An Application of a Delay CSOH Model on the Coconut Farm

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Abstract: In this paper, we introduce the mathematical model that represents the quantity and population dynamics on the coconut farm. The model encompasses the number of coconuts and population of squirrels, barn owls, and squirrel hunters. We study the fundamental properties of the model that include positivity, boundedness, and equilibrium points. We also investigate the effect of the time delay on the stability of the equilibrium points. The results of the analysis show that when the time delay reaches its critical value, the interior equilibrium point lost its stability, and there occurs the Hopf bifurcation.

Index Terms: Samut Songkhram, Thailand, squirrels, coconuts, barn owls, squirrel hunters, CSOH model, time delay.

I. INTRODUCTION

Samut Songkhram, a province of Thailand, where coconut trees are grown extensively, has encountered the problem of squirrels eating the coconuts of farmers. The invasion of squirrels leads to a significant reduction of coconut yields and income of the farmers. Although the coconut farmers can prevent their coconuts from squirrels by trapping and hunting practice, they still searching for alternative practices to control squirrels to a particular density that allow them to have more stable or predictable coconut outputs and income. To identify controlling strategies, mathematical modeling becomes a popular technique. The models that are constructed and studied are often based on the concept of prey-predator relationship proposed in 1931 by Lotka-Volterra [1]. The works in this area could be found in, e.g., [2]-[6]. These kinds of the model are studied and investigate for the stability of the solutions and also the conditions for bifurcation. This bifurcation can be emerged by various parameters. However, the time delay is a parameter that is intensively studied for the bifurcation phenomenon purpose. The works that were conducted to investigate the effect of the time delay on the stability of solution could be found in, e.g., [7] which analyze the effect of time delay in the following model:

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$$\frac{dS}{dt} = rS \left(1 - \frac{S+I}{k} \right) - \theta IS,$$

$$\frac{dI}{dt} = \theta IS - \mu I - m I y_m,$$

$$\frac{dy_i}{dt} = \alpha y_m - \alpha y_m(t-\tau) e^{-r\tau} - \gamma y_i,$$

$$\frac{dy_m}{dt} = \alpha y_m(t-\tau) e^{-r\tau} - \beta y_m^2 + \pi m I y_m,$$

where $S(t)$ and $I(t)$ denote the population of susceptible prey and infected prey respectively. $y_i(t)$ is immature predator population and $y_m(t)$ represents mature predator population. Here a constant τ represent the time delay in transferring to a new stage. An application of time delay to explain the effect of gestation time delay can be found in [8] which integrates the time delay in the following model:

$$\frac{dN(t)}{dt} = (a + \xi_1(t) - bN(t))N(t) - \frac{cN(t)P(t)}{mP(t)+1},$$

$$\frac{dP(t)}{dt} = \left(-d + \xi_2(t) + \frac{kN(t-\tau)}{mP(t-\tau)+1} \right) P(t),$$

where $N(t)$ and $P(t)$ describe the prey and predator population. τ is the gestation period or reaction time of the predators. Alternatively, time delay in convert food to the growth rate can be found in [9] which present the time delay in the following model:

$$\frac{dN(t)}{dT} = RN(t) \left(1 - \frac{N(t-\tau)}{K} \right) - \frac{CN(t)P(t)}{A+N(t)} - \frac{FEN}{m_1E+m_2N},$$

$$\frac{dP(t)}{dT} = \frac{C_1N(t)P(t)}{A_1+N(t)} - D_1P(t) - \frac{C_2P(t)Q(t)}{A_2+P(t)+B_1Q(t)},$$

$$\frac{dQ(t)}{dT} = D_2Q(t)^2 - \frac{C_3Q(t)^2}{A_3+P(t)},$$

where $N(t)$, $P(t)$ and $Q(t)$ are the prey, middle-level predator and top-level predator population respectively. Here τ denote time delay to convert the food into growth of the pray. In addition, the model that consists of two time delays could be found in [10] which study the effect of time delay in the following model:

$$\frac{dx}{dt} = x \left(r_1 - b_1 x(t-\tau_1) \right) - \frac{a_1 xy}{x+k_1},$$

$$\frac{dy}{dt} = y \left(r_2 - \frac{a_2 y(t-\tau_2)}{x(t-\tau_2)+k_2} \right),$$

where $x(t)$ and $y(t)$ denote the population densities of the prey and the predator. τ_1 is the feedback delay of the prey to the growth of the species itself. τ_2 is the feedback delay of the predator.

Other applications of two time delay model could be found in, e.g., [11]-[14].

Motivate by the squirrel situation on the coconut farm in Samut Songkhram and the effect of the time delay on the stability of solutions, we thus propose the delay CSOH model to explain the effect of time delay on the stability of the squirrel density and especially the coconut outputs. This paper is organized as follows. In Section 2, we explain our model. We analyze the basic properties of the model in Section 3 and Section 4. The conditions for stability and Hopf bifurcation are derived in Section 5. Finally, the numerical simulations and policy discussion are conducted and discussed in Section 6.

II. CSOH MODEL FORMULATION

The interaction between coconut yields, squirrels, barn owls, and squirrel hunters within the coconut farm can be represented as follow:

$$\begin{aligned} \frac{dC(t)}{dt} &= \beta C(t) \left(1 - \frac{C(t)}{K}\right) - \gamma C(t)S(t) - \mu C(t), \\ \frac{dS(t)}{dt} &= \eta S(t) \left(1 - \frac{S(t)}{\omega C(t) + Q}\right) + \lambda C(t - \tau_1)S(t) \\ &\quad - \delta S(t)O(t) - \sigma S(t)H(t), \\ \frac{dO(t)}{dt} &= \xi O(t) \left(1 - \frac{O(t)}{\kappa S(t) + B}\right) + \zeta S(t - \tau_2)O(t), \\ \frac{dH(t)}{dt} &= cS(t) - dH(t), \end{aligned} \quad (1)$$

with the following history

$$\begin{aligned} [C(t), S(t), O(t), H(t)]^T &= [C(0), S(0), O(0), H(0)]^T \\ , \max\{\tau_1, \tau_2\} &\leq t \leq 0. \end{aligned} \quad (2)$$

where $C(t)$, $S(t)$, $O(t)$ and $H(t)$ denote respectively, the number of coconut yields, the densities of squirrels, barn owls, and squirrel hunters at time t . β , γ , μ , η , ω , λ , δ , σ , ξ , κ , ζ , c , and d are positive constants. τ_1 and τ_2 are non-negative constants which represent the time delay of coconut convert to new squirrel biomass and time delay of squirrel convert to new barn owls biomass respectively.

III. NON-NEGATIVITY AND BOUNDEDNESS OF SOLUTIONS

The following theorems ensure the positivity and boundedness of system (1).

Theorem: All solutions of (1) with positive initial conditions are positive and ultimately bounded for $\forall t \geq 0$.

Proof: [15], [16] From the first equation of (1), it provides

$$\frac{dC(t)}{C(t)} = \left(\beta \left(1 - \frac{C(t)}{K}\right) - \gamma S - \mu \right) dt,$$

which gives

$$C(t) = C(0)e^{\int_0^t \left(\beta \left(1 - \frac{C(s)}{K}\right) - \gamma S(s) - \mu \right) ds} > 0.$$

Similarly, from the second and third equation of (1) we can obtain

$$S(t) = S(0)e^{\int_0^t \left(\eta \left(1 - \frac{S(s)}{C(s)\omega + Q}\right) + \lambda C(s - \tau_1) - \delta O(s) - \sigma \right) ds} > 0,$$

$$O(t) = O(0)e^{\int_0^t \left(\xi \left(1 - \frac{O(s)}{S(s)\kappa + B}\right) + \zeta S(s - \tau_2) \right) ds} > 0.$$

Therefore $C(t) > 0$, $S(t) > 0$, and $O(t) > 0$, $\forall t \in [0, \infty)$.

