Mathematical Models for Calculation of Mixing Zones for Coastal Effluent Discharges

Anton Purnama, Huda A. Al-Maamari, Easwaran Balakrishnan

Abstract: *The potential long-term environmental impacts of coastal effluent discharges can be addressed and regulated using a mixing zone concept to control and manage the mixing capacity of the receiving coastal waters. The standard regulatory guidelines consist of two key elements: a concentration limit and a point of compliance expressed as a radius centered at the end of the outfall pipe. Modeling studies of the effect of seabed depth upon dispersion of coastal effluent discharges into the sea in the far-field are investigated analytically using the solutions of two-dimensional advection-diffusion equations with a point source on the simple depth profiles of a flat seabed and a uniformly sloping seabed. Solutions are then illustrated graphically by plotting contours of concentration, showing that the effluent discharged plumes are spreading downstream and heading towards the shoreline. Based on the location of maximum value of concentration at the shoreline, the imposed radius of and limit of concentration level within the circular mixing zone are formulated to be used as a measure for assessing the impact of effluent discharges in the coastal environment. The results found agreed with the mixing zones implemented in practice as the environmental quality standards by the regulatory authorities to minimize the impact. As the sea outfall's long pipeline may go beyond the continental shelf, a sudden step change in the seabed depth profiles is introduced, where the solutions of the advection-diffusion equations are obtained using the method of images. An extended formulation of the mixing zones radius and concentration limit are also presented.*

Index Terms: advection diffusion equation, flat seabed, sloping seabed, water quality model.

I. INTRODUCTION

For coastal industrial plants, it is a safe, economical and reliable solution to discharge pretreated effluent wastewater into the sea through an effective, well-designed outfall system [1], [2]. A long sea outfall is a pipeline constructed for discharging effluent streams into the ocean. Extensive experiences around the world have reported that an outfall system installed with multiport diffusers has minimal environmental impacts since it apparently prevents the effluent discharged plumes from reaching coastal areas of human usage [1], [2]. For the near-shore discharges, it is observed in the far-field that the bent-over effluent plumes

are spreading towards the shoreline and may impair coastal water quality and affect marine life, and may also create a health concern to the public who use the nearby beaches for swimming and other recreational purposes [3], [4]. The extent and magnitude of changes in the receiving water quality due to effluent discharges define a mixing zone. Thus, the main objective of the coastal discharges is to meet the water quality standards set at a certain safety level (within a prescribed circular mixing zone) by the regulatory authorities to protect the receiving coastal and marine environment and human health [1], [4]-[6].

A regulatory mixing zone (RMZ) is an allocated impact zone in the receiving waters around the outfall beyond which the imposed water quality criteria must be met [4], [5], [7]-[9]. Conventionally, this standard regulatory guidelines consists of a concentration limit C^* and a point of compliance expressed as a distance R^* from the outfall's discharge port. In order to minimize the impacts of coastal effluent discharges, the mixing zones should be kept to the smallest area as practicable. It is recommended that the excess concentration limit at the mixing zone boundary normally should not exceed 5% (or correspond to a minimum dilution of about 20:1) of that occurring naturally in the waters around the discharge within a radius R^* that varies from 50 to 300 m [4], [7], [10]. Note that for acutely toxic effluents discharge, both the allowable concentration limit and mixing zone size are smaller than that of the specified values for conventional effluents.

The field monitoring and predictive models are widely used in order to demonstrate the compliance of operating an outfall system to discharge effluents with the regulatory standards. Due to a unique nature and local conditions of the receiving water, no unified method can be used to establish a mixing zone, and each discharge should be analyzed separately on a case-by-case basis. Thus, no specific mathematical models are recommended to assess the dilution and dispersion capacities of the receiving waters, and in searching for baseline quality standards for sophisticated, computational (full) three-dimensional hydrodynamic and water quality models, the analytical solutions of simple far-field models are presented to perform preliminary worst-case assessments [7]-[9], [11].

