 = \text{Variance}\left(\sum_{i=1}^N x_i\right) = \sum_{i=1}^N \text{Variance}(x_i) = \sum_{i=1}^N p(1-p)f_i^2 = p(1-p)\sum_{i=1}^N f_i^2$$

Since $F = (f_i)$ is totally arbitrary, it is difficult to know the exact distribution of $\sum_{i=1}^N x_i$. For large “N” however, it can be approximated by the Normal distribution ⁷, so that:

$$\text{VAR}_{\mathbf{a}} = \mathbf{m} + z_{\mathbf{a}}\mathbf{s} = pV + z_{\mathbf{a}}\sqrt{p(1-p)\sum_{i=1}^N f_i^2} \dots\dots(3.1)$$

Again, if $\text{VAR}_{\alpha} \leq K$, after a bit of algebra one arrives at the following expression:

$$\boxed{\frac{\sum_{i=1}^N f_i^2}{\left(\sum_{i=1}^N f_i\right)^2} \leq \frac{(\mathbf{y}-p)^2}{z_{\mathbf{a}}^2 p(1-p)} = \Theta(p, \mathbf{y}, \mathbf{a})} \dots\dots (3.2)$$

Note that this bound $\Theta(p, \psi, \alpha)$ is exactly the same as in (2.5) above. The difference is that, instead of having portfolio concentration measured by the single obligor limit θ , it is now measured by:

$$\boxed{\text{Concentration} = H(F) = \frac{\sum_{i=1}^N f_i^2}{\left(\sum_{i=1}^N f_i\right)^2}}$$

Readers familiar with the literature of industrial organization will have recognized that the above measure is the “*Herfindahl-Hirshman*” concentration index.⁸

⁷ See for example DeGroot (1988), p. 263.

4. Analysis of the value at risk inequality and Capital Adequacy.

The first observation is that, with the obvious limitations, it seems that portfolio concentration risk can be managed using a very general measure of concentration, other than the single obligor limit. Apart from the Herfindahl index, all concepts are totally familiar to the most orthodox of bankers; i.e. the capitalization ratio and the observed and/or estimated default rate; regardless of how the latter is calculated.

Next, it is interesting to note that capital adequacy as represented by the capitalization ratio $\mathbf{y}$ requires that

$$\mathbf{y} \geq p + z_{\mathbf{a}} \sqrt{p(1-p)H(\mathbf{F})} \dots\dots(4.1)$$

This inequality relates capital adequacy to the probability of default, the confidence level used for value at risk, and the concentration index. Furthermore, it shows that there is a direct relation between Herfindahls' index and the variance of the default probability p . Since the index takes on values which approach zero in large very diversified portfolios, and one when high concentration is present, the variance of the default probability will vary between zero and $\sqrt{p(1-p)}$ depending on the the degree of concentration of the portfolio as measured by $H(\mathbf{F})$.

In what follows, it is seen that everything behaves as it should. The following theorem summarizes the main implications for risk managers of the previous analysis. These results are introduced early because they remain basically unchanged throughout all future generalizations.

Theorem 4.1.

The bound $\mathbf{Q}(p, \mathbf{y}, \mathbf{a})$ on the concentration measure, be it the single obligor limit θ or the Herfindahl-Hirshman index $H(\mathbf{F})$ has the following properties:

1. $\mathbf{Q}(p, \mathbf{y}, \mathbf{a})$ varies in direct proportion to the capitalization ratio $\mathbf{y}$ and inversely to the default probability “ p ” and the value at risk confidence level “ $z_{\mathbf{a}}$ ”.
2. If the concentration measure exceeds the bound (i.e. $H(\mathbf{F}) > \mathbf{Q}(p, \mathbf{y}, \mathbf{a})$), then the capital of the bank is at risk, for the given confidence level.
3. If the probability of default increases by “ Δp ”, the corresponding *local increase* in the capitalization ratio “ $\Delta \mathbf{y}$ ” required to restore the value at risk inequality, is given by:

⁸ See for example Shy (1995) or Tirol (1995)

$$\Delta \mathbf{y} \geq \left[1 + \frac{z_{\mathbf{a}}(q-p)\sqrt{H(\mathbf{F})}}{2\sqrt{pq}} \right] \Delta p$$

Where $q = 1-p$.

4. If the default probability “ p ” exceeds the capitalization ratio “ $\mathbf{y}$ ”, then the capital of the bank is at risk for any confidence level, regardless of the concentration of the loan portfolio.
5. If $\mathbf{Q}(p, \mathbf{y}, \mathbf{a}) > 1$, no degree of concentration of the loan portfolio, places the capital of the bank at risk.

Proof.

Point one is obvious from the form of $\mathbf{Q}(p, \mathbf{y}, \mathbf{a})$. The second point is easily verified; that is: If $\mathbf{q}(F) > \mathbf{Q}(p, \mathbf{y}, \mathbf{a})$ then,

$$VAR_{\mathbf{a}} = (p + z_{\mathbf{a}}\sqrt{\mathbf{q}(F)pq})V > (p + z_{\mathbf{a}}\sqrt{\Theta pq})V = \left(p + \frac{z_{\mathbf{a}}\sqrt{pq}(\mathbf{y}-p)}{z_{\mathbf{a}}\sqrt{pq}} \right)V = K$$

Point three follows directly from 4.1:

$$VAR_{\mathbf{a}} \leq K \iff \mathbf{y} \geq h(p) = p + z_{\mathbf{a}}\sqrt{p(1-p)H(\mathbf{F})}$$

Therefore, a local measure of the increase in $\mathbf{y}$ required to maintain capital adequacy is:

$$d\mathbf{y} \approx h'(p) dp$$

Since

$$h'(p) = \left[1 + \frac{z_{\mathbf{a}}(q-p)\sqrt{\mathbf{q}}}{2\sqrt{pq}} \right]$$

one obtains the desired result.

Point four is also verified easily. If $p > \mathbf{y}$, then 4.1 is violated:

$$VAR_{\mathbf{a}} = (p + z_{\mathbf{a}}\sqrt{H(\mathbf{F})pq})V > (\mathbf{y} + z_{\mathbf{a}}\sqrt{H(\mathbf{F})pq})V = K + z_{\mathbf{a}}V\sqrt{H(\mathbf{F})pq} > K$$

As for point five, it is well known that $H(F) \leq 1$ for any arbitrary F . ■

Theorem 4.1 provides a wealth of useful rules for the risk manager and for the regulator. First, one has the means to determine capital adequacy because one obtains precise

⁹ See Encaoua and Jacquemin 1980.

measures of the adjustments in the capitalization ratio required by variations in the default rates and/or the concentration of the loan portfolio. Furthermore, depending on the amount of control that banks have on the default ratio and loan concentration, adjustments in the default probability and the concentration of the loan portfolio necessary to maintain capital adequacy can also be calculated. Thus, if the concentration of the loan portfolio exceeds the bound at the desired confidence level, inequality (3.2) provides a convenient means of fine tuning the adjustments required in $\mathbf{y}$, p and $H(F)$ so that the value at risk does not place the capital of the bank in jeopardy. Another interesting result is that if the default rate of the portfolio exceeds the capitalization ratio, a loud signal of alarm is sent to the risk manager and the financial authorities, that the banks' capital is at risk regardless of the concentration of the loan portfolio and the confidence level adopted.

5. A closer look at Herfindahls' index: *Two properties important for managing loan concentration risk.*

The preceding results provide a very convenient analytical framework for evaluating loan concentration risk, as it relates to the capitalization ratio and the default rates of a bank. One of the attractive features of the approach taken is that a measure of loan concentration as it relates to concentration risk arises naturally; namely Herfindahls' index. In particular, it shows that under certain assumptions, the single obligor limit and Herfindahls' index are both measures of portfolio concentration that have the same bound, if the value at risk due to loan concentration is to avoid placing the capital of the bank at risk. A few loose ends remain however, with respect to how Herfindahls' index relates to the intuitive notion that concentration is related to the minimum number of obligors where credit is more concentrated. Furthermore, a better understanding of the relation between the single obligor limit and the concentration index is important.