To prove $H(t) > 0$ on $[0, \infty)$, suppose that there exists $t_H > 0$ such that $H(t_H) = 0$, $S(t_H) \geq 0$ and $\frac{d}{dt}H(t_H) < 0$.

From the fourth equation of (1) we have.

$$\frac{d}{dt}H(t_H) = cS(t_H) - dH(t_H),$$

that provides

$$\frac{d}{dt}H(t_H) = cS(t_H) > 0,$$

which is contradicted. Therefore $H(t) > 0$, $\forall t \in [0, \infty)$.

Now, we show that all solutions of (1) are uniformly ultimately bounded. From the first equation of (1), it provides

$$\frac{d}{dt}C(t) \leq \beta C(t) \left(1 - \frac{C(t)}{K}\right) - \mu C(t),$$

which gives

$$C(t) \leq \max \left\{ \frac{(\beta - \mu)K}{\beta}, C_0 \right\} := M_1,$$

and

$$\frac{d}{dt}C(t) \leq \beta M_1.$$

Also applying this process to second equation of (1), we have

$$\frac{d}{dt}S(t) \leq \eta S(t) \left(1 - \frac{S(t)}{C\omega + Q}\right) + \lambda C(t - \tau_1)S(t),$$

$$S(t) \leq \max \left\{ \frac{M_1^2 \lambda \omega + (Q\lambda + \eta\omega)M_1 + Q\eta}{\eta}, S_0 \right\} := M_2,$$

$$\frac{d}{dt}S(t) \leq M_1 M_2 \lambda + M_2 \eta.$$

Similarly, from the rest of the equations of (1), we obtain

$$O(t) \leq \max \left\{ \frac{M_2^2 \zeta \kappa + (B\zeta + \kappa\xi)M_2 + B\xi}{\xi}, O_0 \right\} := M_3,$$

$$\frac{d}{dt}O(t) \leq M_2 M_3 \zeta + M_3 \xi,$$

and

$$H(t) \leq \frac{M_2 c}{d} := M_4,$$

$$\frac{d}{dt}H(t) \leq cM_2.$$

The proof is complete.

IV. EQUILIBRIUM POINTS: EXISTENCE AND ANALYSIS

To find equilibrium points of system (1), we set the derivative equal to zero and then solve for C^* , S^* , O^* , and H^* . It is easily to find that the first five equilibrium points of system (1) are as follow:

$$E_1 = (0, 0, 0, 0),$$

$$E_2 = (C^*, 0, 0, 0), \text{ where } C^* = \frac{(\beta - \mu)K}{\beta}, \beta > \mu,$$

$$E_3 = (0, S^*, 0, H^*), \text{ where } S^* = \frac{\eta Q d}{Q c \sigma + d \eta}, H^* = \frac{\eta Q c}{Q c \sigma + d \eta},$$

$$E_4 = (0, 0, O^*, 0), \text{ where } O^* = B,$$

$$E_5 = (C^*, 0, O^*, 0), \text{ where } C^* = \frac{K(\beta - \mu)}{\beta}, O^* = B.$$

To solve for the rest of equilibrium points, i.e., $E_6 = (C^*, S^*, 0, H^*)$, $E_7 = (0, S^*, O^*, H^*)$, and $E_8 = (C^*, S^*, O^*, H^*)$ which are characterized by 3 and 4 systems of equation, we begin by writing C , O , and H as functions of S and then substitute them into the S equation so that we can solve for S .

For E_6 which has $O=0$, we set the derivatives equal zero in the first, second, and fourth equation of (1) and obtain

$$0 = \beta C^* \left(1 - \frac{C^*}{K} \right) - \gamma C^* S^* - \mu C^*, \quad (3)$$

$$0 = \eta S^* \left(1 - \frac{S^*}{C^* \omega + Q} \right) + \lambda C^* S^* - \sigma S^* H^*, \quad (4)$$

$$0 = c S^* - d H^*. \quad (5)$$

Therefore, solving for H^* and C^* as functions of S^* , we obtain

$$C^* = \frac{K(\beta - S^* \gamma - \mu)}{\beta},$$

$$H^* = \frac{c S^*}{d}.$$

Then substituting these formulas into (4), we find

$$A_1 S^{*2} + A_2 S^* + A_3 = 0, \quad (6)$$

where

$$A_1 = K^2 d \gamma^2 \lambda \omega + K \beta c \gamma \omega \sigma,$$

$$A_2 = 2K^2 d \gamma \lambda \mu \omega + K \beta c \mu \omega \sigma - (2K^2 \beta d \gamma \lambda \omega + K Q \beta d \gamma \lambda + K \beta^2 c \omega \sigma + K \beta d \eta \gamma \omega + Q \beta^2 c \sigma + \beta^2 d \eta),$$

$$A_3 = K^2 \beta^2 d \lambda \omega + K^2 d \lambda \mu^2 \omega + K Q \beta^2 d \lambda + K \beta^2 d \eta \omega + Q \beta^2 d \eta - (2K^2 \beta d \lambda \mu \omega + K Q \beta d \lambda \mu - K \beta d \eta \mu \omega).$$

$$\text{The } S \text{ solution can be found from } S^* = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1 A_3}}{2A_1}.$$

Since A_1 is always positive, thus if $A_2, A_3 > 0$, S does not

exist. If $A_2 < 0, A_3 > 0$, two S^* exists when $A_2^2 > 4A_1 A_3$.

If $A_2 A_3 < 0$, unique S^* exists.

Repeating the same process, we obtain S for E_7 and E_8 respectively as below

$$A_1 S^{*2} + A_2 S^* + A_3 = 0, \quad (7)$$

where $A_1 = Q \zeta d \delta \kappa$, $A_2 = B Q \zeta d \delta + Q d \delta \kappa \xi + Q c \sigma \xi + d \eta \xi$, and $A_3 = B Q d \delta \xi - \eta \xi d Q$.

and

$$A_1 S^{*2} + A_2 S^* + A_3 = 0, \quad (8)$$

where

$$A_1 = B K \zeta \beta d \delta \gamma \omega + K^2 d \gamma^2 \lambda \omega \xi + K \zeta \beta d \delta \kappa \mu \omega + K \beta d \delta \gamma \kappa \omega \xi + K \beta c \gamma \omega \sigma \xi - (K \zeta \beta^2 d \delta \kappa \omega + Q \zeta \beta^2 d \delta \kappa),$$

$$A_2 = K \beta c \mu \omega \sigma \xi + B K \zeta \beta d \delta \mu \omega + K \beta d \delta \kappa \mu \omega \xi + 2K^2 \gamma d \lambda \mu \omega \xi + B K \gamma \beta d \delta \omega \xi - (B K \zeta \beta^2 d \delta \omega + K \gamma \beta d \eta \omega \xi + Q \beta^2 d \delta \kappa \xi + 2K^2 \gamma \beta d \lambda \omega \xi + K \beta^2 d \delta \kappa \omega \xi + Q \beta^2 c \sigma \xi + K \beta^2 c \omega \sigma \xi + \beta^2 d \eta \xi + K \gamma Q \beta d \lambda \xi + B Q \zeta \beta^2 d \delta),$$

$$A_3 = B K \beta d \delta \mu \omega \xi + K^2 \beta^2 d \lambda \omega \xi + K^2 d \lambda \mu^2 \omega \xi + K Q \beta^2 d \lambda \xi + K \beta^2 d \eta \omega \xi + Q \beta^2 d \eta \xi - (B K \beta^2 d \delta \omega \xi + 2K^2 \beta d \lambda \mu \omega \xi + B Q \beta^2 d \delta \xi + K Q \beta d \lambda \mu \xi + K \beta d \eta \mu \omega \xi).$$