Many large cities are located on the coastal areas, and many operated outfall systems are built predominantly on sandy beaches. Physically, the coastal area is a dynamic region where land and sea meet. In some places, it takes the form of a sloping sandy beach, but in other places it is a mountainous coast with rocky sea cliffs, where the water depth gets deep very rapidly

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within a short offshore distance [3], [12], [13]. There are no beaches with seabed's sloping uniformly towards the sea. Going further seaward from the shore, the first submerged region is termed continental shelf. The seaward limits of the shelf are determined by the distinct change in slopes between the shelf and its adjacent continental slope. Thus a seabed depth profile is typically depicted in the oceanography textbooks as a gentle slope coming in contact with a steeper one.

To study the effect of water depth variations upon dispersion of coastal effluent discharges into the sea, we first consider simple profiles of a flat seabed with a constant water depth and a uniformly sloping seabed [14]-[16]. Solutions of a two-dimensional advection-diffusion equation with a point source are presented graphically by plotting contours of concentration showing that the discharged effluent plumes are spreading downstream towards the shoreline. However, it seems impractical to impose the concentration limit value obtained at the shoreline to the operators of marine outfall systems. The difficulties in adopting a mixing zone concept are mainly related to setting up the concentration limit C^* to be met at the circular mixing zone boundary of radius R^* [5], [8], [9], [11]. We proposed to use the location of the maximum value of concentration at the shoreline to set up the radius (from the point source) of the RMZ and then use it to formulate the concentration limit. Finally, using the method of image, the solutions are extended to a point discharge on a step change in seabed depth profiles formed by joining a shallow flat seabed to a deeper flat seabed and, for a sloping seabed, to another sloping seabed with a steeper slope [13].

II. MODEL SOLUTIONS FOR COASTAL DISCHARGES ON CONTINUOUS SEABED DEPTH

Modeling the environmental impact of effluent discharges into the sea is a difficult and complex task due to the variety and variability of the mixing processes, and the change in the sea currents, temperature and density. Conventionally, a local mixing zone should be determined based on the worst-case scenario of calm sea conditions drifting with a steady current. As we are only concerned with the effect of the seabed depth profile (perpendicular to the coastline), for simplicity, the other complexities such as tidal motions, temperature and density are ignored. The sea is assumed to be wide, the shoreline straight, and the effluent is discharged with a rate Q from a point source (from the end of outfall pipe) at $(0, \alpha h_0)$, where h_0 is an arbitrary reference water depth. The marine outfall systems are commonly installed with multiport diffusers designed to rapidly mix and dilute the discharged effluents with the receiving sea currents. It is also assumed that the effluent concentration is vertically well-mixed over the water depth.

As waves at sea approach the beach at an angle, the shallow seabed refracts and generates the (net coast parallel-component) longshore currents. The current is assumed to be steady with speed U and remain in the x -direction (positive to right of the outfall) at all times. The dispersion processes are represented by eddy diffusivities, and diffusion in the x -direction is neglected, as the

discharged plumes in steady currents become very elongated in the x -direction. The variations in the y -direction (positive towards the sea) of current speed U and coefficient of diffusivity D are assumed as the power functions only of water depth h [3], [14]-[16], and for application, we take U to be proportional to $h^{1/2}$ and D to $h^{3/2}$, which are the appropriate scaling for a turbulent shallow-water flow over a smooth bed [17], [18].