In what follows, it will be convenient to adopt vector-matrix notation in order to simplify the analysis. Thus, we will change the notation as follows:

$$H(F) = \frac{\sum_{i=1}^N f_i^2}{\left(\sum_{i=1}^N f_i\right)^2} = \frac{\|F\|^2}{(\mathbf{1}^T F)^2}$$

In the above, $\|F\| = \sqrt{\sum_{i=1}^N f_i^2}$ is the *Euclidean norm of vector* $F \in E^N$. Furthermore,

$\mathbf{1}^T F = \sum_{i=1}^N f_i$ where “ $\mathbf{1}$ ” denotes the *sum vector in* E^N ; that is:

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

Note that $\mathbf{1}^T F = V$ always, and can be considered a constant normalizing factor, so that what really differentiates concentration in a loan portfolio is in fact $\|F\|$. In order to examine how concentration relates to the notion that more credit in less hands means more concentration, it must be consistent with the notion that maximum concentration occurs when all credit is held by a single obligor and the minimum is when all debtors owe the same amount. Formally:

- a) The maximum concentration occurs when for some “ i ”, one has that:

$$f_j = \begin{cases} V & \text{for } j = i \\ 0 & \text{for } j \neq i; j = 1, 2, \dots, N \end{cases}$$

i.e. $\mathbf{F}^{\max} = V\mathbf{e}^i$, where $\mathbf{e}^i \in \mathbf{E}^N$ is the i^{th} unit vector.

- b) The minimum concentration occurs when $f_i = \frac{V}{N}$ for $i = 1, 2, \dots, N$

The Herfindahl-Hirschman index has been extensively studied, mainly in relation to industrial concentration, and it is known to have several important properties. Thus, it is well known that the index takes values between the reciprocal of “ N ” and one¹⁰, and it has also been established that the index behaves well in terms of “*the five properties of inequality measures*”.¹¹

Now concentration has to do with numbers, and it is indeed the case that Herfindahls’ index has several interesting *numbers related properties*. The best known of these is *Adelman’s “numbers-equivalent”*¹², which states that its inverse can be interpreted as “*the minimum number of firms of equal size that would result in a specific value of the index*”. In the very

¹⁰ A simple normalization is possible, where it is easily seen that $\phi(F)$ as defined below, satisfies $0 \leq \phi \leq 1$

$$\mathbf{f}(F) = \frac{N - \frac{1}{\mathbf{q}(F)}}{N - 1}$$

¹¹ See Cowell 1995 and Encaoua and Jacquemin 1980.

¹² See Adelman 1969 and Kelly Jr. 1981. For other interesting numbers related properties, see Weinstock 1984.

first derivation of the bound $\mathbf{Q}(p, \mathbf{y}, \mathbf{a})$, based on purely intuitive notions, it was stated that the maximum concentration possible under the single obligor rule, was when the minimum possible number of obligors were loaned up to the limit. It is now shown that Herfindahl's index is consistent with this intuitive notion; that is: *The value of the index is maximized under the single obligor limit, when all credit is concentrated in the minimum number of obligors, and each obligor holds credit up to the limit.* The theorem establishes the relation between the single obligor limit and the Herfindahl-Hirshman measure of concentration, and in so doing, it shows that Adelman's numbers-equivalent is in fact the maximum concentration possible, when loans are constrained by a certain limit¹³.

Theorem 5.1.

If the loan portfolio $\mathbf{F}$ complies with the single obligor limit $\mathbf{F} \leq \theta \mathbf{V} \mathbf{1}$, then $H(\mathbf{F}) \leq \theta$ and the maximum concentration under Herfindahl-Hirschman concentration measure occurs, if and only if, $\mathbf{F}$ is some permutation of the following loan distribution:

$$f_k^* = \begin{cases} \mathbf{q}V ; k = 0, 1, 2, 3, \dots, n \\ 0 ; k = n+1, n+2, \dots, N \end{cases}$$

$$\text{where } n\mathbf{q}V = V \mathbf{P} n\mathbf{q} = 1 \text{ or } n = 1/\mathbf{q}$$

Proof:

The detailed proof of the theorem is a bit cumbersome so it has been relegated to the appendix. Here, we will simply provide a sketch of the proof.

We first prove that the maximum concentration possible under the single obligor rule, is achieved when the minimum possible number of obligors are loaned up to the limit θV , and all others are zero. Note that this is equivalent to showing that the aforementioned distribution of loans, solves the following optimization problem:

$$\begin{aligned} \max \quad & \mathbf{F}^T \mathbf{F} \\ \text{s.t. : } & \mathbf{1}^T \mathbf{F} = V \\ & \mathbf{F} \leq \mathbf{q}V \mathbf{1} \\ & \mathbf{F} \geq 0 \end{aligned}$$

To prove necessity we show in the appendix that the proposed solution satisfies the Karush-Kuhn-Tucker conditions for this problem. In order to prove sufficiency, we argue that since we are maximizing a convex function under linear constraints, the maximum lies at an extreme point of the polyhedron defined by the constraints. It can be seen that all extreme

¹³ Although the result conforms to intuition, no formal proof has been detected by the authors in the more frequent references, such as Sleuwaegen et. al. 1989, Weinstock or Encaoua and Jaquemin op. cit..

points of this polyhedron are vectors with “k” elements equal to θV , and the remaining elements are zeroes; $k \leq n$. Thus the maximum is obtained when $k = n$.

From here, it is easy to verify that $F \leq \theta V \mathbf{1}$ implies $H(F) \leq \theta$ as follows: Since under the single obligor limit the maximum concentration has exactly “n” loans in the limit $\mathbf{q}V$, then:

$$\|F^*\| = \sqrt{\sum_{k=1}^N (f_k^*)^2} = \sqrt{\sum_{k=1}^n (\mathbf{q}V)^2} = \sqrt{n\mathbf{q}V} = \sqrt{\mathbf{q}V}$$

because $n\theta = 1$ and this means that $H(F) \leq H(F^*) = \frac{\|F^*\|^2}{(\mathbf{1}^T F^*)^2} = \frac{\mathbf{q}V^2}{V^2} = \mathbf{q}$ ■

This result has important implications for risk management and regulation since *de facto*, it states that by placing a limit on individual loans, one is also placing a limit on concentration as measured by Herfindahls’ index by the same amount θ . Therefore, it is simple to check for capital adequacy by comparing the single obligor parameter θ to

$\frac{(\mathbf{y} - p)^2}{z_{\mathbf{a}}^2 p(1-p)} = \Theta(p, \mathbf{y}, \mathbf{a})$. Alternatively, from (4.1), one can obtain the capital adequacy relation in terms of the single obligor limit; that is:

$$\boxed{\mathbf{y} \geq p + z_{\mathbf{a}} \sqrt{p(1-p)\mathbf{q}}} \text{(5.1)}$$

Thus, (5.1) provides a very simple means to check for capital adequacy, without doing complicated calculations.

As will be seen later the result also comes in handy in the general model when correlations of default probabilities are included in the model.

It should be realized however, that this condition is sufficient but not necessary. As will be shown in the following theorem, if one chooses to explicitly constrain the portfolio to satisfy $y H(F) \leq \theta$, it is possible to have specific loans that as a proportion of the total value of the loan portfolio represent a quantity larger than θ . The proof is cumbersome as it involves looking at the KKT conditions for an equivalent non-convex optimization problem and has been relegated to the appendix so as not to distract the reader from the main subject of this paper. Those interested in the details can turn to the appendix at their leisure.

Theorem 5.2.