Similarly E_7 and E_8 will exist if there exist positive solutions of S^* in (7) or (8). Therefore we have the following equilibrium points:

$$E_6 = (C^*, S^*, 0, H^*), \text{ where } C^* = \frac{K(\beta - S^* \gamma - \mu)}{\beta},$$

$$H^* = \frac{c S^*}{d}, \beta > (S^* \gamma + \mu),$$

$$E_7 = (0, S^*, O^*, H^*), \text{ where } O^* = \frac{\zeta \kappa S^{*2} + (B \zeta + \kappa \xi) S^* + B \xi}{\xi},$$

$$H^* = \frac{S^* c}{d},$$

$$E_8 = (C^*, S^*, O^*, H^*), \text{ where } C^* = \frac{K(\beta - S^* \gamma - \mu)}{\beta},$$

$$O^* = \frac{\zeta \kappa S^{*2} + (B \zeta + \kappa \xi) S^* + B \xi}{\xi}, H^* = \frac{S^* c}{d}.$$

V. LOCAL STABILITY AND HOPF BIFURCATION

As a first example, we will consider the system without barn owls to make adjustment about the squirrel hunters capacity in controlling squirrels when there do not exist barn owls and also to consider the sensitivity of solutions on changing the value of time delay τ , i.e., we consider the system with only one time delay τ . Thus system (1) becomes

$$\begin{aligned}\frac{dC(t)}{dt} &= \beta C(t) \left(1 - \frac{C(t)}{K}\right) - \gamma C(t) S(t) - \mu C(t), \\ \frac{dS(t)}{dt} &= \eta S(t) \left(1 - \frac{S(t)}{\omega C(t) + Q}\right) + \lambda C(t - \tau) S(t) - \sigma S(t) H(t), \\ \frac{dH(t)}{dt} &= c S(t) - d H(t),\end{aligned}\quad (9)$$

To meet the biological sense, the coexist equilibrium point $E_{80} = (C^*, S^*, H^*)$ of (9) will be analyzed. Thus the Jacobian matrix of (9) is given by

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned}a_{11} &= \beta \left(1 - \frac{C^*}{K}\right) - \frac{\beta C^*}{K} - \lambda S^* - \mu, \\ a_{12} &= -\lambda C^*, a_{21} = \frac{\eta S^{*2} \omega}{(C^* \omega + Q)^2} + e^{-\lambda \tau} \lambda S^*, \\ a_{22} &= \eta \left(1 - \frac{S^*}{C^* \omega + Q}\right) - \frac{\eta S^*}{C^* \omega + Q} + \lambda C^* - \sigma H^*, \\ a_{23} &= -\sigma S^*, a_{32} = c, a_{33} = -d.\end{aligned}$$

The characteristic equation of (10) is

$$V_1 \Lambda^2 + V_2 \Lambda + V_3 + V_4 e^{-\Lambda \tau} = 0, \quad (11)$$

where

$$\begin{aligned}V_1 &= \sigma H^* + \gamma S^* + d + \mu (2a_{11}\beta + 2a_{21}\eta - \lambda C^* - \beta - \eta), \\ V_2 &= \left(\gamma a_2^2 \eta \omega C^* + 2\gamma a_2 \eta S^* - \gamma \lambda C^* S^*\right) + (4a_{11}a_{21}\beta \eta - 2a_{21}\beta \eta) \\ &\quad + (d\sigma H^* + \mu\sigma H^* - \beta\sigma H^*) + (\gamma d S^* + c\sigma S^* - \gamma \eta S^*) \\ &\quad + (2a_{11}\beta d - 2a_{11}\beta \eta) + (\beta \lambda C^* 2a_{21}\eta - d\eta - d\lambda C^*) + (2a_{21}\eta \mu - \eta \mu) \\ &\quad + (\beta \eta + d\mu - \beta d) + (\gamma S^* H^* \sigma + 2a_{11}\beta \sigma H^* - 2a_{11}\beta \lambda C^* - \mu \lambda C^*), \\ V_3 &= (\gamma c \sigma S^* - \gamma d \eta S^*) + (c\sigma \mu S^* + \beta d \lambda C^* - \mu d \lambda C^* - c\sigma \beta S^*) \\ &\quad + (2a_{21}d\eta \mu + d\eta \beta - d\eta \mu - 2a_{21}d\eta \beta) + (d\sigma \mu H^* + 4a_{11}a_{21}\beta d\eta - d\sigma \beta H^*) \\ &\quad + (\gamma a_2^2 d\eta \omega C^* + 2a_{11}\beta c\sigma S^* - 2a_{11}\beta d\eta - \gamma d \lambda C^* S^*) \\ &\quad + (2a_{11}\beta d\sigma H^* + 2\gamma a_{21}d\eta S^* + \gamma d \sigma S^* H^* - 2a_{11}\beta d \lambda C^*), \\ V_4 &= \gamma d \lambda C^* S^* + \gamma \lambda^2 C^* S^*.\end{aligned}$$

By choosing τ as the bifurcation parameter, we study the stability of the equilibrium and the Hopf bifurcation of system (9) by separating it into two cases.

For case 1: $\tau=0$. The characteristic equation (11) reduces to a third order polynomial equation,

$$\Lambda^3 + V_1 \Lambda^2 + V_2 \Lambda + G_1 = 0, \quad (12)$$

where $G_1 = V_3 + V_4$.

According to Routh-Hurwitz criterion, if

$$V_i > 0, \quad i=1,2,3,4 \quad \text{and} \quad V_1 V_2 - G_1 > 0, \quad (13)$$

then all roots have negative real parts.

Thus, the equilibrium $E_{80} = (C^*, S^*, H^*)$ of system (9) is locally asymptotically stable if the conditions in (13) are satisfied.

For case 2: $\tau > 0$, the characteristic equation (11) becomes

$$\Lambda^3 + V_1 \Lambda^2 + V_2 \Lambda + V_3 + V_4 e^{-\Lambda \tau} = 0. \quad (14)$$

We now look for a possible Hopf bifurcation which could occur if a purely imaginary complex conjugate pair of eigenvalues exists. Defining $\lambda = \pm i\omega$, where ω is real and positive, and separating real and imaginary parts of the characteristic equation (14), we have

$$\begin{aligned}-V_1 \omega^2 + V_4 \cos(\omega \tau) + V_3 &= 0, \\ -\omega^3 - \sin(\omega \tau) V_4 + V_2 \omega &= 0.\end{aligned}\quad (15)$$

Squaring both sides of the real and imaginary parts in (15) and adding, we obtain a formula for ω as follows.

$$\omega^6 + G_2 \omega^4 + G_3 \omega^2 + G_4 = 0, \quad (16)$$

where $G_2 = V_1^2 - 2V_2$, $G_3 = V_2^2 - 2V_1 V_3$, $G_4 = V_3^2 - V_4^2$.

A Hopf bifurcation can only exist if ω^2 is real and positive. By Descartes Rule of Signs, if one of the following sets of conditions [17]:

$$G_2 > 0, \quad G_4 < 0,$$

$$G_3 < 0, \quad G_4 < 0,$$

hold. Then (16) has at least one positive real root ω^2 .