A. Semi-Infinite Flat Seabed

For coastal areas where the variations in water depth becomes insignificant, we consider the highly simplified profile of a flat seabed with a constant water depth, $h = h_0$ (Fig. 1a), with a vertical beach at $y = 0$. The advection-diffusion equation for the discharged effluent concentration in the far field $c_f(x, y)$ is

$$h_0 U_0 \frac{\partial c_f}{\partial x} - h_0 D_0 \frac{\partial^2 c_f}{\partial y^2} = Q \delta(x) \delta(y - \alpha h_0), \quad (1)$$

with the boundary condition $h_0 D_0 \frac{\partial c_f}{\partial y} = 0$ at the shoreline

$y = 0$, and c_f is assumed to be completely dissolved into the sea. $\delta(*)$ is the Dirac delta function used to represent the point at the end of outfall pipe where the discharge is made. Using dimensionless quantities

$y = Y h_0$, $x = X h_0$, and $c_f(x, y) = C_f(X, Y) Q / h_0^2 U_0$, and for sufficiently long sea outfalls, the analytical solution of (1) for $X \geq 0$ is given by

$$C_f = \sqrt{\frac{\lambda}{4\pi X}} \exp\left(-\frac{\lambda(Y - \alpha)^2}{4X}\right), \quad (2)$$

where the model parameter $\lambda = U_0 h_0 / D_0$ represents the effluent plume elongation in the x -direction [3], [9], [14], [15].

Fig. 1. Continuous seabed depth profile of a: (a) flat seabed and (b) sloping seabed

Contours of concentration (2) for a point source at $\alpha = 12$ were plotted in Fig. 2 for two values of $\lambda = 0.1$ and $\lambda = 0.2$ to illustrate the variability of sea conditions. The discharged effluent plumes appear to be spreading downstream parallel to the shoreline, and the larger the values of λ , the more elongated the plumes, which are mostly due to a stronger current U_0 with less longitudinal dispersivity D_0 . The maximum concentration of 0.020 at the shoreline is shown by the blue contour.

Fig. 2. Contours of concentration for a point source at $\alpha = 12$ on a flat seabed for (a) $\lambda = 0.1$ and (b) $\lambda = 0.2$

The sensitive areas for assessing the environmental impact of effluent discharges into the sea would be along the coastal areas [3], [14]–[16]. Thus, on substituting $Y = 0$ in concentration (2), and then by differentiating, it is straightforward to see that the maximum value of concentration at the shoreline is $C_{f \max} = 1/\alpha\sqrt{2\pi e}$, which occurs at $X_{f \max} = \lambda\alpha^2/2$.

Table- I: The RMZ for discharges on a flat seabed

α	11.5	12	12.5
C_{*f} (%)	3.469	3.325	3.192
λ	R_{*f}		
0.050	3.306	3.600	3.906
0.075	4.959	5.400	5.859
0.100	6.613	7.200	7.813
0.125	8.266	9.000	9.766
0.150	9.919	10.800	11.719
0.175	11.572	12.600	13.672
0.200	13.225	14.400	15.625

Using the location of the maximum concentration at the shoreline, we proposed to set the size of the RMZ by $Y = \alpha$ and $X = R_{*f} = \lambda\alpha^2/2$. From concentration (2), we obtain the concentration limit value at the edge of the RMZ as $C_{*f} = 1/\sqrt{2\pi}\alpha$. The effect of uncertainty in sea conditions

on the radius R_{*f} is analyzed in Table I. Since R_{*f} is linearly proportional with λ and quadratically with α , the value of R_{*f} increases significantly for larger the values of λ . However, C_{*f} is inversely proportional only with α , and for $\alpha \geq 10$, the value of $C_{*f} < 4\%$. We conclude that the limiting concentration values and imposed radius of the mixing zone agreed with the recommended standard for conventional effluents discharge.