If $H(\mathbf{F}) \leq \theta$ then:

$$f_i \leq \frac{1}{N} \left(1 + \sqrt{(N\mathbf{q} - 1)(N - 1)} \right) V < \sqrt{\mathbf{q}V} \quad \text{for } i = 1, 2, 3, \dots, N$$

Corollary:

Let α^*V be the largest individual loan possible under the constraint $\|a\|^2 \leq \mathbf{q}$. Then:

- i) $\mathbf{a}^* = \frac{1}{N} \left(1 + \sqrt{(N\mathbf{q} - 1)(N - 1)} \right)$
- ii) $\mathbf{q} < \mathbf{a}^* < \sqrt{\mathbf{q}}$
- iii) $H(F) \leq \mathbf{q}V \Rightarrow f_k^* < \sqrt{\mathbf{q}}V \quad \forall k.$

No proof for the corollary is provided because the results are obvious from the proof of the theorem as presented in the appendix. The results show that if the risk manager chooses to regulate loan concentration by placing a limit θ on $H(F)$ as opposed to doing so on individual loans, by (iii) he would automatically be limiting the maximum possible loan to the square root of that limit. Note that since θ is a positive quantity less than unity, its square root is a larger number. Thus, there is a certain symmetry with the results of theorem 5.2, where limiting the size of individual loans also limits the size of $H(F)$ in the same amount. The difference is that the tradeoff for the portfolio manager to concentrate the maximum possible loan in a single debtor, is at the expense of the amount that could be loaned to all other debtors, which tends to zero as N increases.

It is necessary to emphasize that concentration measures that may be appropriate in some applications are not necessarily good for others. Having a good concentration index is desirable however, particularly from the regulatory point of view, since it facilitates comparisons of loan concentration between different institutions, and it leads to an assessment of concentration risk for the financial system as a whole. For the risk manager, besides measuring his own risk, it provides benchmarks for setting business strategy and goals, and allows comparisons with the competition. As has been shown, Herfindahl's index seems particularly well suited for the task, since besides measuring concentration it is directly related to risk, and provides a quick means to check capital adequacy. In the following section it will be seen that the concept is robust under more general conditions.

6. Generalizations.

The results obtained so far, rely on the following assumptions:

- a) Default probabilities are homogenous and independent from each other, for all loans along the dimension where loan concentration can occur.
- b) There is only one dimension of possible loan concentration.
- c) Nothing is recovered from defaulting loans.

Except for very particular cases, such as certain leasing companies or perhaps savings and loans, none of the above is totally realistic. In this section, the assumptions are relaxed and the results obtained are generalized to account for situations, which one is more likely to encounter in practice. The first generalization completely relaxes the homogeneity assumption to examine the case where default probabilities can be different and are correlated.

Intuitively, there is no reason why different types of loans, in different geographical regions, different industries, different countries, or to individuals of different social strata, should have the same default probabilities, nor why default probabilities within and between different groups should be uncorrelated. In this latter case however, it is this authors' feeling that if an increase in the default rate of the loans in a group of similar characteristics is observed, distinguishing between an increase in default probabilities and correlation could prove to be a real challenge. Whatever the practical problems however, there are no theoretical limitations for including correlations in the analysis, as will be shown in the following section.

6.1. A General Model.

From the particular situations examined in the previous sections, we now go to the other extreme. Thus, consider the general case where all loans have different default probabilities, which can be correlated to each other. Assume that default probabilities can be characterized by a vector of *default probabilities* $\boldsymbol{\pi}$ and the corresponding “*covariance matrix* $\mathbf{M}$ ”. The value at risk relation is now:

$$VAR_a = \mathbf{p}^T \mathbf{F} + z_a \sqrt{\mathbf{F}^T \mathbf{M} \mathbf{F}} \leq K \quad \dots(6.1)$$

Since $\mathbf{M}$ is positive definite, it is well known that there exists a Matrix “ $\mathbf{Q}$ ” such that,

$$\mathbf{M} = \mathbf{Q}^T \boldsymbol{\Lambda} \mathbf{Q} \quad \dots\dots(6.2)$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of characteristic values of $\mathbf{M}$, and $\mathbf{Q}$ is an *orthogonal matrix* with the property that $\mathbf{Q}^{-1} = \mathbf{Q}^T$.¹⁴ Let $\mathbf{S} = \sqrt{\boldsymbol{\Lambda}} \mathbf{Q}$, where $\sqrt{\boldsymbol{\Lambda}}$ is the diagonal matrix of the square roots of the eigen-values of $\mathbf{M}$, so that $\mathbf{M} = \mathbf{S}^T \mathbf{S}$. Now change the variable to $\mathbf{G} = \mathbf{S} \mathbf{F}$ and divide the value at risk relation by $\mathbf{1}^T \mathbf{G}$, to obtain:

$$H(\mathbf{G}) = \frac{\mathbf{G}^T \mathbf{G}}{(\mathbf{1}^T \mathbf{G})^2} \leq \left(\frac{K - \mathbf{p}^T \mathbf{F}}{z_a \mathbf{1}^T \mathbf{G}} \right)^2$$

Finally, dividing the numerator and denominator of the right-hand side by V , one arrives at:

¹⁴ Any intermediate text on matrix theory can be consulted. See for example Strang G. 1980.

$$H(\mathbf{G}) = \leq \left(\frac{\mathbf{y} - \bar{p}}{z_a \mathbf{S}} \right)^2 \dots (6.3)$$

It is easy to verify that the previous reduces to the simple one-dimensional case with independent and equal default probabilities.¹⁵ In this relation,

$$\bar{p} = \frac{\mathbf{p}^T \mathbf{F}}{V} \dots (6.4)$$

is the average default probability of the loan portfolio, and

$$\mathbf{S} = \frac{\mathbf{1}^T \mathbf{G}}{V} = \frac{\mathbf{1}^T \mathbf{G}}{\mathbf{1}^T \mathbf{F}} \dots (6.5)$$

is a measure of its standard deviation. In fact, (6.3) has exactly the same form except that now, Herfindahl's index is calculated in " $\mathbf{G}$ " as opposed to the original distribution of loans " $\mathbf{F}$ ". This change of variable transformation effectively *rescales* " $\mathbf{F}$ " in terms of the matrix " $\mathbf{S}$ " which in turn is representative of the "*square root*" of the covariance matrix " $\mathbf{M}$ ". In terms of risk, what this means is that the loans in the portfolio are redimensioned according to the covariances of the default probabilities between the loans. Thus, the loans that are more correlated and/or have larger default rate variances will be magnified in relation to those that are less correlated or have smaller variances.

It should also be noted that rescaling implies *that concentration by numbers is not the only thing that must be taken into account*. Although much credit in a few hands is potentially dangerous, it is even more dangerous when *too much risk is concentrated in a particular group of debtors*, as evidenced by the rescaling of the loan portfolio in terms of " $\mathbf{S}$ ". Thus, in a given moment a numerically large portfolio of highly diversified small loans that exhibit large variances and are highly correlated, may be riskier than a numerically small portfolio of large loans that are uncorrelated and have low default probabilities. The rescaling of " $\mathbf{F}$ " by " $\mathbf{S}$ " provides a means for addressing this problem.

6.2. Setting Single Obligor Limits on Individual Loans and Capital Adequacy.

Since concentration risk is usually managed by limiting the amount of credit to a single debtor, the implications of the above are now examined in the context of the *single obligor limit*. Although Theorem 5.1 provides $\mathbf{G} \leq \theta \mathbf{1}(\mathbf{1}^T \mathbf{G})$ implies $H(\mathbf{G}) \leq \theta$, this would be difficult to implement in practice. For practical purposes, a useful bound would have to be expressed as a limit on $\mathbf{F}$.¹⁶ For example, since $\mathbf{F} \leq \theta \mathbf{1}V$ implies $H(\mathbf{F}) \leq \theta$ from Theorem 5.1 then:

¹⁵ Simply set $\boldsymbol{\pi} = \mathbf{p}\mathbf{1}$ and $\mathbf{M} = \mathbf{p}(\mathbf{1}-\mathbf{p})\mathbf{I}$.

