Generalized Gaussian Jacobsthal and Generalized Gaussian Jacobsthal Lucas Polynomial

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Abstract: The aim of the present paper is to present a generalization of Gaussian Jacobsthal polynomial and Gaussian Jacobsthal Lucas polynomial. Present paper extends the work of Asci and Gurel [3]. Some important generalizations of Generating function, Binet formula, Explicit formula, Q matrix and determinantal representations of these polynomials are also produced.

Keywords: Generalized Gaussian Jacobsthal; Generalized Gaussian Jacobsthal Lucas Polynomial; Binet formula; Cassini Identity

I. INTRODUCTION

In 1977, Berzsenyi [4] initiated the theory of Gaussian Fibonacci numbers. Afterwards different authors formulated various interesting results using Gaussian Fibonacci and Gaussian Lucas polynomials. On the other hand, Asci and Gurel [1] introduced Bivariate Gaussian Fibonacci and Bivariate Gaussian Lucas polynomials. In the same work, authors [Asci and Gurel] gave generating functions, Binet formulas, explicit formulas and Q matrix for these polynomials. Similarly, Asci and Gurel [2] defined Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers and using these notions they discussed some interesting results. Recently, Asci and Gurel [3] defined the Gaussian Jacobsthal polynomial and Gaussian Jacobsthal Lucas polynomials and formulated generating function, Binet formula, Explicit formulas and determinantal representations for these polynomials.

Horadam [7] introduced the Jacobsthal and the Jacobsthal Lucas sequences under the following recurrence relations:

$$J_n = J_{n-1} + 2J_{n-2}, n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n = j_{n-1} + 2j_{n-2}, n \geq 2$$

where $j_0 = 2$ and $j_1 = 1$.

In the same work Horadam [7], Jacobsthal and the Jacobsthal Lucas polynomial sequences were defined by using the following recurrence relations:

$$J_n(x) = J_{n-1}(x) + 2J_{n-2}(x), n \geq 2$$

where $J_0 = 0$ and $J_1 = 1$, and

$$j_n(x) = j_{n-1}(x) + 2j_{n-2}(x), n \geq 2$$

where $j_0 = 2x$ and $j_1 = 1$.

Subsequently, the Gaussian Jacobsthal sequence and Gaussian Jacobsthal Lucas sequence were defined by Asci and Gurel [2] as follows:

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, n \geq 1$$

where $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$, and

$$Gj_{n+1} = Gj_n + 2Gj_{n-1}, n \geq 1$$

where $Gj_0 = 2 - \frac{i}{2}$ and $Gj_1 = 1 + 2i$.

The Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomial in [3] were defined by

$$GJ_{n+1}(x) = GJ_n(x) + 2GJ_{n-1}(x), n \geq 1$$

where $GJ_0(x) = \frac{i}{2}$ and $GJ_1(x) = 1$, and

$$Gj_{n+1}(x) = Gj_n(x) + 2Gj_{n-1}(x), n \geq 1$$

where $Gj_0(x) = 2 - \frac{i}{2}$ and $Gj_1(x) = 1 + 2i$.

Further, in [3] the authors have also formulated the following notions:

Generating functions:

$$(i) g(t, x) = \sum_{n=0}^{\infty} GJ_n(x) \cdot t^n = \frac{2t + i(1-t)}{2-2t-4t^2x},$$

$$(ii) h(t, x) = \sum_{n=0}^{\infty} Gj_n(x) \cdot t^n = \frac{4-2t + i(t-1+4xt)}{2-2t-4t^2x}.$$

Binet Formula:

For $n \geq 0$,

$$(i) GJ_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i \cdot x \frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)},$$

$$(ii) Gj_n(x) = \frac{\alpha^n(x) + \beta^n(x)}{i \cdot x(\alpha^{n-1}(x) + \beta^{n-1}(x))} +$$

Explicit Formula:

$$(i) GJ_n(x) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-r-1}{r} (2, x)^r + i \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-r-2}{r} (2)^r \cdot (x)^{r+1},$$

$$(ii) Gj_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-r} \binom{n-r}{r} (2, x)^r + i \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-r-1} \binom{n-r-1}{r} (2)^r \cdot (x)^{r+1}.$$