B. Uniformly Sloping Seabed

Coastal industrial plants are built predominantly on the sloping sandy beaches, where the water depth varies increasingly linear with the offshore distance y . If m is the beach slope (Fig. 1b), then $h(y) = my$. The advection-diffusion equation for the discharged effluent concentration in the far field $c_s(x, y)$ is

$$\frac{\partial}{\partial x}(hUc_s) - \frac{\partial}{\partial y}\left(hD\frac{\partial c_s}{\partial y}\right) = Q\delta(x)\delta(y - \alpha h_0). \quad (3)$$

Similar to discharges on a flat seabed, the boundary condition is $hD\frac{\partial c_s}{\partial y} = 0$ at the shoreline $y = 0$, and c_s is assumed to be completely dissolved into the sea. Assuming $U = U_0Y^{1/2}$ and $D = D_0Y^{3/2}$, in its dimensionless form, the analytical solution of (3) for $X \geq 0$ can be written as [3], [14], [15]

$$C_s = \frac{\lambda}{mX} \left(\frac{1}{\alpha Y}\right)^{3/4} \exp\left(-\frac{\lambda(Y + \alpha)}{X}\right) I_{3/2}\left(\frac{2\lambda\sqrt{\alpha Y}}{X}\right), \quad (4)$$

where $I_{3/2}(\ast)$ is a modified Bessel function of order $3/2$.

Fig. 3. Contours of concentration for a point source at $\alpha = 12$ on a sloping seabed with $m = 0.1$ for (a) $\lambda = 0.1$ and (b) $\lambda = 0.2$

Contours of concentration (4) were plotted in Fig. 3 with $m = 0.1$ and $\alpha = 12$ for two values of $\lambda = 0.1$ and $\lambda = 0.2$. Unlike discharges on a flat seabed, the discharged effluent plumes are spreading downstream and heading towards the shoreline, and

cause a higher build up in concentration levels at the beach. The maximum concentration of 0.012 at the shoreline is shown by the blue contour.

Next, letting $Y \rightarrow 0$, and using the asymptotic expansion $I_{3/2}(z) = 4(z/2)^{3/2}/3\sqrt{\pi}$ in (4), we obtain an approximate effluent discharged concentration at the shoreline as $C_s(X, 0) = (4/3m\sqrt{\pi})(\lambda/X)^{5/2} \exp(-\lambda\alpha/X)$. Next by differentiating, it has a maximum value of approximately $C_{s\max} = 0.61/m\alpha^{5/2}$, which occurs at $X_{s\max} = 2\lambda\alpha/5$.

To formulate the RMZ, using the modified Bessel function by the asymptotic expansion $I_{3/2}(z) = \exp(z)/\sqrt{2\pi z}$, and for sufficiently long sea outfalls, we first approximate concentration (4) as

$$C_s = \frac{1}{m\alpha Y} \sqrt{\frac{\lambda}{4\pi X}} \exp\left(-\frac{\lambda(\sqrt{Y} - \sqrt{\alpha})^2}{X}\right).$$

Then on setting $Y = \alpha$ and $X = R_{*s} = 2\lambda\alpha/5$, we obtain the concentration limit value at the edge of the RMZ as $C_{*s} = \sqrt{5/8\pi}/m\alpha^{5/2}$. R_{*s} is linearly proportional to both λ and α ; whereas C_{*s} is inversely proportional to the slope m and α to power 2.5.

Table- II: The RMZ for discharges on a sloping seabed with $m = 0.1$

α	11.5	12	12.5
$C_{*s} (\%)$	0.995	0.894	0.807
λ	R_{*s}		
0.050	0.230	0.240	0.25
0.075	0.345	0.60	0.375
0.100	0.460	0.480	0.500
0.125	0.575	0.600	0.625
0.150	0.690	0.720	0.750
0.175	0.805	0.840	0.875
0.200	0.920	0.960	1.000
0.225	1.035	1.080	1.125
0.250	1.150	1.200	1.250

The variability in sea conditions on the radius R_{*s} are investigated in Table II for $m = 0.1$. In comparison with discharges on a flat seabed, both the concentration limit C_{*s} and the radius R_{*s} are much smaller than that of the flat seabed's values. In fact, for calm sea conditions, the radius R_{*s} is less than a tenth of the value R_{*f} ; and for $\alpha \geq 10$, the value of $C_{*s} < 1.5\%$. This is due to the fact that the relatively low currents in the coastal areas delays the effluent transport towards the shore, whereas diffusion away from the shore continues to reduce the concentration [16], [17]. Thus, the larger the value of α , the larger the distance travelled for the discharged effluent plumes to reach the shore (or the plumes do not feel the presence of the shoreline at $y = 0$).