¹⁶ We would like to point out that $\mathbf{F} \leq \theta \mathbf{V}\mathbf{S}^{-1}\mathbf{1}$, has serious technical complications since, among other things, the inequality does not necessarily hold for arbitrary $\mathbf{S}$.

$$H(\mathbf{G}) \leq \mathbf{q} \frac{H(\mathbf{G})}{H(\mathbf{F})} \leq \left(\frac{\mathbf{y} - \bar{p}}{z_a \mathbf{s}} \right)^2$$

So let

$$\mathbf{q} \leq \frac{H(\mathbf{F})}{H(\mathbf{G})} \left(\frac{\mathbf{y} - \bar{p}}{z_a \mathbf{s}} \right)^2 \dots\dots(6.6)$$

Now note that $\left(\frac{\|\mathbf{G}\|}{\|\mathbf{F}\|} \right)^2$ is “Rayleighs quotient”, and it is well known that $\mathbf{I}_{\min} \leq \left(\frac{\|\mathbf{G}\|}{\|\mathbf{F}\|} \right)^2 \leq \mathbf{I}_{\max}$, where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest eigenvalues of the matrix $\mathbf{M}$. Thus, after some algebra, due to (6.5) it can be verified that:

$$\mathbf{q} \leq \frac{1}{\mathbf{I}_{\max}} \left(\frac{\mathbf{1}^T \mathbf{G}}{\mathbf{1}^T \mathbf{F}} \right)^2 \left(\frac{\mathbf{y} - \bar{p}}{z_a \mathbf{s}} \right)^2 = \frac{1}{\mathbf{I}_{\max}} \left(\frac{\mathbf{y} - \bar{p}}{z_a} \right)^2 \dots\dots(6.7)$$

is also a bound for the single obligor limit, but it is stricter than (6.6) (see footnote 17 on the next page).

Thus, choosing θ to satisfy (6.6) or (6.7) guarantees that $\mathbf{F} \leq \theta \mathbf{1V}$ implies that (6.3) is satisfied. If limits on loans in relation to capital are preferred, in order to conform to established practice, simply set $\mathbf{d} = \frac{\mathbf{q}}{\mathbf{y}}$ as indicated by (2.1).

Finally, it can be verified that, adjusting for the transformation of “ $\mathbf{F}$ ” through “ $\mathbf{S}$ ”, all the results of Theorem 4.1 are still true under this generalization, along with the implications for regulation and risk management. Specifically, the capital adequacy inequality (4.1) generalizes to:

$$\boxed{\mathbf{y} \geq \bar{p} + z_a \mathbf{s} \sqrt{H(\mathbf{G})}} \dots\dots(6.8)$$

Furthermore, (6.7) allows an analogy to (5.1) where capital adequacy can be related to the single obligor limit as:

$$\boxed{\mathbf{y} \geq \bar{p} + z_a \sqrt{\mathbf{I}_{\max} \mathbf{q}}} \dots\dots(6.9)$$

It is interesting to note that the only difference between (6.9) and (5.1) is the appearance of the largest eigenvalue of “ $\mathbf{M}$ ” in the expression. It can be shown that $\mathbf{I}_{\max} \mathbf{q}^3 \mathbf{s}^2 H(\mathbf{G})$ and in

fact increases with positive correlation, which is consistent with the notion that correlation increases the capital requirement¹⁷.

6.3. Dealing with Different Dimensions of Concentration.

When one establishes a partition of a loan portfolio into sub-portfolios or “*buckets*” according to some classification criterion such as geographical region, industry, product, or income brackets for consumer loans, there is the implicit assumption that the default behavior is somehow similar for each loan within the group. Furthermore, unless macro-economic conditions change drastically, it is assumed that there is a certain amount of independence between the default characteristics of the different segments. Therefore, if one could partition the portfolio into segments where the default rates and the covariance structure were homogenous within each group but different and independent from the others, one would obtain a very desirable classification criterion for loans as relates to risk management. In what follows, *class*, *segment* and *dimension of concentration* are synonymous.

Obviously there could be many ways of doing this, and dealing with all of them would be impossible without more empirical research. So for purely illustrative purposes, consider the possibility of identifying groups of loans where the default behavior of all loans within each group are similar, and different from those of the other groups. Formally, suppose that $\mathbf{F}$ is partitioned into “ h ” segments, $\mathbf{F}^T = (F_1, \dots, F_h)$, where $\mathbf{F}_i$ is a vector whose elements are the amounts outstanding of the loans in the group. Suppose that default probabilities for all loans within a particular segment are the same and similarly correlated. Furthermore, suppose that default probabilities are independent between groups. This means that the expected default rates vector and the associated covariance matrix can be partitioned according to the following:

- a) $\mathbf{p} = (\mathbf{p}_i)$; $\mathbf{p}_i = p_i \mathbf{1}_i$; for each class $i = 1, 2, 3, \dots, h$ where p_i = default probability for each class i , and $\mathbf{1}_i$ is the sum vector of size N_i corresponding to the number of loans in each class.

- b) $\mathbf{M} = (\mathbf{s}_{j,k})$ such that $\mathbf{s}_{j,k} \begin{cases} > 0 \text{ if } k, j \in \text{class } i \\ = 0 \text{ else.} \end{cases}$

The covariance matrix is “*block diagonal*”. Each block $\mathbf{M}_i$ is $(N_i \times N_i)$ and positive definite the same as $\mathbf{M}$. Therefore, each submatrix can be factorized accordingly by its corresponding $\mathbf{S}_i$.

6.3.1. Analysis of the Individual Segments.

¹⁷ Simply divide (6.6) by (6.7) and using the bounds on Rayleighs quotient establish that the ratio is greater than one.

For the analysis that follows let $V_i = \sum_{j \in F_i} f_j$, the value of the portfolio associated to class “i”, and $\sum_{i=1}^h V_i = V$. Now define $\mathbf{g}_i = \frac{V_i}{V}$, and $K_i = \mathbf{g}K$. Assuming the number of loans in each segment is sufficiently large, and proceeding in the usual way, the value at risk inequality for each class is:

$$VAR_{\mathbf{a}}^i = p_i V_i + z_{\mathbf{a}} \sqrt{\mathbf{F}_i^T \mathbf{M}_i \mathbf{F}_i} \leq K_i = \mathbf{y} V_i ; i=1,2,3,\dots,h$$

Dividing both sides by V_i , performing the change of variable $\mathbf{G}_i = \mathbf{S}_i \mathbf{F}_i$ and dividing by $\mathbf{1}_i^T \mathbf{G}_i$ where appropriate, after some algebraic manipulation one obtains:

$$H(\mathbf{G}_i) = \leq \left(\frac{\mathbf{y} - p_i}{z_{\mathbf{a}} \mathbf{s}_i} \right)^2 = \Theta_i \dots\dots(6.10)$$

where $\mathbf{s}_i = \frac{\mathbf{1}_i^T \mathbf{G}_i}{V_i}$.

Apart from the fact that the expression is of the same usual form, it is interesting to note that the bound on concentration depends on the capitalization ratio for the total portfolio $\mathbf{y}$ and not the fraction $\mathbf{y}_i$ as may have been expected. Having the same form means that all the previous results go through for this particular case, and one can obtain single obligor limits $\mathbf{q}$ on the loans within each class so that:

$$\mathbf{F}_i \leq \mathbf{q} \mathbf{1}_i V_i \Rightarrow H(\mathbf{G}_i) \leq \Theta_i \dots\dots(6.11)$$

by having θ_i satisfy (6.6) or (6.7) within each individual group of loans.