Cassini Identity:

For $n \geq 1$,

$$GJ_{n-1}(x) \cdot Gj_{n+1}(x) - GJ_n^2(x) = (-1)^n (2)^{n-2} \cdot x^{n-1} (2+x-i).$$

In this paper, we generalized the work of Asci and Gurel in [13] by introducing the Generalized Gaussian Jacobsthal polynomial, Generalized Gaussian Lucas polynomial and thereby we obtain generating function and Cassini identity and introduce Q matrix for these polynomials. We also present explicit formula and determinantal representation, and give various identities using these polynomials.

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II. GENERALIZED GAUSSIAN JACOBSTHAL AND GENERALIZED GAUSSIAN JACOBSTHAL LUCAS POLYNOMIALS.

In this section we give our notions and formulate our main results.

Definition 2.1. The Generalized Gaussian Jacobsthal polynomial $\{G_{j,k,n}(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$G_{j,k,n+1}(x) = G_{j,k,n}(x) + 2^k \cdot x \cdot G_{j,k,n-1}(x)$, $n \geq 1$ and k is non – negative integer with initial conditions

$$G_{j,k,0}(x) = \frac{i}{2} \quad \text{and} \quad G_{j,k,1}(x) = 1.$$

Definition 2.2. The Generalized Gaussian Jacobsthal Lucas polynomial $\{G_{j,k,n}(x)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$G_{j,k,n+1}(x) = G_{j,k,n}(x) + 2^k \cdot x \cdot G_{j,k,n-1}(x)$, $n \geq 1$, and k is non – negative integer with initial conditions

$$G_{j,k,0}(x) = 2 - \frac{i}{2} \quad \text{and} \quad G_{j,k,1}(x) = 1 + 2xi.$$

It is interesting to note that

$$G_{j,k,n}(1) = G_{j,k,n} \quad \text{and} \quad G_{j,k,n}(1) = G_{j,k,n}$$

where $G_{j,k,n}$ and $G_{j,k,n}$ are the Generalized Gaussian Jacobsthal and Generalized Gaussian Jacobsthal Lucas number.

Next, we give few number of terms of these sequences in the following tables:

Generalized Gaussian Jacobsthal Polynomials:

N	$G_{j,k,n}$
0	$\frac{i}{2}$
1	1
2	$1 + i2^{k-1} \cdot x$
3	$(1 + 2^k \cdot x) + i2^{k-1} \cdot x$
4	$(1 + 2^{k+1}x) + i(2^{k-1} + 2^{2k-1}) \cdot x$
5	$1 + (2^k + 2^{k+1})x + 2^{2k} \cdot x^2 + i(2^{k-1} + 2^{2k-1} \cdot x + 2^{2k-1}) \cdot x$
6	$1 + 2^{k+2} \cdot x + (2^{2k} + 2^{2k+1}) \cdot x^2 + ix(2^{k-1} + 2^{2k-1} \cdot x + 2^{2k} \cdot x + 2^{3k-1}) \cdot x$
$\vdots$	$\vdots$

Generalized Gaussian Jacobsthal Lucas Polynomials:

N	$G_{j,k,n}$
0	$2 - \frac{i}{2}$
1	$1 + 2xi$
2	$(1 + 2^{k+1} \cdot x) + ix(2 - 2^{k-1})$
3	$(1 + 2^{k+1} + 2^k \cdot x) + ix(2 - 2^{k-1} + 2^{k+1} \cdot x)$
4	$(1 + 2^{k+2}x + 2^{2k-1}x^2) + ix(2 - 2^{k-1} + 2^{k+2}x - 2^{2k-1} \cdot x)$
5	$(1 + 2^{k+2}x + 2^{2k+2} \cdot x^2 + 2^{2k} \cdot x^2) + ix(2 - 2^{k-1} + 2^{k+2}x - 2^{2k} \cdot x + 2^{k+1}x - 2^{2k+1} \cdot x^2)$
6	$(1 + 2^{k+2}x + 2^{2k+3} \cdot x^2 + 2^k \cdot x + 2^{3k+1} \cdot x^3) + ix(2 - 2^{k-1} + 2^{k+2}x + 2^{2k+1} \cdot x^2 - 2^{2k-1}x + 2^{2k+3} \cdot x^2 - 2^{3k-1} \cdot x^2)$
$\vdots$	$\vdots$