Finally, we conclude that the imposed radius of the mixing zone and allowable concentration limits are suitable for the RMZ of acutely toxic coastal effluents discharge.

III. MODEL SOLUTIONS FOR COASTAL DISCHARGES ON A STEP SEABED DEPTH

The seabed depth profile of eroding sandy beaches is changing as a result of coastal sediment transport [3], [12], [13]. Erosion is causing sediment particles to move alongshore and/or drift out to sea. As the sediment particles are deposited to the offshore region, a step beach profile is formed [13]. To account for the presence of an abrupt depth change in the profile (Fig. 4), we assume that the sudden water depth change is occurring only across the line $y = \ell h_0$ (parallel to the straight shoreline at $y = 0$), where $r = h_1/h_0$ is the ratio of water depths across the discontinuity line $y = \ell h_0$. By writing the discharged effluent concentration in the far-field

$$c(x, y) = \begin{cases} c_0(x, y), & 0 \leq y < \ell h_0 \\ c_1(x, y), & y \geq \ell h_0 \end{cases},$$

the solution of the advection-diffusion equation can be obtained using the method of image [14], where the depth discontinuity at the line $y = \ell h_0$ will be considered as a (reflecting or absorbing) barrier (Fig. 5), beyond which the water depth is zero (or infinite).

Fig. 4. Discontinuous step seabed depth profile in a: (a) flat seabed and (b) sloping seabed

To make use of the presence of shallow water depth, we consider the case where the effluent discharge is made from a point source at $(x = 0, y = \alpha_1 h_0)$ with $\alpha_1 > \ell$, i.e. the effluent is released in the deeper offshore region

$y \geq \ell h_0$ at a distance $\alpha_1 h_0$ from the shoreline (Fig. 5). Treating the discontinuity line $y = \ell h_0$ as a reflecting boundary, the solution $c_1(x, y)$ is obtained due to a point source at $(0, \alpha_1 h_0)$ discharging with a rate Q and an image source at $(0, (2\ell - \alpha_1) h_0)$ discharging with a rate aQ . In the shallow nearshore region $0 \leq y < \ell h_0$, the line $y = \ell h_0$ is treated as an absorbing boundary, and the solution $c_0(x, y)$ is obtained due to a virtual source at $(0, \beta h_0)$ discharging with a rate bQ , where $\beta > \ell$. However, as there can be no discontinuities in either the effluent concentration or its gradient across the discontinuity line $y = \ell h_0$, the matching conditions to be satisfied are

$$\lim_{y \rightarrow \ell h_0} c_0 = \lim_{y \rightarrow \ell h_0} c_1$$

and

$$\lim_{y \rightarrow \ell h_0} h_0 D_0 \frac{\partial c_0}{\partial y} = \lim_{y \rightarrow \ell h_0} h_1 D_1 \frac{\partial c_1}{\partial y}.$$

Fig. 5. Plan view of the method of image for a point source in the deeper region $y \geq \ell h_0$

A. Flat-Step Seabed

For a step discontinuity on a flat seabed depth profile (Fig. 4a), the water depth h_f varies according to

$$h_f(y) = \begin{cases} h_0, & 0 \leq y < \ell h_0 \\ h_1, & y \geq \ell h_0 \end{cases}.$$

Note that if $r=1$, there is no depth change across the discontinuity line $y = \ell h_0$, and this flat-step depth profile is exactly that of a simple flat seabed.