6.3.2. Analysis of the Total Portfolio

Under the particular structure described, the value at risk inequality takes the following form for the total portfolio:

$$VAR_{\mathbf{a}} = \sum_{i=1}^h p_i V_i + z_{\mathbf{a}} \sqrt{\sum_{i=1}^h \mathbf{G}_i^T \mathbf{G}_i} \leq K$$

Which leads to:

$$z_{\mathbf{a}} \sqrt{\sum_{i=1}^h \|\mathbf{G}_i\|^2} \leq K - \sum_{i=1}^h p_i V_i$$

Dividing both sides by V , multiplying and dividing inside the summation sign by $(\mathbf{1}_i^T \mathbf{G}_i)^2$ and squaring the terms one obtains:

$$\sum_{i=1}^h \|\mathbf{G}_i\|^2 = \sum_{i=1}^h (\mathbf{1}_i^T \mathbf{G}_i)^2 H(\mathbf{G}_i) \leq \frac{\left(\mathbf{y} - \sum_{i=1}^h \mathbf{g}_i p_i \right)^2}{\frac{z_{\mathbf{a}}^2}{V^2}}$$

Dividing both sides of the inequality by $\sum_i (\mathbf{1}_i^T \mathbf{G}_i)^2$ and the numerator and the denominator of the right-hand side by V^2 one arrives at:

$$\boxed{\overline{H}(\mathbf{G}) = \sum_{i=1}^h \mathbf{I}_i H(\mathbf{G}_i) \leq \frac{\left(\mathbf{y} - \sum_{i=1}^h \mathbf{g}_i p_i \right)^2}{z_{\mathbf{a}}^2}} \quad \text{.....(6.12)}$$

where $\mathbf{I}_i = \frac{(\mathbf{1}_i^T \mathbf{G}_i)^2}{\sum_j (\mathbf{1}_j^T \mathbf{G}_j)^2}$.

Again one obtains an expression which essentially has the same form. Here, $\sum_{i=1}^h \mathbf{g}_i p_i$ is the weighted average of the default probabilities of the portfolio,

$$\mathbf{s} = \frac{\sqrt{\sum_j (\mathbf{1}_j^T \mathbf{G}_j)^2}}{V}$$

is as in the general case and $\overline{H}(\mathbf{G})$ is a measure of the concentration of the portfolio, as a weighted average of the individual indices $H(\mathbf{G}_i)$. The following result shows that placing individual limits on each class as derived in the preceding section, is a sufficient condition for satisfying the above inequality.

Proposition 6.1.

Under the assumptions made, and assuming $p_i \neq \mathbf{y}$ for $i = 1, 2, 3, \dots, h$, then:

$$H(\mathbf{G}_i) \leq \left[\frac{\mathbf{y} - p_i}{z_{\mathbf{a}} \mathbf{s}_i} \right]^2 \quad \forall i = 1, 2, \dots, h \text{ implies } \overline{H}(\mathbf{G}) \leq \left[\frac{\mathbf{y} - \sum_{i=1}^h \mathbf{g}_i p_i}{z_{\mathbf{a}} \mathbf{s}} \right]^2$$

Proof:

Taking the left-hand side of the inequality first, because of (6.7) one arrives at:

$$\sum_{i=1}^h \mathbf{l}_i H(\mathbf{G}_i) \leq \sum_{i=1}^h \frac{\mathbf{l}_i (\mathbf{y} - p_i)^2}{z_{\mathbf{a}}^2 \mathbf{s}_i^2} = \sum_{i=1}^h \frac{\mathbf{l}_i \mathbf{g}_i^2 (\mathbf{y} - p_i)^2}{z_{\mathbf{a}}^2 \mathbf{g}_i^2 \mathbf{s}_i^2} = \sum_{i=1}^h \frac{\mathbf{g}_i^2 (\mathbf{y} - p_i)^2}{z_{\mathbf{a}}^2 \frac{\mathbf{g}_i^2 \mathbf{s}_i^2}{\mathbf{l}_i}} = \sum_{i=1}^h \frac{\mathbf{g}_i^2 (\mathbf{y} - p_i)^2}{z_{\mathbf{a}}^2 \mathbf{s}^2} \quad \dots (6.13)$$

$$\text{Since } \mathbf{l}_i = \frac{(\mathbf{1}_i^T \mathbf{G}_i)^2}{\sum_j (\mathbf{1}_j^T \mathbf{G}_j)^2} ; \mathbf{s}_i = \frac{\mathbf{1}_i^T \mathbf{G}_i}{V_i} ; \mathbf{g}_i = \frac{V_i}{V} \text{ and } \mathbf{s} = \frac{\sqrt{\sum_j (\mathbf{1}_j^T \mathbf{G}_j)^2}}{V}. \text{ Now,}$$

$$\text{since } \sum_{i=1}^h \mathbf{g}_i = 1 \text{ then:}$$

$$\left[\frac{\sum_{i=1}^h \mathbf{g}_i (\mathbf{y} - p_i)}{z_{\mathbf{a}} \mathbf{s}} \right]^2 = \left[\frac{\mathbf{y} - \sum_{i=1}^h \mathbf{g}_i p_i}{z_{\mathbf{a}} \mathbf{s}} \right]^2$$

Finally, since $\mathbf{y} \geq p_i, \forall i$, it is always true that $\sum_{i=1}^h \mathbf{g}_i^2 (\mathbf{y} - p_i)^2 \leq \left(\sum_{i=1}^h \mathbf{g}_i (\mathbf{y} - p_i) \right)^2$ and from (6.10) and (6.12), the following inequality holds completing the proof:

$$\sum_{i=1}^h \frac{\mathbf{g}_i^2 (\mathbf{y} - p_i)^2}{z_{\mathbf{a}}^2 \mathbf{s}^2} \leq \left[\frac{\sum_{i=1}^h \mathbf{g}_i (\mathbf{y} - p_i)}{z_{\mathbf{a}} \mathbf{s}} \right]^2 \quad \blacksquare$$

The result means that individual limits on the loans of each segment can be set according to $\mathbf{F}_i \leq \mathbf{q}_i \mathbf{1}_i V_i$, with θ_i satisfying (6.6) Or (6.7) within each group while maintaining capital adequacy. It is also clear that capital adequacy under this framework can be established by calculating $\overline{H}(\mathbf{G})$ and comparing it with

$$\left[\frac{\mathbf{y} - \sum_{i=1}^h \mathbf{g}_i p_i}{z_{\mathbf{a}} \mathbf{s}} \right]^2.$$

It should be noted however that the condition is only sufficient, and inequality (6.12) could be satisfied in many other ways which perhaps would be less restrictive in individual terms but equally effective for the overall portfolio. Indeed, there are several interesting possibilities for dealing with the capital adequacy issue, under the segmentation structure described in this section, a couple of which will be explored in the following subsection.

6.3.3. Alternative conditions for Capital Adequacy.

An analysis of inequalities (6.6) and (6.7) lead to very simple rules for establishing capital adequacy. The relations derived in this section are based on (6.7) and (6.10). Thus, from (6.7) one obtains the following capital adequacy condition:

$$\mathbf{y} \geq p_i + z_{\mathbf{a}} \sqrt{\mathbf{I}_{\max}^i \mathbf{q}_i} \quad \text{for } i = 1, 2, \dots, h \quad \dots (6.14)$$

Since this should hold for each group or dimension of concentration considered, then it must also hold for:

$$\boxed{\mathbf{y} \geq \max_i \{p_i + z_{\mathbf{a}} \sqrt{\mathbf{I}_{\max}^i \mathbf{q}_i}\}} \quad \dots (6.15)$$

Thus, if one computes $p_i + z_{\mathbf{a}} \sqrt{\mathbf{I}_{\max}^i \mathbf{q}_i}$ for every segment in the portfolio, then (6.15) indicates that if the capitalization ratio $\mathbf{y}$ is at least as much as the largest of these quantities, capital adequacy is established.

Another possibility is to add up both sides of (6.14) along all dimensions of concentration. Doing this leads to the following capital adequacy requirement:

$$\boxed{\mathbf{y} \geq \frac{1}{h} \sum_{i=1}^h \{p_i + z_{\mathbf{a}} \sqrt{\mathbf{I}_{\max}^i \mathbf{q}_i}\}} \quad \dots (6.16)$$

That is, capital adequacy is established if the capitalization ratio is at least as big as the arithmetic average of the quantities $p_i + z_{\mathbf{a}} \sqrt{\mathbf{I}_{\max}^i \mathbf{q}_i}$.

If a relation depending on the individual concentration measures $H(\mathbf{G}_i)$ of each segment of the portfolio is preferred, (6.10) leads to:

$$\mathbf{y} \geq \max_i \{p_i + z_a \mathbf{s}_i \sqrt{H(\mathbf{G}_i)}\} \dots (6.17)$$

And adding up both sides of the inequality leads to:

$$\mathbf{y} \geq \frac{1}{h} \sum_{i=1}^h \{p_i + z_a \mathbf{s}_i \sqrt{H(\mathbf{G}_i)}\} \dots (6.18)$$

So again, the analysis leads to fairly simple relations by which capital adequacy can be established. Furthermore (6.15) through (6.18) can be used as policy instruments to determine single obligor limits, changes in portfolio composition and/or, adjustments to capital necessary to maintain capital adequacy to account for changes in the probabilistic behaviour of defaults along different dimensions of concentration.