The main recursive properties of the generalized Gaussian Jacobsthal and Generalized Gaussian Lucas polynomial are determined in this section. Initially, we can begin with the characteristic equation, that is

$$t^2 - t - 2^k x = 0$$

$$t = \frac{1 \pm \sqrt{1 + 4 \cdot 2^k x}}{2}$$

$$t = \frac{1 \pm \sqrt{1 + 2^{k+2} x}}{2}$$

$$\alpha(x) = \frac{1 + \sqrt{1 + 2^{k+2} x}}{2} \quad \text{and} \quad \beta(x) = \frac{1 - \sqrt{1 + 2^{k+2} x}}{2}$$

$$\alpha(x) + \beta(x) = 1, \quad \alpha(x) \cdot \beta(x) = -2^k \cdot x, \quad \alpha(x) - \beta(x) = \sqrt{1 + 2^{k+2} x}$$

Some Properties:

Theorem 2.1. (i) The Generating function for Generalized Gaussian Jacobsthal polynomial is

$$g(t, x) = \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^n = \frac{2t + i(1-t)}{2 - 2t - 2^{k+1}t^2x}$$

and

(ii) For Generalised Gaussian Jacobsthal Lucas polynomial is

$$h(t, x) = \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^n = \frac{4-2t+i(t-1+4xt)}{2-2t-2^{k+1}t^2x}$$

Proof: (i) Let $g(t, x)$ be the generating function of generalized Gaussian Jacobsthal polynomial sequence $G_{j,k,n}(x)$ then

$$g(t, x) - t \cdot g(t, x) - 2^k \cdot x \cdot t^2 \cdot g(t, x) = \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^n - \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^{n+1} - 2^k \cdot x \cdot \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^{n+2}$$

$$\begin{aligned} &= G_{j,k,0}(x) + G_{j,k,1}(x) \cdot t - G_{j,k,0}(x) \cdot t \\ &+ \sum_{n=2}^{\infty} t^n [G_{j,k,n}(x) - G_{j,k,n-1}(x) - 2^k G_{j,k,n-2}(x)] \\ &= \frac{i}{2} + (1 - \frac{i}{2})t \\ &= \frac{2i + i(1-t)}{2} \end{aligned}$$

By taking $g(t, x)$ parenthesis, we get

$$\begin{aligned} (1-t-2^k \cdot x \cdot t^2) g(t, x) &= \frac{2i + i(1-t)}{2} \\ g(t, x) &= \frac{2i + i(1-t)}{2(1-t-2^k \cdot x \cdot t^2)} \\ g(t, x) &= \frac{2i + i(1-t)}{(2-2t-2^{k+1} \cdot x \cdot t^2)} \end{aligned}$$

(ii) Let $h(t, x)$ be the generalized function of generalized Gaussian Jacobsthal Lucas polynomial sequence $G_{j,k,n}(x)$ then

$$\begin{aligned} h(t, x) - t \cdot h(t, x) - 2^k \cdot x \cdot t^2 \cdot h(t, x) &= \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^n - \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^{n+1} - 2^k \cdot x \cdot \sum_{n=0}^{\infty} G_{j,k,n}(x) \cdot t^{n+2} \\ &= G_{j,k,0}(x) + G_{j,k,1}(x) \cdot t - G_{j,k,0}(x) \cdot t \\ &+ \sum_{n=2}^{\infty} t^n [G_{j,k,n}(x) - G_{j,k,n-1}(x) - 2^k G_{j,k,n-2}(x)] \\ &= 2 - \frac{i}{2} + (1+2xi) - (2 - \frac{i}{2})t \\ &= 2 - \frac{i}{2} + 1 + 2xi - 2t + \frac{i}{2}t, \end{aligned}$$