Similar to the previous section on a flat seabed, the advection-diffusion equation for concentration $c_{f0}(x, y)$ in the nearshore region $0 \leq y < \ell h_0$ is given by (1) but with a virtual point source at $(0, \beta_f h_0)$ discharging with a rate $b_f Q$, and thus the solution, in its dimensionless form, is

$$C_{f0} = b_f \sqrt{\frac{\lambda}{4\pi X}} \exp\left(-\frac{\lambda(Y - \beta_f)^2}{4X}\right).$$

In the deeper offshore region $y \geq \ell h_0$, on applying the superposition principle, the advection-diffusion equation for concentration $c_{f1}(x, y)$ is

$$h_1 U_1 \frac{\partial c_{f1}}{\partial x} - h_1 D_1 \frac{\partial^2 c_{f1}}{\partial y^2} = Q \delta(x) \delta(y - \alpha_1 h_0) + a_f Q \delta(x) \delta(y - 2\ell h_0 + \alpha_1 h_0)$$

and the solution, in its dimensionless form, is

$$C_{f1} = \frac{1}{r^2} \sqrt{\frac{\lambda}{4\pi X}} \left[\exp\left(-\frac{\lambda(Y - \alpha_1)^2}{4rX}\right) + a_f \exp\left(-\frac{\lambda(Y - 2\ell + \alpha_1)^2}{4rX}\right) \right], \quad (5)$$

where $U_1 = U_0 r^{1/2}$ and $D_1 = D_0 r^{3/2}$. The dimensionless form of the matching conditions

$$\lim_{Y \rightarrow \ell} C_{f0} = \lim_{Y \rightarrow \ell} C_{f1} \text{ and } \lim_{Y \rightarrow \ell} \frac{\partial C_{f0}}{\partial Y} = \lim_{Y \rightarrow \ell} r^{5/2} \frac{\partial C_{f1}}{\partial Y}$$

may now be applied in order to calculate a_f , b_f and β_f , and we obtain

$$\beta_f = \ell + \frac{\alpha_1 - \ell}{\sqrt{r}}, \quad a_f = \frac{r^2 - 1}{r^2 + 1} \text{ and } b_f = \frac{2}{r^2 + 1}.$$

Fig. 6. A flat-step seabed when $\ell = 11$: (a) the coefficients a_f and b_f ; and (b) β_f

It is easy to verify, $a_f + b_f = 1$ and if there is no depth change, i.e. $r=1$, then $\beta_f = \alpha_1$, $a_f = 0$ and $b_f = 1$. The variations of these coefficients for $r \geq 1$ when $\ell = 11$ are shown in Fig. 6. The value of a_f is increasing $0 \leq a_f < 1$, whereas the value of b_f is decreasing $0 < b_f \leq 1$. The

value of β_f is also decreasing $\ell < \beta_f \leq \alpha_1$.

To calculate the RMZ, by substituting $Y = \alpha_1$ and $X = R_{*f1} = \lambda\beta_f^2/2$ in concentration (5), we obtain the concentration limit at the edge of the RMZ as

$$C_{*f1} = \frac{1}{\sqrt{2\pi}\beta_f r^2} \left[1 + \frac{r^2 - 1}{r^2 + 1} \exp \left\{ -2 \left(1 - \frac{\ell}{\beta_f} \right)^2 \right\} \right].$$

In comparison with discharges on a flat seabed, we obtain

$$\frac{R_{*f1}}{R_{*f}} = \left(\frac{\beta_f}{\alpha_1} \right)^2$$

and

$$\frac{C_{*f1}}{C_{*f}} = \frac{\alpha_1}{\beta_f r^2} \left[1 + \frac{r^2 - 1}{r^2 + 1} \exp \left\{ -2 \left(1 - \frac{\ell}{\beta_f} \right)^2 \right\} \right].$$

Since $\beta_f \leq \alpha_1$ for $r \geq 1$, the radius $R_{*f1} \leq R_{*f}$ and the concentration limit $C_{*f1} \leq C_{*f}$.