6.4. Accounting for Recovery Rates.

It is simple to extend all the relations so far obtained, to include loan recovery rates. Doing so leads to less restrictive limits in terms of tolerable concentration along the different dimensions where concentration can occur. Basically, there are two ways to account for recovery rates. The first is to define $\mathbf{F}$ directly as the vector of “*loss given default*” (*LGD*), *as opposed to the outstanding balance*, where it is assumed that nothing is recovered if loans default. This would be very much in line with current practice.¹⁸ Thus if an estimation of the LGD vector is at hand, one can simply use this in the relations derived, and no changes are necessary. The other alternative is to include recovery rates *explicitly* in the analysis. This permits the explicit analysis of the impact of recovery rates in concentration risk, and facilitates the evaluation through assumptions in order to compensate for deficiencies in the available information. Under the framework of the previous section, let “ $\mathbf{r}_i$ ” be the recovery rate for defaulted loans in segment “ i ”. Proceeding in the usual manner leads to:

$$\sum_{i=1}^h (1 - \mathbf{r}_i)_i \mathbf{I}_i H(\mathbf{G}_i) \leq \left[\frac{\mathbf{y} - \sum (1 - \mathbf{r}_i) \mathbf{g}_i p_i}{z_a \mathbf{s}} \right]^2$$

The expression clearly shows that any change in recovery rates has a double impact. On the one hand an increase (decrease) in the recovery rate of any particular segment reduces (increases) the importance in this segments contribution to concentration on the left-hand side of the inequality, because its “*weight*”, $(1 - \mathbf{r}_i) \mathbf{I}_i^2$ *decreases (increases) as the recovery rate increases*. Additionally, its contribution to the expected loss also decreases (increases)

¹⁸ See the document on *Credit Risk Modeling* by the Basle Committee on Banking Supervision. April 1999.

in the numerator of the right-hand side, increasing (decreasing) the established bound on concentration.

7. Concluding remarks.

The results obtained are very appealing for managing risk, since they permit a precise quantification of the measures that should be adopted in order to keep concentration from getting out of hand. Since both the single obligor limit and the Herfindahl measure are subject to the same bound, either indicator can be used as a policy instrument. Furthermore, there is no reason why both measures cannot be used in conjunction. For example, if an overzealous branch manager decides to grant a certain loan disregarding the single obligor limit established by policy, the gravity of the transgression may be ultimately assessed using Herfindahls index. It may very well be that apart from the misbehavior of disregarding orders, the infraction is not serious in terms of increasing concentration risk. Finally, if default or recovery rates change, or concentration along a particular dimension is seen to be excessive, the relations can be used to determine the adjustments to the capitalization ratio and/or concentration composition of the portfolio that would reestablish capital adequacy. Obviously, to the degree to which banks have control over default or recovery rates, these can be part of the mix of management instruments that can be used to reestablish the value at risk inequalities.

Clearly the authors would favor using both indicators in conjunction, as both have their merits and their problems. Whereas the single obligor limit is easy to implement and supervise, it may lead to overly constrained loan distributions. On the other hand, although Herfindahls index allows more flexibility in the configuration of the loan portfolio, it is much more difficult to implement, and the logistics could prove difficult to manage. To illustrate this point, consider the simple case of concentration along only one dimension of uncorrelated loans with equal default probabilities. Suppose that management is considering granting a new loan. Even for this simple one-dimensional case, one would have to check that the value at risk inequality is not violated. Whereas in the single obligor rule it is only necessary to verify that the size of the loan does not exceed the established limit, using Herfindahls index would imply verifying that the inequality $\frac{1}{V}\|F\| \leq \sqrt{\mathbf{q}}$ is maintained. If only one loan “ f ” is involved, the condition that must be satisfied so as not to violate the inequality is:

$$\|F\|^2 + f^2 \leq \mathbf{q}(V + f)^2$$

This means that a quadratic equation must be solved to arrive at:

$$f \leq 2\mathbf{q} + V \sqrt{(1-\mathbf{q}) \left[\mathbf{q} - \frac{\|F\|^2}{V^2} \right]}$$

Although manageable, this is clearly much more complicated than the traditional rule. Now imagine the situation in a traditional credit committee, where many loans are being considered for approval simultaneously, for correlated loans along different segments: The logistics can prove to be a real challenge.

Under the approach taken, it is inferred that perhaps *the best way to classify loans into different dimensions of concentration is by homogeneity of default behavior, in terms of the default probabilities and their covariation within each segment*. This may lead to significant changes in the way it is usually done, by industry, geographical region, product and so on, *unless empirical research confirms that this type of classification leads to groups of loans with similar default behavior*. This is due to the results obtained where it is made clear that concentration in numbers is not necessarily indicative of risk, but rather, *one must look at how risk is concentrated* through the transformation $\mathbf{G} = \mathbf{S}\mathbf{F}$. Furthermore, it is pointed out that the rating agencies have been doing this for a long time: Similarly rated companies are automatically assumed to have the same default behavior, regardless of geographical region or industry. In fact, the rating agencies are the only readily available sources of information on default and recovery rates, transition probabilities from one category to another and so on. Thus, the results here obtained would be directly applicable to portfolios of rated debt instruments.

Finally, it is clear that default probability distributions as well as recovery rates exhibit random behaviors through time, depending on many economic and financial factors. As opposed to market risk, where risk factors can be modeled using continuous processes, because loan defaults are discrete events in time, default behavior in a particular group can also change in pronounced discrete jumps. This introduces different types of technical difficulties. Obviously, there is no reason why the results obtained cannot be imbedded in simulation models which generate scenarios of trajectories of these variables through time that exhibit the discontinuities typical of default related events. With the scenario's generated, one could obtain the distribution of the adequate capitalization ratio that must be maintained in order to keep concentration under control, and/or the distribution of the bound to place a limit on concentration that has a high probability of being satisfied. Indeed, this type of experiment may be precisely what is needed to set the proper policy on the capitalization ratio and the adequate level of concentration. Whatever the dynamics, it is always possible to make the necessary adjustments through time by monitoring only a few variables. All of these considerations point to new directions for future research, which to our mind will prove challenging and hopefully, useful.

Appendix.

Theorem 5.1.

If the loan portfolio is constrained by the single obligor limit $F \leq \theta V \mathbf{1}$, then $H(F) \leq \theta$ and the maximum concentration under Herfindahl-Hirschman concentration measure, occurs if and only if F is some permutation of the following loan distribution:

$$f_k^* = \begin{cases} \mathbf{q}V ; k = 0, 1, 2, 3, \dots, n \\ 0 ; k = n+1, n+2, \dots, N \end{cases}$$

where $n\mathbf{q}V = V \mathbf{P} n\mathbf{q} = 1$ or $n = 1/\mathbf{q}$

Proof:

Only the optimization issue will be proved, since the first part is proved in the text. To prove *necessity*, note that the maximum concentration under the single obligor limit in the sense of definition 1, can be obtained by solving the following optimization problem:

$$\begin{aligned} \max \quad & \mathbf{F}^T \mathbf{F} \\ \text{s.t. : } & \mathbf{1}^T \mathbf{F} = V \\ & \mathbf{F} \leq \mathbf{q}V \mathbf{1} \\ & \mathbf{F} \geq 0 \end{aligned}$$

Let $F = aV$, where a is such that $\mathbf{1}^T a = 1$, then the above is equivalent to the following problem:

$$(P.1) \dots \dots \dots \left\{ \begin{array}{l} \max \mathbf{a}^T \mathbf{a} \\ \text{s.t. } \mathbf{1}^T \mathbf{a} = 1 \\ \mathbf{q}\mathbf{1} - \mathbf{a} \geq 0 \\ \mathbf{a} \geq 0 \end{array} \right.$$