Substituting ω^* into (15), we obtain

$$\tan(\omega^* \tau^c) = \frac{-\omega^{*3} + V_2 \omega^*}{V_1 \omega^{*2} - V_3},$$

which leads to

$$\tau_h^c = \frac{1}{\omega^*} \arctan \frac{-\omega^{*3} + V_2 \omega^*}{V_1 \omega^{*2} - V_3} + \frac{1}{\omega^*} 2h\pi, \quad h=0,1,2.$$

Next, we will prove $\Lambda(\tau)$ meets the transversality conditions; taking the derivative of system (14) with respect to τ , one derives that

$$\left[\frac{d\Lambda}{d\tau} \right]^{-1} = \frac{-2V_1 \Lambda - 3\Lambda^2 - V_2}{\Lambda(V_1 \Lambda^2 + \Lambda^3 + V_2 \Lambda + V_3)} \cdot \frac{\tau}{\Lambda}.$$

So, if

$$\operatorname{Re} \left[\frac{d\Lambda}{d\tau} \right]_{\Lambda=i\omega}^{-1} = \frac{3\omega^5 + A_1 \omega^3 + A_2 \omega}{\omega^7 + A_3 \omega^5 + A_4 \omega^3 - A_5 \omega} > 0,$$

where $A_1 = -2V_1^2 - 4V_2$, $A_2 = 2V_1 V_3 + V_2^2$, $A_3 = -V_1^2 - 2V_2$,

$A_4 = 2V_1 V_3 + V_2^2$, $A_5 = V_3^2$,

holds, then the root Λ of (14) crosses the imaginary axis at $\tau = \tau_h^c$ from left to right.

These two conditions prove that a Hopf bifurcation could exist at τ_h^c . However, to prove that the point is actually a Hopf bifurcation it would also be necessary to prove that the $E_{80}=(C^*, S^*, H^*)$ equilibrium is stable for all $\tau < \tau_h^c$.

We now consider the system with barn owl (1) with the non-zero equilibrium E_8 . By linearization, the corresponding characteristic equation of system (1) becomes

$$\Lambda^4 + A_1\Lambda^3 + A_2\Lambda^2 + A_3\Lambda + A_4 + (A_5\Lambda^2 + A_6\Lambda - A_7)e^{-\lambda\tau_1} + (A_8\Lambda^2 + A_9\Lambda + A_{10})e^{-\lambda\tau_2} = 0, \quad (17)$$

where

$$\begin{aligned} A_1 &= 2a_3\xi + b_1 + b_2 + d - (b_3 + b_4 + \beta), \\ A_2 &= a_2^2\eta\omega C^* + a_3^2\delta\kappa\xi S^* + c\sigma S^* + 2a_3b_1\xi + 2a_3b_2\xi \\ &\quad + 2a_3d\xi + b_1b_2 + b_2d_4 + b_3\beta + b_1d + b_4\beta \\ &\quad - (2a_3b_3\xi + 2a_3\beta\xi + b_1b_3 + b_1b_4 + b_2b_4 + b_2\beta + b_3d + b_4d + \beta d), \\ A_3 &= 2\gamma a_2^2a_3\eta\omega\xi C^* + db_2b_1 + 2a_3b_1b_2\xi + 2a_3b_3\beta\xi + 2da_3b_1\xi \\ &\quad + 2da_3b_2\xi + b_1c\sigma S^* + db_3b_4 + db_3\beta + db_4\beta + b_1b_3b_4 + b_2b_4\beta \\ &\quad + 2S^*a_3c\sigma\xi + da_3^2\delta\kappa\xi S^* + d\gamma a_2^2\eta\omega C^* + a_3^2b_1\delta\kappa\xi S^* (b_1b_2b_4 \\ &\quad + db_3b_1 + db_1b_4 + db_2b_4 + db_2\beta + b_3b_4\beta + \gamma a_2^2b_4\eta\omega C^* + a_3^2\beta\delta\kappa\xi S^* \\ &\quad + 2a_3b_1b_3\xi + 2a_3b_2\beta\xi + 2da_3b_3\xi + 2da_3\beta\xi + b_4c\sigma S^* + \beta c\sigma S^*), \\ A_4 &= +c\sigma b_4\beta S^* + 2da_3b_3\beta\xi + da_3^2b_1\delta\kappa\xi S^* + db_1b_3b_4 + db_2b_4\beta \\ &\quad + 2d\gamma a_2^2a_3\eta\omega\xi C^* + 2da_3b_1b_2\xi + 2c\sigma a_3b_1\xi S^* - (2da_3b_1b_3\xi \\ &\quad + 2c\sigma a_3\beta\xi S^* + da_3^2\beta\delta\kappa\xi S^* + 2da_3b_2\beta\xi + d\gamma a_2^2b_4\eta\omega C^* \\ &\quad + db_1b_2b_4 + c\sigma b_1b_4 S^* + db_3b_4\beta), \\ A_5 &= \gamma\lambda C^* S^* \Lambda^2, \quad A_6 = 2a_3\gamma\lambda\xi C^* S^* + d\gamma\lambda C^* S^* - b_4\gamma\lambda C^* S^*, \\ A_7 &= 2d\gamma a_3\lambda\xi C^* S^* - d\gamma b_4\lambda C^* S^*, \quad A_8 = \delta\xi O^* S^*, \\ A_9 &= b_1\delta\xi O^* S^* + d\delta\xi O^* S^* - \beta\delta\xi O^* S^*, \\ A_{10} &= db_1\delta\xi O^* S^* - d\beta\delta\xi O^* S^*, \quad b_1 = \gamma S^* + a_1\beta + \mu, \\ b_2 &= \sigma H^* + \delta O^* + na_2, \quad b_3 = \lambda C^* + \eta, \quad b_4 = \xi S^* + \xi. \end{aligned}$$

Next, we consider the local stability of positive equilibrium $E_8=(C^*, S^*, O^*, H^*)$ and the Hopf bifurcation of system (1) for different combinations of τ_1 and τ_2 .

Case 1: $\tau_1=\tau_2=0$. (17) becomes

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0, \quad (18)$$

where $B_1=A_1$, $B_2=A_2+A_5+A_8$, $B_3=A_3+A_6+A_9$, $B_4=A_{10}+A_4+A_7$.

By Routh-Hurwitz criteria, $E_8=(C^*, S^*, O^*, H^*)$ is locally asymptotically stable if the following conditions $B_1>0, B_1B_2-B_3>0, B_3(B_1B_2-B_3)-B_1^2B_4>0, B_4>0$, hold.

Case 2: $\tau_1>0, \tau_2=0$. On substituting $\tau_2=0$, (17) becomes

$$\Lambda^4 + A_1\Lambda^3 + A_2\Lambda^2 + A_3\Lambda + A_4 + V_1e^{-\Lambda\tau_1} + A_8\Lambda^2 + A_9\Lambda + A_{10} = 0. \quad (19)$$

Then setting $\lambda=\pm i\omega_1$ and separating real and imaginary parts, we obtain

$$\begin{aligned} A_6 \sin(\omega_1\tau_1)\omega_1 - A_5 \cos(\omega_1\tau_1)\omega_1^2 + \omega_1^4 - A_2\omega_1^2 - A_8\omega_1^2 \\ + A_7 \cos(\omega_1\tau_1) + A_{10} + A_4 = 0, \\ -A_1\omega_1^3 + A_5 \sin(\omega_1\tau_1)\omega_1^2 + A_6 \cos(\omega_1\tau_1)\omega_1 + A_3\omega_1 \\ - A_7 \sin(\omega_1\tau_1) + A_9\omega_1 = 0. \end{aligned} \quad (20)$$

Squaring both sides of the real and imaginary parts and adding, we have the equation for ω_1 as

$$\omega_1^8 + B_7\omega_1^6 + B_8\omega_1^4 + B_9\omega_1^2 + B_{10} = 0, \quad (21)$$

where

$$\begin{aligned} B_7 &= A_1^2 - 2(A_2 + A_8), \\ B_8 &= A_2^2 + A_8^2 + 2(A_{10} + A_4 + A_2A_8) - 2(A_1A_3 + A_1A_9) - A_5^2, \\ B_9 &= A_3^2 + A_9^2 + 2(A_3A_9 + A_5A_7) - 2(A_{10}A_2 + A_{10}A_8 + A_2A_4 + A_4A_8) - A_6^2, \\ B_{10} &= A_{10}^2 + 2A_{10}A_4 + A_4^2 - A_7^2. \end{aligned}$$

If one set of the following sets of conditions

$$\begin{aligned} B_7 > 0, \quad B_8 > 0, \quad B_{10} < 0, \\ B_7 > 0, \quad B_9 < 0, \quad B_{10} < 0, \\ B_8 < 0, \quad B_9 < 0, \quad B_{10} < 0, \end{aligned}$$

holds, then (21) has at least one positive root ω_1^* .