that is,

$$(1-t-2^k \cdot x \cdot t^2) h(t, x) = \frac{4-i-2t-4xti-4t+it}{2}$$

which implies,

$$\begin{aligned} h(t, x) &= \frac{4-2t+i(4xt-1+t)}{2(1-t-2^k \cdot x \cdot t^2)} \\ \text{and hence } h(t, x) &= \frac{4-2t+i(4xt-1+t)}{2-2t-2^{k+1} \cdot x \cdot t^2} \end{aligned}$$

The following results hold immediately by applying induction:

Theorem 2.2. (Generalized Binet Formula)

For $n \geq 0$

$$(i) G_{j,k,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + i \cdot x \cdot 2^{k-1} \frac{\alpha^{n-1}(x) - \beta^{n-1}(x)}{\alpha(x) - \beta(x)}$$

$$(ii) \begin{aligned} (i) G_{j,k,n}(x) &= \frac{\alpha^n(x) + \beta^n(x)}{i \cdot x \cdot 2^{k-1}(\alpha^{n-1}(x) + \beta^{n-1}(x))} + \end{aligned}$$

Theorem 2.3. The explicit formula of Generalized Gaussian Jacobsthal polynomial is

$$G_{j,k,n}(x) = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-r-1}{r} (2^k \cdot x)^r + i \sum_{r=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-r-2}{r} (2^k)^r \cdot (x)^{r+1}$$

Theorem 2.4. The explicit formula of Generalized Gaussian Jacobsthal Lucas polynomial is

$$G_{j,k,n}(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-r} \binom{n-r}{r} (2^k \cdot x)^r + i \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-r-1} \binom{n-r-1}{r} (2^k)^r \cdot (x)^{r+1}$$

Theorem 2.5. Let $D_n(x)$ denote the $n \times n$ tridiagonal matrix as

$$D_n(x) = \begin{bmatrix} 1 & i2^{k-1} & 0 & \cdots & 0 \\ -x & 1 & 2^k x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 2^k x \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, n \geq 1$$

and let $D_0(x) = \frac{i}{2}$, Then $\det D_n(x) = G_{j,k,n}(x)$, $n \geq 1$

Proof: We shall prove the result by using mathematical induction on n .

For $n = 1$ and $n = 2$

$$\det D_1(x) = 1 = G_{j,k,1}(x)$$

and

$$\det D_2(x) = 1 + i2^{k-1}x = G_{j,k,2}(x)$$

Assume that the theorem is true for $n-1$ and $n-2$, that is

$$\det D_{n-1}(x) = G_{j,k,n-1}(x)$$

and

$$\det D_{n-2}(x) = G_{j,k,n-2}(x)$$

Now, we shall prove that the theorem for n

For,

$$\begin{aligned} \det D_n(x) &= \det D_{n-1}(x) + 2^k \cdot x \det D_{n-2}(x) \\ &= G_{j,k,n-1}(x) + 2^k \cdot x G_{j,k,n-2}(x) \\ &= G_{j,k,n}(x) \end{aligned}$$

Theorem 2.6. Let $H_n(x)$ denote the $n \times n$ tridiagonal matrix as

$$H_n(x) = \begin{bmatrix} 2 - \frac{i}{2} & \frac{x}{2} - 1 & 0 & \cdots & 0 \\ 1 & ix & 2^k x & \ddots & \vdots \\ 0 & -1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 2^k x \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, n \geq 1$$

Then, $\det H_n(x) = G_{j,k,n-1}(x)$, $n \geq 0$.

Proof: We shall prove the result by using mathematical induction on n .