In order to examine the Karush-Kuhn-Tucker (KKT) conditions define the Lagrangean function:

$$L(\mathbf{a}, \mathbf{I}, \mathbf{u}, \mathbf{v}) = \mathbf{a}^T \mathbf{a} + \mathbf{I}(1 - \mathbf{1}^T \mathbf{a}) + \mathbf{v}^T (\mathbf{q}\mathbf{1} - \mathbf{a}) + \mathbf{u}^T \mathbf{a}.$$

From which the first order optimality conditions yield:

$$\nabla_{\mathbf{a}} L = 2\mathbf{a} - \mathbf{1}\mathbf{1} - \mathbf{v} + \mathbf{u} = \mathbf{0} \Rightarrow \mathbf{a}^* = \frac{1}{2}(\mathbf{1}\mathbf{1} + \mathbf{v} - \mathbf{u}) \dots (5.1)$$

Note that for any vector “ $\mathbf{x}$ ” such that $\mathbf{x}^T \mathbf{x} = \theta$, then for any other vector “ $\mathbf{y}$ ” which is simply a permutation of the elements of “ $\mathbf{x}$ ”, it is also true that $\mathbf{y}^T \mathbf{y} = \theta$. Thus, if $n\theta=1$, without loss of generality it will be shown that any permutation of:

$$a_i = \begin{cases} \mathbf{q} & \text{if } i = 1, \dots, n \\ 0 & \text{if } i = n+1, \dots, N \end{cases}$$

is an optimal solution of problem P.1.

Clearly the solution is feasible since all components are none-negative and add up to one. Furthermore, none of the elements is larger than θ . It is therefore only necessary to find values of the multipliers λ , u and v consistent with the proposed solution and satisfying the complementarity conditions; that is:

$$\left. \begin{aligned} u_i a_i &= 0 \\ v_i (\mathbf{q} - a_i) &= 0 \end{aligned} \right\} \forall i$$

Let $u_i = \mathbf{q} - a_i$ and $v_i = a_i$, and complementarity is easily verified.

In order to prove sufficiency, *introduce slack variables* “ h ”, and rewrite the problem as follows:

$$\left. \begin{aligned} \max \quad & \mathbf{a}^T \mathbf{a} \\ \text{s. t.} \quad & \mathbf{1}^T \mathbf{a} = 1 \\ & \mathbf{Ia} + \mathbf{Ih} = \mathbf{q1} \\ & \mathbf{a} \geq \mathbf{0}, \mathbf{h} \geq \mathbf{0} \end{aligned} \right\} \dots (P'.1.)$$

Since a convex function is being maximized over a polihedron defined by a set of linear constraints, the optimal solution lies at an extreme point of the polihedron¹⁹. The extreme points of the polihedron are associated to *bases* of the constraint matrix. In what follows, it is shown that these bases and their inverses can be characterized so that all optimal extreme points of the polihedron can also be identified and are in fact permutations of the proposed solution. Rewriting the constraints in matrix form, one obtains:

¹⁹ See for example Márquez (1987) or Nash and Sofer (1996)

$$\mathbf{A} \begin{bmatrix} \mathbf{a} \\ \mathbf{h} \end{bmatrix} = \mathbf{b}$$

$$\mathbf{a} \geq \mathbf{0}; \quad \mathbf{h} \geq \mathbf{0}$$

Where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{1}_n^T & \vdots & \mathbf{0}_n^T \\ \dots & \dots & \dots \\ \mathbf{I}_N & \vdots & \mathbf{I}_N \end{bmatrix}; \quad \mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \quad \mathbf{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ \mathbf{q} \\ \vdots \\ \mathbf{q} \end{bmatrix}$$

It is easy to verify that every basic solution is associated to a basis composed of $N+1$ linearly independent columns of $\mathbf{A}$, and that they necessarily have the following form:

$$\mathbf{B} = \begin{bmatrix} \mathbf{1}_n^T & \vdots & \mathbf{0}_{N-n}^T & \vdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ & & & \vdots & \\ & \mathbf{I}_N & & \vdots & \mathbf{e}_1 \\ & & & \vdots & \end{bmatrix}$$

$\mathbf{1}_n$ is an n -component sum vector, $\mathbf{0}_{N-n}$ is an $(N-n \times N-n)$ matrix of zero's, $\mathbf{e}_1$ is a unit vector in which the i^{th} element is unity and the remaining elements are zero. Note that no generality is lost by assuming it is always $\mathbf{e}_1$. It can also be verified that the inverse of the above has the following form:

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & \vdots & 0 & \vdots & -\mathbf{1}_{n-1}^T & \vdots & \mathbf{0}_{N-n}^T \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ & \vdots & & \vdots & \mathbf{I}_{n-1} & \vdots & \mathbf{0}_{n-1 \times N-n} \\ \mathbf{0}_{N-n} & \vdots & \mathbf{0}_{N-n} & \vdots & \dots & \vdots & \dots \\ & \vdots & & \vdots & \mathbf{0}_{N-n \times n-1} & \vdots & \mathbf{I}_{N-n} \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ -1 & \vdots & 1 & \vdots & \mathbf{1}_{n-1}^T & \vdots & \mathbf{0}_{N-n}^T \end{bmatrix}$$

Multiplying the inverse by the vector of independent terms one obtains:

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & \vdots & 0 & \vdots & -\mathbf{1}_{n-1}^T & \vdots & \mathbf{0}_{N-n}^T \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ & \vdots & & \vdots & \mathbf{I}_{n-1} & \vdots & \mathbf{0}_{n-1 \times N-n} \\ \mathbf{0}_{N-n} & \vdots & \mathbf{0}_{N-n} & \vdots & \dots & \vdots & \dots \\ & \vdots & & \vdots & \mathbf{0}_{N-n \times n-1} & \vdots & \mathbf{I}_{N-n} \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ -1 & \vdots & 1 & \vdots & \mathbf{1}_{n-1}^T & \vdots & \mathbf{0}_{N-n}^T \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{q} \\ \mathbf{q} \\ \mathbf{q} \\ \vdots \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{q} \\ \vdots \\ \mathbf{q} \\ 0 \end{bmatrix}$$

Rearranging rows and columns as is convenient, it is seen that $\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \mathbf{q}\mathbf{1}_N \\ 0 \end{bmatrix}$, for any basis

$\mathbf{B}$ of $\mathbf{A}$. As was to be proved, this means that all extreme points of the polyhedron are some permutation of:

$$a_i^n = \begin{cases} \mathbf{q} & \text{if } i = 1, \dots, n \\ 0 & \text{if } i = n+1, \dots, N \end{cases} \quad \blacksquare$$

Theorem 5.2.

If $H(\mathbf{F}) \leq \theta$ then:

$$f_i \leq \frac{1}{N} \left(1 + \sqrt{(N\mathbf{q} - 1)(N - 1)} \right) V < \sqrt{\mathbf{q}} V \quad \text{for } i = 1, 2, 3, \dots, N$$

Proof:

The theorem will be proved by finding the maximum loan that could be allowed while satisfying the constraint. As in the proof of the previous theorem, redefining $\mathbf{F}$ in terms of proportions of the total value of the portfolio “a” means that the maximum size loan under the constraint as a proportion of the value of the portfolio, is obtained by solving the following optimization problem:

$$\begin{aligned} & \max \mathbf{e}_i^T \mathbf{a} \\ \text{s. a. } & \mathbf{a}^T \mathbf{a} = \mathbf{q} \\ & \mathbf{1}^T \mathbf{a} = 1 \\ & \mathbf{a} \geq \mathbf{0} \end{aligned}$$

Again, first order necessary conditions are examined by forming the Lagrangean:

$$L(\mathbf{a}, \mathbf{I}, \mathbf{mu}) = \mathbf{e}_i^T \mathbf{a} - \mathbf{I}(\mathbf{a}^T \mathbf{a} - \mathbf{q}) - \mathbf{m}(\mathbf{1}^T \mathbf{a} - 1) - \mathbf{u}^T \mathbf{a}$$