Substituting ω_1^* into (20), we obtain

$$\tau_{1h}^c = \frac{1}{\omega_1^*} \arccos \frac{D_1\omega_1^{*6} + D_2\omega_1^{*4} + D_3\omega_1^{*2} + D_4}{D_5\omega_1^{*4} + D_6\omega_1^{*2} + D_7} + \frac{2\pi h}{\omega_1^*}$$

where $h=0,1,2,\dots$ and

$$\begin{aligned} D_1 &= A_5, \quad D_2 = A_1A_6 - A_2A_5 - A_5A_8 - A_7, \\ D_3 &= A_{10}A_5 + A_2A_7 + A_4A_5 + A_7A_8 - A_3A_6 - A_6A_9, \\ D_4 &= -A_{10}A_7 - A_4A_7, \quad D_5 = A_5^2, \quad D_6 = A_6^2 - 2A_5A_7, \quad D_7 = A_7^2. \end{aligned}$$

Differentiating Λ w.r.t. τ_1 and using (19), we obtain

$$\left[\frac{d\Lambda}{d\tau_1} \right]^{-1} = \frac{2A_5\Lambda + A_6}{(A_5\Lambda^2 + A_6\Lambda + A_7)\Lambda} - \frac{3A_1\Lambda^2 + 4\Lambda^3 + 2A_2\Lambda + 2A_8\Lambda + A_3 + A_9}{\Lambda(A_1\Lambda^3 + \Lambda^4 + A_2\Lambda^2 + A_8\Lambda^2 + A_3\Lambda + A_9\Lambda + A_{10} + A_4)} \frac{\tau_1}{\Lambda},$$

which lead to

$$Re\left[\frac{d\Lambda}{d\tau_1}\right]_{\Lambda=i\omega_1^*}^{-1} = \frac{P_{2I}Q_{2I} + P_{2R}Q_{2R}}{Q_{2I}^2 + Q_{2R}^2},$$

where

$$P_{2I} = (A_1A_5 + 3A_6)\omega^4 + (A_3A_5 + A_5A_9 - (3A_1A_7 + A_2A_6 + A_6A_8))\omega^2 + A_3A_7 + A_7A_9 - (A_{10}A_6 + A_4A_6),$$

$$P_{2R} = -2A_5\omega^5 + (2A_1A_6 + 4A_7)\omega^3 + (2A_{10}A_5 + 2A_4A_5 - (2A_2A_7 + 2A_7A_8))\omega,$$

$$Q_{2I} = (A_1A_5 + A_6)\omega^6 + (-A_1A_7 - A_2A_6 - A_3A_9 - A_5A_9 - A_6A_8)\omega^4 + (A_{10}A_6 + A_3A_7 + A_4A_6 + A_7A_9)\omega^2,$$

$$Q_{2R} = -A_5\omega^7 + (A_1A_6 + A_2A_5 + A_5A_8 + A_7)\omega^5 + (-A_{10}A_5 - A_2A_7 - A_3A_6 - A_4A_9 - A_1A_8)\omega^3 + (A_{10}A_7 + A_4A_7)\omega.$$

If

$$P_{2I}Q_{2I} + P_{2R}Q_{2R} > 0,$$

holds, then the root Λ of (19) crosses the imaginary axis at $\tau_1 = \tau_1^c$ from left to right.

Case 3: $\tau_1=0, \tau_2>0$. On substituting $\tau_1=0$, (17) becomes

$$\Lambda^4 + A_1\Lambda^3 + A_2\Lambda^2 + A_3\Lambda + A_4 + A_5\Lambda^2 + A_6\Lambda + A_7 + V_2e^{-\Lambda\tau_2} = 0. \quad (22)$$

Defining $\Lambda = i\omega_2$ and separating real and imaginary parts, we have

$$\begin{aligned} & A_9 \sin(\omega_2\tau_2)\omega_2 - A_8 \cos(\omega_2\tau_2)\omega_2^2 + \omega_2^4 - A_2\omega_2^2 \\ & - A_5\omega_2^2 + A_{10} \cos(\omega_2\tau_2) + A_4 + A_7 = 0, \\ & -A_1\omega_2^3 + A_8 \sin(\omega_2\tau_2)\omega_2^2 + A_9 \cos(\omega_2\tau_2)\omega_2 \\ & - A_{10} \sin(\omega_2\tau_2) + A_3\omega_2 + A_6\omega = 0. \end{aligned} \quad (23)$$

Squaring both sides of the real and imaginary parts and adding, we obtain

$$\omega_2^8 + B_{11}\omega_2^6 + B_{12}\omega_2^4 + B_{13}\omega_2^2 + B_{14} = 0, \quad (24)$$

where

$$B_{11} = A_1^2 - (2A_2 + 2A_5), B_{12} = A_2^2 + 2A_2A_5 + A_5^2 + 2A_4 + 2A_7 - (2A_1A_3 + 2A_1A_6 + A_8^2),$$

$$B_{13} = 2A_{10}A_8 + A_3^2 + 2A_3A_6 + A_6^2 - (2A_2A_4 + 2A_2A_7 + 2A_4A_5 + 2A_5A_7 + A_9^2),$$

$$B_{14} = A_4^2 + 2A_4A_7 + A_7^2 - A_{10}^2.$$

If one set of the following sets of conditions

$$B_{11} > 0, B_{12} > 0, B_{14} < 0,$$

$$B_{11} > 0, B_{13} < 0, B_{14} < 0,$$

$$B_{12} < 0, B_{13} < 0, B_{14} < 0,$$

hold, then (24) has at least one positive root ω_2^* .

Substituting ω_2^* into (23), we obtain

$$\tau_{2h}^c = \frac{1}{\omega_2^*} \arccos \frac{D_8\omega_2^{*6} + D_9\omega_2^{*4} + D_{10}\omega_2^{*2} + D_{11}}{D_{12}\omega_2^{*4} + D_{13}\omega_2^{*2} + D_{14}} + \frac{2\pi h}{\omega_2^*}, \quad (25)$$

where

$$D_8 = A_8, D_9 = A_1A_8 - A_2A_8 - A_5A_8 - A_{10},$$

$$D_{10} = A_{10}A_2 + A_{10}A_5 + A_4A_8 + A_7A_8 - A_3A_9 - A_6A_9,$$

$$D_{11} = -A_{10}A_4 - A_{10}A_7, D_{12} = A_8^2,$$

$$D_{13} = A_9^2 - 2A_{10}A_8, D_{14} = A_{10}^2.$$

Differentiating Λ w.r.t. τ_2 and using (22), we obtain

$$\left[\frac{d\Lambda}{d\tau_2}\right]_{\Lambda=i\omega_2^*}^{-1} = \frac{2A_8\Lambda + A_9}{(A_8\Lambda^2 + A_9\Lambda + A_{10})\Lambda} - \frac{3A_1\Lambda^2 + 4\Lambda^3 + 2A_2\Lambda + 2A_5\Lambda + A_3 + A_6}{\Lambda(A_1\Lambda^3 + \Lambda^4 + A_2\Lambda^2 + A_5\Lambda^2 + A_3\Lambda + A_6\Lambda + A_4 + A_7)} \frac{\tau_2}{\Lambda},$$