For $n = 1$ and $n = 2$

$$\det H_1(x) = 2 - \frac{i}{2} = G_{j,k,0}(x)$$

and

$$\det H_2(x) = 1 + 2xi = G_{j,k,1}(x)$$

Assume that the theorem is true for $n-1$ and $n-2$

$$\det H_{n-1}(x) = G_{j,k,n-2}(x)$$

and

$$\det H_{n-2}(x) = G_{j,k,n-3}(x)$$

Now, we shall prove that the theorem for n

For,

$$\begin{aligned} \det H_n(x) &= \det H_{n-1}(x) + 2^k \cdot x \det H_{n-2}(x) \\ &= G_{j,k,n-1}(x) + 2^k \cdot x G_{j,k,n-2}(x) \\ &= G_{j,k,n}(x) \end{aligned}$$

Now, we introduce the matrices $Q(x)$, P and R that plays the role of the Q -Matrix of Fibonacci numbers. Let $Q(x)$, P and R denotes the 2×2 matrices defined as

$$\begin{aligned} Q(x) &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix}, R = \\ &\begin{bmatrix} 1 + 2^{k+1} \cdot x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix}. \end{aligned}$$

Theorem 2.7. Let $n \geq 1$, Then

$$Q^n(x) \cdot P = \begin{bmatrix} G_{j,k,n+2}(x) & G_{j,k,n+1}(x) \\ G_{j,k,n+1}(x) & G_{j,k,n}(x) \end{bmatrix}$$

where $G_{j,k,n}(x)$ is the n th Generalized Gaussian Jacobsthal Polynomial.

Proof:- We shall prove the result by using mathematical induction on n .

For $n = 1$

$$\begin{aligned} Q(x).P &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 + i2^{k-1}.x + 2^k x & 1 + i2^{k-1}x \\ 1 + i2^{k-1}x & 1 \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,3}(x) & G_{j,k,2}(x) \\ G_{j,k,2}(x) & G_{j,k,1}(x) \end{bmatrix} \end{aligned}$$

Assume that the theorem hold for $n = t$, that is

$$\begin{aligned} [Q(x)]^t.P &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \\ G_{j,k,t+1}(x) & G_{j,k,t}(x) \end{bmatrix} \end{aligned}$$

Now, we shall prove the theorem for $n = t+1$, we have

$$\begin{aligned} [Q(x)]^{t+1}.P &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^{t+1} \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + ix2^{k-1} & 1 \\ 1 & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \\ G_{j,k,t+1}(x) & G_{j,k,t}(x) \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,t+3}(x) & G_{j,k,t+2}(x) \\ G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \end{bmatrix} \end{aligned}$$

Theorem 2.8. Let $n \geq 1$, Then

$$Q^n(x).R = \begin{bmatrix} G_{j,k,n+2}(x) & G_{j,k,n+1}(x) \\ G_{j,k,n+1}(x) & G_{j,k,n}(x) \end{bmatrix}$$

where $G_{j,k,n}(x)$ is the n th Generalized Gaussian Jacobsthal Lucas Polynomial.

Proof: We shall prove the result by using mathematical induction on n .

For $n = 1$

$$\begin{aligned} Q(x).R &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 + 2^{k+1}.x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2^{k+1}.x + 2^k.x + ix(2 - 2^{k-1} + 2^{k+1}x) & 1 + 2ix + 2^{k+1}x - i.2^{k-1}.x \\ 1 + 2ix + 2^{k+1}x - i.2^{k-1}.x & 1 + 2xi \end{bmatrix} \\ &= \begin{bmatrix} 1 + 2^{k+1}.x + 2^k.x + ix(2 - 2^{k-1} + 2^{k+1}x) & 1 + 2^{k+1}x + i.x(2 - 2^{k-1}) \\ 1 + 2^{k+1}x + i.x(2 - 2^{k-1}) & 1 + 2xi \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,3}(x) & G_{j,k,2}(x) \\ G_{j,k,2}(x) & G_{j,k,1}(x) \end{bmatrix} \end{aligned}$$

Assume that the theorem holds for $n = t$, that is

$$\begin{aligned} [Q(x)]^t &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + 2^{k+1}.x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \\ G_{j,k,t+1}(x) & G_{j,k,t}(x) \end{bmatrix} \end{aligned}$$