The reductions in the concentration limit C_{*f1} and in the ratio of radius R_{*f1}/R_{*f} for $r \geq 1$ on a flat-step seabed when $\ell = 11$ are shown in Fig. 7 for three values of $\alpha_1 = 11.5, 12$ and 12.5 . In particular for the case of $\alpha_1 = 12$, a reduction in value of C_{*f1} (relative to C_{*f}) is about 38% for $r = 1.5$ to about 72% for $r = 2.5$. However, the reduction in R_{*f1} (relative to R_{*f}) is less significant, and it is only about 5% for $r = 2$ to about 7% for $r = 3$.

Fig. 7. The RMZ for discharges on a flat-step seabed when $\ell = 11$: (a) the concentration limit C_{*f1} and (b) the ratio of radius R_{*f1}/R_{*f}

B. Slope-Step Seabed

For a step discontinuity on a sloping seabed depth profile (Fig. 4b), the water depth h_s varies as

$$h_s(y) = \begin{cases} m_0 y, & 0 \leq y < \ell h_0 \\ m_1 y, & y \geq \ell h_0 \end{cases},$$

where the slope in the nearshore region $m_0 = 1/\ell$ and in the offshore region $m_1 = r/\ell = r m_0$. Note that if $r = 1$, there is no depth change across the discontinuity line $y = \ell h_0$, and this slope-step depth profile is exactly that of a uniformly sloping seabed.

Similar to the previous section on a sloping seabed, the advection-diffusion equation for concentration $c_{s0}(x, y)$ in the nearshore region $0 \leq y < \ell h_0$ is given by (3) but with a virtual point source at $(0, \beta_s h_0)$ discharging with a rate $b_s Q$, and thus the solution, in its dimensionless form, is

$$C_{s0} = \frac{b_s \lambda}{m_0 X} \left(\frac{1}{\beta_s Y} \right)^{3/4} \exp \left(-\frac{\lambda(Y + \beta_s)}{X} \right) I_{3/2} \left(\frac{2\lambda\sqrt{\beta_s Y}}{X} \right).$$

Next, on applying the superposition principle, the advection-diffusion equation for concentration $c_{s1}(x, y)$ in the deeper offshore region $y \geq \ell h_0$ is

$$h_1 U_1 \frac{\partial c_{s1}}{\partial x} - \frac{\partial}{\partial y} (h_1 D_1 c_{s1}) = Q \delta(x) \delta(y - \alpha_1 h_0) + a_s Q \delta(x) \delta(y - 2\ell h_0 + \alpha_1 h_0)$$

and the solution, in its dimensionless form, is

$$C_{s1} = \frac{\lambda}{m_1 r^{3/2} X} \left[\left(\frac{1}{\alpha_1 Y} \right)^{3/4} \exp \left(-\frac{\lambda(Y + \alpha_1)}{rX} \right) I_{3/2} \left(\frac{2\lambda\sqrt{\alpha_1 Y}}{rX} \right) + a_s \left(\frac{1}{(2\ell - \alpha_1)Y} \right)^{3/4} \exp \left(-\frac{\lambda(Y + 2\ell - \alpha_1)}{rX} \right) \times I_{3/2} \left(\frac{2\lambda\sqrt{(2\ell - \alpha_1)Y}}{rX} \right) \right], \quad (6)$$

where $U_1 = U_0 r^{1/2} Y^{1/2}$ and $D_1 = D_0 r^{3/2} Y^{3/2}$. Note that the arguments of modified Bessel functions are approximately the same provided $2\ell - \alpha_1 \approx \alpha_1$, i.e. when $\alpha_1/\ell \approx 1$.