which leads to

$$Re\left[\frac{d\Lambda}{d\tau_2}\right]_{\Lambda=i\omega_2^*}^{-1} = \frac{P_{3I}Q_{3I} + P_{3R}Q_{3R}}{Q_{3I}^2 + Q_{3R}^2}$$

where

$$P_{3I} = (-5A_1A_8 - 5A_9)\omega^4 + (3A_1A_{10} + 3A_2A_9 + 3A_3A_8 + 3A_5A_9 + 3A_6A_8)\omega^2 - A_{10}A_3 - A_{10}A_6 - A_4A_9 - A_7A_9,$$

$$P_{3R} = 6A_8\omega^5 + (-4A_1A_9 - 4A_2A_8 - 4A_{10})\omega^3 + (2A_{10}A_2 + 2A_{10}A_5 + 2A_3A_9 + 2A_4A_8 + 2A_6A_9 + 2A_7A_8)\omega,$$

$$Q_{3I} = (-A_1A_8 - A_9)\omega^6 + (A_1A_{10} + A_2A_8 + A_3A_9 + A_6A_8)\omega^4 + (-A_{10}A_3 - A_{10}A_6 - A_4A_9 - A_7A_9)\omega^2,$$

$$Q_{3R} = A_8\omega^7 + (-A_1A_9 - A_2A_8 - A_5A_8 - A_{10})\omega^5 + (A_{10}A_2 + A_{10}A_5 + A_3A_8 + A_6A_8 + A_1A_8)\omega^3 + (-A_{10}A_4 - A_{10}A_7)\omega.$$

If the following condition

$$P_{3I}Q_{3I} + P_{3R}Q_{3R} > 0,$$

holds, then the root Λ of (22) crosses the imaginary axis at $\tau_2 = \tau_2^c$ from left to right.

Case 4: $\tau_1 = \tau_2 = \tau > 0$. By substituting $\tau_1 = \tau_2 = \tau$, (17) becomes

$$\Lambda^4 + A_1\Lambda^3 + A_2\Lambda^2 + A_3\Lambda + A_4 + V_1e^{-\Lambda\tau} + V_2e^{-\Lambda\tau} = 0. \quad (26)$$

Defining $\Lambda = \pm i\omega_3$ and separating real and imaginary parts, we obtain

$$\begin{aligned} & A_6 \sin(\omega_3\tau)\omega_3 + A_0 \sin(\omega_3\tau)\omega_3 - A_5 \cos(\omega_3\tau)\omega_3^2 \\ & - A_8 \cos(\omega_3\tau)\omega_3^2 + \omega_3^4 - A_2\omega_2^2 + A_{10} \cos(\omega_3\tau) \\ & + A_1 \cos(\omega_3\tau) + A_4 = 0, \end{aligned}$$

$$\begin{aligned} & -A_1 \omega_2^3 + A_5 \sin(\omega_3 \tau) \omega_3^2 + A_8 \sin(\omega_3 \tau) \omega_3^2 \\ & + A_6 \cos(\omega_3 \tau) \omega_3 + A_9 \cos(\omega_3 \tau) \omega_3 - A_{10} \sin(\omega_3 \tau) \\ & + A_3 \omega_3 - A_1 \sin(\omega_3 \tau) = 0. \end{aligned}$$

Squaring both sides and adding, we get

$$\omega_3^8 + B_{13} \omega_3^6 + B_{14} \omega_3^4 + B_{15} \omega_3^2 + B_{16} = 0, \quad (28)$$

where

$$B_{13} = A_1^2 + 2A_2,$$

$$B_{14} = A_2^2 - (2A_1 A_3 + A_5^2 + 2A_5 A_8 + A_8^2 + 2A_4),$$

$$\begin{aligned} B_{15} = & 2A_{10} A_5 + 2A_{10} A_8 + A_5^2 + 2A_5 A_7 + 2A_7 A_8 \\ & - (2A_2 A_4 + A_6^2 + 2A_6 A_9 + A_9^2), \end{aligned}$$

$$B_{16} = -A_{10}^2 + A_4^2 - (2A_{10} A_7 + A_7^2).$$

If one set of the following sets of conditions

$$B_{15} > 0, B_{16} > 0, B_{18} < 0,$$

$$B_{15} > 0, B_{17} < 0, B_{18} < 0,$$

$$B_{16} < 0, B_{17} < 0, B_{18} < 0,$$

hold, then (28) has at least one positive root ω_3^* .

Substituting ω_3^* into (27), we obtain

$$\tau_{3h}^c = \frac{1}{\omega_3^*} \arccos \frac{D_{15} \omega_3^{*6} + D_{16} \omega_3^{*4} + D_{17} \omega_3^{*2} + D_{18}}{D_3 \omega_3^{*4} + D_{20} \omega_3^{*2} + D_{21}} + \frac{2\pi h}{\omega_3^*}, \quad (29)$$

where

$$D_{15} = A_5 + A_8, D_{16} = A_1 A_6 + A_1 A_9 - (A_2 A_5 + A_2 A_8 + A_{10} + A_7),$$

$$D_{17} = A_{10} A_2 + A_2 A_7 + A_4 A_5 + A_4 A_8 - (A_3 A_6 + A_3 A_9),$$

$$D_{18} = -A_{10} A_4 - A_4 A_7 D_{19} = A_5^2 + 2A_5 A_8 + A_8^2,$$

$$D_{20} = A_6^2 + 2A_6 A_9 + A_9^2 - (2A_{10} A_5 + 2A_{10} A_8 + 2A_5 A_7 + 2A_7 A_8),$$

$$D_{21} = A_{10}^2 + 2A_{10} A_7 + A_7^2.$$

Differentiating Λ w.r.t. τ_1 and using (26), we obtain

$$\begin{aligned} \left[\frac{d\Lambda}{d\tau} \right]^{-1} = & \frac{2A_5 \Lambda + 2A_8 \Lambda + A_6 + A_9}{\Lambda (A_5 \Lambda^2 + A_8 \Lambda^2 + A_6 \Lambda + A_9 \Lambda + A_{10} + A_7)} \\ & - \frac{3A_1 \Lambda^2 + 4\Lambda^3 + 2A_2 \Lambda + A_3}{\Lambda (A_1 \Lambda^3 + \Lambda^4 + A_2 \Lambda^2 + A_3 \Lambda + A_4)} \frac{\tau}{\Lambda}, \end{aligned}$$

which leads to

$$\text{Re} \left[\frac{d\Lambda}{d\tau} \right]_{\Lambda=i\omega_3^*}^{-1} = \frac{P_{4I} Q_{4I} + P_{4R} Q_{4R}}{Q_{4I}^2 + Q_{4R}^2},$$

where

$$\begin{aligned} P_{4I} = & (2A_5 + 2A_8) \omega^5 + (-2A_1 A_6 - 2A_1 A_9 - 4A_{10} - 4A_7) \omega^3 \\ & + (2A_{10} A_2 + 2A_2 A_7 - 2A_4 A_5 - 2A_4 A_8) \omega, \end{aligned}$$

$$\begin{aligned} P_{4R} = & (A_1 A_5 + A_1 A_8 + 3A_6 + 3A_9) \omega^4 + (-3A_1 A_{10} - 3A_1 A_7 - A_2 A_6 \\ & - A_2 A_9 + A_3 A_5 + A_3 A_8) \omega^2 + A_{10} A_3 + A_3 A_7 - A_4 A_6 - A_4 A_9, \end{aligned}$$

$$\begin{aligned} Q_{4I} = & (-A_5 - A_8) \omega^7 + (A_1 A_6 + A_1 A_9 + A_2 A_5 + A_2 A_8 + A_{10} + A_7) \omega^5 \\ & + (-A_{10} A_2 - A_2 A_7 - A_3 A_6 - A_3 A_9 - A_4 A_5 - A_4 A_8) \omega^3 \\ & + (A_{10} A_4 + A_4 A_7) \omega, \end{aligned}$$

$$\begin{aligned} Q_{4R} = & (A_1 A_5 - A_1 A_8 - A_6 - A_9) \omega^6 + (A_1 A_{10} + A_1 A_7 + A_2 A_6 \\ & + A_2 A_9 + A_3 A_5 + A_3 A_8) \omega^4 \\ & + (-A_{10} A_3 - A_3 A_7 - A_4 A_6 - A_4 A_9) \omega^2. \end{aligned}$$

If the following condition

$$P_{4I} Q_{4I} + P_{4R} Q_{4R} > 0,$$

holds, then the root Λ of (26) crosses the imaginary axis at $\tau = \tau_3^c$ from left to right.