Now, we shall prove that the theorem for $n = t+1$, For,

$$\begin{aligned} [Q(x)]^{t+1} &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^{t+1} \begin{bmatrix} 1 + 2^{k+1}.x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix} \cdot R \\ &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix}^t \begin{bmatrix} 1 + 2^{k+1}.x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2^k x \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \\ G_{j,k,t+1}(x) & G_{j,k,t}(x) \end{bmatrix} \\ &= \begin{bmatrix} G_{j,k,t+3}(x) & G_{j,k,t+2}(x) \\ G_{j,k,t+2}(x) & G_{j,k,t+1}(x) \end{bmatrix} \end{aligned}$$

Theorem 2.9. (Generalized Cassini Identity)

For $n \geq 1$

$$G_{j,k,n-1}(x).G_{j,k,n+1}(x) - G_{j,k,n}^2(x) = (-1)^n (2^k)^{n-2}.x^{n-1} (2^{k-1}x + 2 - i)$$

Proof: We can prove the theorem by matrix method. For, first of all we shall determine the determinants of matrices.

$$\begin{aligned} \det Q^{n-1}(x) &= \begin{vmatrix} 1 & 2^k x \\ 1 & 0 \end{vmatrix}^{n-1} = (-2^k x)^{n-1}. \\ \det P &= \begin{vmatrix} 1 + i.2^{k-1}.x & 1 \\ 1 & \frac{i}{2} \end{vmatrix} = \frac{i}{2} - \frac{2^{k-1}}{2}x - 2 = -\frac{1}{2} [2^{k-1}.x + 2 - i] \end{aligned}$$

By Theorem 2.7,

$$[Q(x)]^{n-1}.P = \begin{bmatrix} G_{j,k,n+1}(x) & G_{j,k,n}(x) \\ G_{j,k,n}(x) & G_{j,k,n-1}(x) \end{bmatrix}.$$

We get the determinants of the matrices

$$\begin{aligned} G_{j,k,n-1}(x).G_{j,k,n+1}(x) - G_{j,k,n}^2(x) &= \det(Q^{n-1}(x).P) \\ &= \det(Q^{n-1}(x)).\det(P) \\ &= (-2^k x)^{n-1} \cdot \left(-\frac{1}{2} [2^{k-1}.x + 2 - i]\right) \\ &= (-1)^n (2^k)^{n-2}.x^{n-1} (2^{k-1}x + 2 - i) \end{aligned}$$

Theorem 2.10. For $n \geq 1$

$$\begin{aligned} G_{j,k,n-1}(x).G_{j,k,n+1}(x) - G_{j,k,n}^2(x) &= (-1)^{n-1}.2^{n-2}.x^{n-1} [(8x)^2 + (2 + 3.2^{k-1})x + 2 - i2^{k+2}.x] \end{aligned}$$

Proof: We can prove the theorem by matrix method. For, first of all we determine the determinants of matrices.

$$\begin{aligned} \det Q^{n-1}(x) &= \begin{vmatrix} 1 & 2^k x \\ 1 & 0 \end{vmatrix}^{n-1} = (-2^k x)^{n-1} \\ \det R &= \begin{vmatrix} 1 + 2^{k+1}.x + ix(2 - 2^{k-1}) & 1 + 2ix \\ 1 + 2ix & 2 - \frac{i}{2} \end{vmatrix} \\ &= [(8x)^2 + (2 + 3.2^{k-1})x + 2 - i - i2^{k+2}.x] \end{aligned}$$

By Theorem 2.8,

$$Q^{n-1}(x).R = \begin{bmatrix} G_{j,k,n+1}(x) & G_{j,k,n}(x) \\ G_{j,k,n}(x) & G_{j,k,n-1}(x) \end{bmatrix}$$