Taking partial derivatives with respect to $\mathbf{a}$, it can be seen that the solution to the problem must have the following form:

$$\begin{aligned}\nabla_{\mathbf{a}} L &= \mathbf{e}_i - \mathbf{m} - \mathbf{u} - 2\mathbf{l}\mathbf{a} = 0 \\ \Rightarrow \mathbf{a}^* &= \frac{\mathbf{e}_i - \mathbf{m}^* - \mathbf{u}^*}{2\mathbf{l}^*} \dots\dots\dots (5.1)\end{aligned}$$

Successively taking the inner product of $\mathbf{a}^*$ with $\mathbf{1}$, $\mathbf{e}_i$, $\mathbf{e}_k$ after a bit of algebra one arrives at:

$$\begin{aligned}\mathbf{1}^T \mathbf{a}^* &= \frac{1 - \mathbf{m}^* N - \mathbf{1}^T \mathbf{u}^*}{2\mathbf{l}^*} = 1 \\ \Rightarrow 2\mathbf{l}^* &= 1 - \mathbf{m}^* N - \mathbf{1}^T \mathbf{u}^* \dots\dots\dots (5.2)\end{aligned}$$

(Since $\mathbf{1}^T \mathbf{1} = N$)

$$\begin{aligned}(\mathbf{a}^*)^T \mathbf{a}^* &= \frac{a_i^* - \mathbf{m}^* - (\mathbf{u}^*)^T \mathbf{a}^*}{2\mathbf{l}^*} = \mathbf{q} \\ \Rightarrow a_i^* &= \mathbf{m}^* + 2\mathbf{q}\mathbf{l}^* \dots\dots\dots (5.3)\end{aligned}$$

(Since by complementarity $(\mathbf{u}^*)^T \mathbf{a}^* = 0$; and $\mathbf{1}^T \mathbf{a}^* = 1$).

$$\mathbf{e}_i^T \mathbf{a}^* = \frac{1 - \mathbf{m}^* - u_i^*}{2\mathbf{l}^*} = a_i^* \dots\dots\dots (5.4)$$

$$\mathbf{e}_k^T \mathbf{a}^* = \frac{-\mathbf{m}^* - u_i^*}{2\mathbf{l}^*} = a_k \dots\dots\dots (5.5)$$

Finally, inspection of (1), (2), (3), (4) and (5) indicates that all solutions to the problem have the following form:

$$a_i = \begin{cases} \mathbf{a} > 0 & \text{if } k = i \\ \mathbf{b} \neq \mathbf{a}; \quad \mathbf{b} > 0 & \text{if } k = 1, \dots, n \quad k \neq i \quad \dots\dots (5.6) \\ 0 & \text{otherwise} \end{cases}$$

To verify that the proposed solution satisfies the KKT conditions, first note that due to complementarity $a_k > 0$ requires $u_k = 0$ for $k = 1, 2, 3, \dots, n$ and therefore $u_i = 0$. Substituting $\mathbf{a} = a_i$ in (3) and (4) and solving for $\mathbf{l}^*$ one obtains:

$$\mathbf{l}^* = \frac{1 - \mathbf{m}^*}{2\mathbf{a}} \dots\dots\dots (5.7)$$

$$\mathbf{l}^* = \frac{\mathbf{a} - \mathbf{m}^*}{2\mathbf{q}} \dots\dots\dots (5.8)$$

Equating (6) and (7) and solving for $\mathbf{m}$ yields:

$$\mathbf{m} = \frac{\mathbf{q} - \mathbf{a}^2}{\mathbf{q} - \mathbf{a}} \dots\dots\dots (5.9)$$

Similarly, the proposed solution along with complementarity results in $u_k > 0$ ($k = n+1, \dots, N$), and $u_k > 0 \Rightarrow a_k = 0$. Thus, from (5) one obtains:

$$u_k = -\mathbf{m} \dots\dots\dots (5.10)$$

Since $\mathbf{1}^T \mathbf{u} = (N-n)\mathbf{m}$ the preceding equation along with (5.2) leads to:

$$2\mathbf{l} = 1 - n\mathbf{m} \dots\dots\dots (5.11)$$

Now, substituting $\mathbf{m}$ in (6):

$$\begin{aligned} 2\mathbf{l} &= \frac{1}{\mathbf{a}}(1 - \mathbf{m}) \quad \Rightarrow \\ 2\mathbf{l} &= \frac{\mathbf{a} - 1}{\mathbf{q} - \mathbf{a}} \dots\dots\dots (5.12) \end{aligned}$$

Equating (8) and (9) and substituting $\mathbf{m}$

$$\frac{\mathbf{a} - 1}{\mathbf{q} - \mathbf{a}} = 1 - n \left(\frac{\mathbf{q} - \mathbf{a}^2}{\mathbf{q} - \mathbf{a}} \right) \Rightarrow n\mathbf{a}^2 - 2\mathbf{a} - \mathbf{q}(n-1) + 1 = 0 \dots\dots\dots (5.13)$$

Solving the quadratic equation:

$$\mathbf{a} = \frac{1}{n} \left[1 + \sqrt{(n\mathbf{q} - 1)(n-1)} \right] \dots\dots\dots (5.14)$$

Where the only relevant root is the positive one. Since $\mathbf{b} = \frac{1 - \mathbf{a}}{n-1}$, then:

$$\mathbf{b} = \frac{1}{n-1} - \frac{1}{n(n-1)} \left[1 + \sqrt{(n\mathbf{q} - 1)(n-1)} \right] \dots\dots\dots (5.15)$$

It is important to note that these values are only meaningful when $\mathbf{q}^3 1/n$. It is also interesting to note what happens under extreme values of θ : If $\theta=1$ then $\alpha=1$, $\beta=0$ and if $\theta=1/n$, then $\alpha=1/n=\beta$.

It must now be shown that the values of α and β are associated to Lagrange multipliers $\mathbf{l}^*$, $\mathbf{m}^*$ and u^* that satisfy the KKT conditions. From (5.4) and (5.5), and due to complementarity, one sees that:

$$a_i^* = \frac{1 - \mathbf{m}^*}{2\mathbf{l}^*}, \text{ and } a_j = -\frac{\mathbf{m}^*}{2\mathbf{l}^*} \quad (1 \leq i \leq n, i \neq j),$$

Similarly:

$$u_k^* = -\mathbf{m}^* \quad (k=n+1, \dots, N).$$

From (5.9) it is clear that the KKT conditions are satisfied if and only if

$$\mu^* < 0, \lambda^* \geq 0 \text{ and } 2\lambda^* \geq -\mu^*$$

Therefore, one must show that $\mu^* < 0$, which by (5.9) is the same as proving that:

$$\mathbf{a}^2 \leq \mathbf{q} < \mathbf{a}.$$

Obviously, $\mathbf{a}^2 < \mathbf{q}$ and to prove that $\mathbf{q} < \mathbf{a}$ note that:

$$\mathbf{a} = \frac{1}{n} \left(1 + \sqrt{(n\mathbf{q}-1)(n-1)} \right) > \mathbf{q} \Leftrightarrow \sqrt{(n\mathbf{q}-1)(n-1)} > n\mathbf{q}-1 \Leftrightarrow n-1 > n\mathbf{q}-1$$

which is always true when $\mathbf{q} < 1$.

To conclude the proof, since the problem is non-convex the maximum is at one of the extremes, so one must determine the value of “n” where the maximum is obtained. This is fairly simple since it is easy to see that $\mathbf{a}$ is a strictly increasing function of “n”, if $\mathbf{q} < 1$,

because in this case $\frac{d\mathbf{a}}{dn} > 0$, so that:

$$\mathbf{a}^*(N) = \frac{1}{N} \left(1 + \sqrt{(N\mathbf{q}-1)(N-1)} \right)$$

Finally, note that:

$$\mathbf{a}^*(N) < \sqrt{\mathbf{q}} = \lim_{N \rightarrow \infty} \mathbf{a}^*(N)$$

which concludes the proof. ■

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