Case 5: $\tau_1 \neq \tau_2, \tau_1, \tau_2 > 0$. We consider system (1) with τ_2 in its stable interval and regarding τ_1 as a parameter.

On substituting τ_1, τ_2 into (17), we obtain

$$\Lambda^4 + A_1 \Lambda^3 + A_2 \Lambda^2 + A_3 \Lambda + A_4 + V_1 e^{-\Lambda \tau_1} + V_2 e^{-\Lambda \tau_2} = 0. \quad (30)$$

Defining $\Lambda = \pm i\omega_4$ and separating real and imaginary parts, we have

$$\begin{aligned} & A_6 \sin(\omega_4 \tau_1) \omega_4 + A_0 \sin(\omega_4 \tau_2) \omega_4 - A_5 \cos(\omega_4 \tau_1) \omega_4^2 \\ & - A_8 \cos(\omega_4 \tau_2) \omega_4^2 + \omega_4^4 - A_2 \omega_4^2 + A_{10} \cos(\omega_4 \tau_2) \\ & + A_1 \cos(\omega_4 \tau_1) + A_4 = 0, \\ & -A_1 \omega_4^3 + A_5 \sin(\omega_4 \tau_1) \omega_4^2 + A_8 \sin(\omega_4 \tau_2) \omega_4^2 \\ & + A_6 \cos(\omega_4 \tau_1) \omega_4 + A_9 \cos(\omega_4 \tau_2) \omega_4 - A_{10} \sin(\omega_4 \tau_2) \\ & + A_3 \omega_4 - A_1 \sin(\omega_4 \tau_1) = 0. \end{aligned}$$

Squaring both sides and adding, we obtain

$$\begin{aligned} & \omega_4^8 + B_{17} \omega_4^6 + B_{18} \omega_4^4 + B_{19} \omega_4^2 + B_{20} \\ & + (-2A_8 \omega_4^6 + (2A_2 A_8 + 2A_{10} - 2A_1 A_9) \omega_4^4 \\ & + (2A_3 A_9 - 2A_{10} A_2 - 2A_4 A_8) \omega^2 + 2A_{10} A_4) \cos(\omega_4 \tau_2) \\ & + ((2A_9 - 2A_1 A_8) \omega_4^5 + (2A_1 A_{10} + 2A_3 A_8 - 2A_2 A_9) \omega_4^3 \\ & + (2A_4 A_9 - 2A_{10} A_3) \omega_4) \sin(\omega_4 \tau_2) = 0, \end{aligned}$$

$$\begin{aligned} \text{where } B_{19} = & A_1^2 - 2A_2, \quad B_{20} = A_2^2 - A_5^2 + A_8^2 + 2A_4 - 2A_1 A_3, \\ B_{21} = & A_3^2 + 2A_5 A_7 + A_9^2 - A_6^2 - 2A_{10} A_8 - 2A_2 A_4, \\ B_{22} = & A_{10}^2 + A_4^2 - A_7^2. \end{aligned}$$

If one set of the following sets of conditions hold:

$$\begin{aligned} B_{19} &> 0, B_{20} > 0, B_{22} < 0, \\ B_{19} &> 0, B_{21} < 0, B_{22} < 0, \\ B_{20} &< 0, B_{21} < 0, B_{22} < 0, \end{aligned}$$

then (32) has at least one positive root ω_4^* . Substituting ω_4^* into (31), we obtain

$$\tau_{4h}^c = \frac{1}{\omega_4^*} \arccos \frac{G}{D_{31}\omega_4^{*4} + D_{32}\omega_4^{*2} + D_{33}} + \frac{2\pi h}{\omega_4^*},$$

where

$$\begin{aligned} G &= D_{22}\omega_4^{*6} + D_{23}\omega_4^{*4} + D_{24}\omega_4^{*2} - D_{25} \\ &+ (D_{26}\omega_4^{*2} - D_{27}\omega_4^{*4} - D_{28}) \cos(\omega_4^* \tau_2) \\ &+ (D_{29}\omega_4^{*3} + D_{30}\omega_4^*) \sin(\omega_4^* \tau_2), D_{22} = A_5, \\ D_{23} &= (A_1A_6 - A_2A_5 - A_7), D_{24} = A_2A_7 - A_3A_6 + A_4A_5, \\ D_{25} &= A_4A_7, D_{26} = A_{10}A_5 + A_7A_8 - A_6A_9, D_{27} = A_5A_8, \\ D_{28} &= A_{10}A_7, D_{29} = A_5A_9 - A_6A_8, \\ D_{30} &= A_{10}A_6 - A_7A_9, D_{31} = A_5^2, \\ D_{32} &= A_6^2 - 2A_5A_7, D_{33} = A_7^2. \end{aligned}$$

Differentiating Λ w.r.t. τ_1 and using (30), we obtain

$$\left[\frac{d\Lambda}{d\tau_1} \right]^{-1} = \frac{2A_5\Lambda + A_6}{\Lambda(A_5\Lambda^2 + A_6\Lambda + A_7)} - \frac{G_1e^{-\Lambda\tau_2} + G_2}{(G_3)\Lambda e^{-\Lambda\tau_2} + (G_4)\Lambda},$$

$$\begin{aligned} \text{where } G_1 &= -A_8\Lambda^2\tau_2 - A_9\Lambda\tau_2 - A_{10}\tau_2 + 2A_8\Lambda + A_9, \\ G_2 &= 3A_1\Lambda^2 + 4\Lambda^3 + 2A_2\Lambda + A_3, \quad G_3 = A_8\Lambda^2 + A_9\Lambda + A_{10}, \\ G_4 &= A_1\Lambda^3 + \Lambda^4 + A_2\Lambda^2 + A_3\Lambda + A_4 \end{aligned}$$

which lead to

$$\text{Re} \left[\frac{d\Lambda}{d\tau} \right]_{\Lambda=i\omega_4}^{-1} = \frac{P_{5I}Q_{5I} + P_{5R}Q_{5R}}{Q_{5I}^2 + Q_{5R}^2},$$