We get the determinants of the matrices

$$\begin{aligned} G_{j,k,n-1}(x).G_{j,k,n+1}(x) - G_{j,k,n}^2(x) &= \det(Q^{n-1}(x).R) \\ &= \det(Q^{n-1}(x)).\det(R) \end{aligned}$$

$$= (-2)^k \cdot x^{n-1} \cdot \frac{1}{2} [(8x)^2 + (2 + 3 \cdot 2^{k-1})x + 2 - i - i 2^{k+2} \cdot x]$$

$$= (-1)^{n-1} \cdot 2^{n-2} \cdot x^{n-1} [(8x)^2 + (2 + 3 \cdot 2^{k-1})x + 2 - i - i 2^{k+2} \cdot x]$$

Theorem 2.11. The sums of the Generalized Gaussian Jacobsthal Polynomial and Generalized Gaussian Jacobsthal Lucas Polynomial are given as:

$$(i) \sum_{m=0}^n GJ_{k,m}(x) = \frac{1}{2^k \cdot x} [GJ_{k,n+2}(x) - 1];$$

$$(ii) \sum_{m=0}^n Gj_{k,m}(x) = \frac{1}{2^k \cdot x} [Gj_{k,n+2}(x) - (1 + 2xi)].$$

Proof: (i) For $n \geq 1$, we have

$$GJ_{k,n+1}(x) = GJ_{k,n}(x) + 2^k \cdot x \cdot GJ_{k,n-1}(x),$$

$$GJ_{k,n-1}(x) = \frac{1}{2^k \cdot x} [GJ_{k,n+1}(x) - GJ_{k,n}(x)].$$

For the last equation

$$GJ_{k,0}(x) = \frac{1}{2^k \cdot x} [GJ_{k,2}(x) - GJ_{k,1}(x)],$$

$$GJ_{k,1}(x) = \frac{1}{2^k \cdot x} [GJ_{k,3}(x) - GJ_{k,2}(x)],$$

$$GJ_{k,2}(x) = \frac{1}{2^k \cdot x} [GJ_{k,4}(x) - GJ_{k,3}(x)],$$

$$\vdots$$

$$GJ_{k,n-1}(x) = \frac{1}{2^k \cdot x} [GJ_{k,n+1}(x) - GJ_{k,n}(x)],$$

$$GJ_{k,n}(x) = \frac{1}{2^k \cdot x} [GJ_{k,n+2}(x) - GJ_{k,n+1}(x)].$$

On adding all of the above equation, can get

$$\sum_{m=0}^n GJ_{k,m}(x) = \frac{1}{2^k \cdot x} [GJ_{k,n+2}(x) - GJ_{k,1}(x)]$$

$$= \frac{1}{2^k \cdot x} [GJ_{k,n+2}(x) - 1].$$

(ii) For $n \geq 1$, we have

$$Gj_{k,n+1}(x) = Gj_{k,n}(x) + 2^k \cdot x \cdot Gj_{k,n-1}(x);$$

$$Gj_{k,n-1}(x) = \frac{1}{2^k \cdot x} [Gj_{k,n+1}(x) - Gj_{k,n}(x)].$$

For the last equation

$$Gj_{k,0}(x) = \frac{1}{2^k \cdot x} [Gj_{k,2}(x) - Gj_{k,1}(x)],$$

$$Gj_{k,1}(x) = \frac{1}{2^k \cdot x} [Gj_{k,3}(x) - Gj_{k,2}(x)],$$

$$Gj_{k,2}(x) = \frac{1}{2^k \cdot x} [Gj_{k,4}(x) - Gj_{k,3}(x)],$$

$$\vdots$$

$$Gj_{k,n-1}(x) = \frac{1}{2^k \cdot x} [Gj_{k,n+1}(x) - Gj_{k,n}(x)],$$

$$Gj_{k,n}(x) = \frac{1}{2^k \cdot x} [Gj_{k,n+2}(x) - Gj_{k,n+1}(x)].$$

On adding all of the above equation, can get

$$\sum_{m=0}^n Gj_{k,m}(x) = \frac{1}{2^k \cdot x} [Gj_{k,n+2}(x) - Gj_{k,1}(x)],$$

$$= \frac{1}{2^k \cdot x} [Gj_{k,n+2}(x) - (1 + 2xi)].$$

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