Where

$$\begin{aligned} P_{5I} &= (-A_5A_8\omega_4^4\tau_2 + (A_{10}A_5\tau_2 + A_6A_9\tau_2 + A_7A_8\tau_2 \\ &+ A_5A_9 - A_6A_8)\omega_4^2 + A_7A_9 - A_{10}A_7\tau_2 \\ &- A_{10}A_6)\cos(\omega_4\tau_2) + ((A_5A_9\tau_2 + A_6A_8\tau_2)\omega_4^3 \\ &+ (2A_7A_8 - A_{10}A_6\tau_2 - A_7A_9\tau_2 - 2A_{10}A_5)\omega_4)\sin(\omega_4\tau_2) \\ &+ (A_1A_5 + 3A_6)\omega_4^4 + (A_3A_5 - 3A_1A_7 - A_2A_6)\omega_4^2 + A_3A_7 - A_4A_6, \\ P_{5R} &= ((-A_5A_9\tau_2 - A_6A_8\tau_2)\omega_4^3 + (A_{10}A_6\tau_2 + A_7A_9\tau_2 + 2A_{10}A_5 \\ &- 2A_7A_8)\omega_4)\cos(\omega_4\tau_2) + (A_5A_8\omega_4^4\tau_2 + (A_{10}A_5\tau_2 + A_6A_9\tau_2 \\ &+ A_7A_8\tau_2 + A_5A_9 - A_6A_8)\omega_4^2 + A_7A_9 - A_{10}A_7\tau_2 - A_{10}A_6)\sin(\omega_4\tau_2) \\ &- 2A_5\omega_4^5 + (2A_1A_6 + 4A_7)\omega_4^3 + (2A_4A_5 - 2A_2A_7)\omega_4, \end{aligned}$$

$$\begin{aligned} Q_{5I} &= (-A_5A_8\omega_4^5 + (A_{10}A_5 + A_7A_8)\omega_4^3 - A_{10}A_7\omega_4)\cos(\omega_4\tau_2) \\ &+ (A_5A_9\omega_4^4 - A_7A_9\omega_4^2)\sin(\omega_4\tau_2) + A_5\omega_4^7 + (-A_2A_5 - A_7)\omega_4^5 \\ &+ (A_2A_7 + A_4A_5)\omega_4^3 - A_4A_7\omega_4, \\ Q_{5R} &= ((-A_5A_9 - A_6A_8)\omega_4^4 + (A_{10}A_6 + A_7A_9)\omega_4^2)\cos(\omega_4\tau_2) \\ &+ (-A_5A_8\omega_4^5 + (A_{10}A_5 + A_6A_9 + A_7A_8)\omega_4^3 - A_{10}A_7\omega_4)\sin(\omega_4\tau_2) \\ &+ (A_1A_5 + A_6)\omega_4^6 + (-A_1A_7 - A_2A_6 - A_3A_5)\omega_4^4 \\ &+ (A_3A_7 + A_4A_6)\omega_4^2. \end{aligned}$$

Then, if

$$P_{5I}Q_{5I} + P_{5R}Q_{5R} > 0,$$

holds, the root Λ of (30) crosses the imaginary axis at $\tau_1 = \tau_4^c$ from left to right.

VI. NUMERICAL EXPERIMENT

In the numerical simulations, the following parameter values were used $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.1$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$. These values were based on discussions with farmers and officials in Samut Songkram province.

Firstly we will shows the result when we apply the parameters to (9) which present the barn owl free system, the coconut yields become extinct as shown by Fig. 1.

Fig. 1. Barn owls free system $\sigma=0.1$ with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau=0$.

Since we need to keep coconuts positive, thus we need to increase the predation rate of squirrel hunters, i.e., squirrel hunter must become more efficient in controlling squirrels. Suppose we raise this hunters' predation rate from $\sigma=0.1$ to $\sigma=0.8$, we have the results shown by Fig. 2.

Fig. 2. Barn owls free system $\sigma=0.8$ with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau=0$.

The effect of time delay τ can be investigated by assuming that squirrels prefer to eat the coconuts at particular age τ . The variation of this τ , however, can change the solutions of (9), i.e., it produces different effects when its values are below or above the critical value. In this case, the critical value $\tau_{c0}=0.096$ calculated from $\omega=2.319$. According to the critical value of τ , it indicates that when $\tau \in (0, 0.096)$, $E_{80}=(443, 23, 4)$ became asymptotically stable. However when $\tau=0.095 < 0.096$, the waveforms and the phase plots of system (9) were changed which is shown by Fig. 3. When $\tau=0.097 > 0.096$, E_{80} became bifurcated as shown by Fig. 4.

Fig. 3. CSOH system with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau=0.095$.

Fig. 4. CSOH system with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau=0.097$.

Now we consider the time delay system (1). Although this system have two time delays, i.e., τ_1 and τ_2 , only τ_1 can cause the bifurcation in $E_8=(375,23,11,4)$. This can observe from the values of the ω of τ_2 which provide no real number but $\pm 2.202187149 \pm 0.2442798467I$, $\pm 0.06132045903I$, $\pm 1.013195741I$, and $\pm 2.202187149 \pm 0.2442798467I$. For τ_1 , there are ω value of ± 2.416 which signal the occurrence of the bifurcation. By using the positive value of ω , the critical

value is given by $\tau_{c0}=0.114$. According to critical value of τ_1 , it indicate that when $\tau_1 \in [0, 0.114)$, $E_8=(375,23,11,4)$ became asymptotically stable. By setting $\tau_1=0.113 < 0.114$, the waveforms and the phase plots of system (1) were shown in Fig. 5. When $\tau_1=0.115 > 0.114$, it could be observed that $E_{14}=(375,23,11,4)$ lost its stability and became bifurcated as show by Fig. 6.

Fig. 5. CSOH system with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau_1=0.113$, and $\tau_2=0$.

Fig. 6. CSOH system with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau_1=0.115$, and $\tau_2=0$.

The effect of time delay can be extended by increasing the time delay. In this manner, the squirrel newborn should be slow down, and the coconut yield should be increased. By

setting $\tau_1=0.2>0.114$, the waveforms and the phase plots of system (1) were represented by Fig. 7 which shown a large oscillation of all species.

Fig. 7. CSOH system with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau_1=0.2$, and $\tau_2=0$.

To suppress the fluctuation of coconut yields to meet the objective of a stable income of farmers, we suggest two possible strategies. The first strategy is to reduce the squirrel coconut eating rate γ . By changing $\gamma=0.1$ to $\gamma=0.08$, all solution of (1) become asymptotically stable as shown by Fig. 8. This strategy can be implemented by searching for methods to protect coconut yields from squirrels. The traditional method that the farmers used for this purpose is to cover some part of the coconut tree to make difficulty for squirrel climbing. However, only this method may be less effective since squirrels can jump between the coconut trees. Therefore the range between the coconut trees should be designed properly to avoid squirrels do their jump. The chemical spray on coconut yields may be a possible alternative to protect coconut from squirrel eating them. Nevertheless, the farmers have to recognize the negative side effect of this practice. The second strategy is to reduce the new squirrel conversion rate λ . The results of changing λ from $\lambda=0.005$ to $\lambda=0.004$ were shown by Fig. 9. To implement this strategy, the farmers may utilize the chemical substance that can help to slow down this rate. However, this chemical substance is not known or even recognized by the farmers. Thus the first strategy may be more practical when compared to the second one.

Fig. 8. Policy 1: $\gamma=0.08$ with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\lambda=0.005$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau_1=0.115$, and $\tau_2=0$.

Fig. 9. Policy 2 $\lambda=0.004$ with $C(0)=500$, $S(0)=20$, $O(0)=2$, $H(0)=2$, $\beta=3$, $\gamma=0.1$, $\mu=0.3$, $\eta=1$, $\omega=0.2$, $\delta=0.2$, $\sigma=0.8$, $\xi=1$, $\kappa=0.05$, $\zeta=0.002$, $c=0.01$, $d=0.06$, $K=3500$, $Q=100$, $B=10$, $\tau_1=0.115$, and $\tau_2=0$.

VII. CONCLUSIONS

In this paper, we propose the delay CSOH model to explain the dynamics on the coconut farm. We investigate the time delay effect on the dynamics of coconut yields, squirrels, barn owls, and squirrel hunters. The results of the analysis showed that time delay are a cause of Bifurcation that makes the number of coconut yields unstable. Therefore to stabilize the coconut yields which imply income stability of farmers, we suggest two possible strategies which can lead the farmers to meet this stability objective.

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