After approximating the modified Bessel functions, the matching conditions, in its dimensionless form,

$$\lim_{Y \rightarrow \ell} C_{s0} = \lim_{Y \rightarrow \ell} C_{s1}$$

and

$$\lim_{Y \rightarrow \ell} m_0 Y^{5/2} \frac{\partial C_{s0}}{\partial Y} = \lim_{Y \rightarrow \ell} m_1 r^{3/2} Y^{5/2} \frac{\partial C_{s1}}{\partial Y}$$

may now be applied in order to calculate a_s , b_s and β_s , we obtain

$$\sqrt{\beta_s} = \sqrt{\ell} + \frac{\sqrt{\alpha_1} - \sqrt{\ell}}{\sqrt{r}}, \quad a_s = \left(\frac{2\ell - \alpha_1}{\alpha_1} \right) \left(\frac{r^2 - 1}{r^2 + 1} \right)$$

and

$$b_s = \left(\frac{\beta_s}{\alpha_1} \right) \left(\frac{2}{r^2 + 1} \right).$$

Unlike discharges on a flat-step seabed, both coefficients a_s and b_s are dependent of α_1 . If there is no depth change, i.e. $r=1$, then $\beta_s = \alpha_1$, $a_s = 0$ and $b_s = 1$. The variations of these coefficients for $r \geq 1$ when $\ell = 11$ are shown in Fig. 8. For $\alpha_1 = 12$, similar to that of the flat-step seabed, the value of a_s is increasing $0 \leq a_s < 1$; while the value of b_s is decreasing $0 < b_s \leq 1$. The value of β_s is also decreasing $\ell < \beta_s \leq \alpha_1$.

Following the previous section on a sloping seabed, using the modified Bessel function by the asymptotic expansion, we approximate concentration (6) as

$$C_{s1} = \frac{1}{m_1 r \alpha_1 Y} \sqrt{\frac{\lambda}{4\pi X}} \left[\exp \left(-\frac{\lambda (\sqrt{Y} - \sqrt{\alpha_1})^2}{rX} \right) + \frac{a_s \alpha_1}{2\ell - \alpha_1} \exp \left(-\frac{\lambda (\sqrt{Y} - \sqrt{2\ell - \alpha_1})^2}{rX} \right) \right].$$

Fig. 8. A slope-step seabed when $\ell = 11$: (a) the coefficients a_s and b_s (for $\alpha_1 = 12$); and (b) β_s

Then, by substituting $Y = \alpha_1$ and $X = R_{*s1} = 2\lambda\beta_s/5$, we obtain the concentration limit at the edge of the RMZ as

$$C_{s1} = \sqrt{\frac{5}{8\pi}} \frac{\ell}{\beta_s (r\alpha_1)^2} \left[1 + \frac{r^2 - 1}{r^2 + 1} \exp \left\{ -5 \left(\frac{\ell - \sqrt{(2\ell - \alpha_1)\alpha_1}}{r\beta_s} \right) \right\} \right].$$

In comparison with discharges on a sloping seabed, we obtain

$$\frac{R_{*s1}}{R_{*s}} = \frac{\beta_s}{\alpha_1}$$

and

$$\frac{C_{*s1}}{C_{*s}} = \frac{1}{r^2} \sqrt{\frac{\alpha_1}{\beta_s}} \left[1 + \frac{r^2 - 1}{r^2 + 1} \exp \left\{ -5 \left(\frac{\ell - \sqrt{(2\ell - \alpha_1)\alpha_1}}{r\beta_s} \right) \right\} \right].$$

We note that the radius R_{*s1} is exactly the same to that of discharges on a sloping seabed radius R_{*s} (with a virtual source at $(0, \beta_s h_0)$). Thus, the presence of a step water depth on a sloping seabed does not change the radius R_{*s} . For $r \geq 1$, the concentration limit C_{*s1} is smaller than that of discharges on a sloping seabed C_{*s} .

The concentration limit C_{*s1} for discharges on a slope-step seabed when $\ell = 11$ are shown in Fig. 9 for three values of $\alpha_1 = 11.5, 12$ and 12.5 . The value C_{*s1} is less than 1.5% for $r \geq 1$. Similar to a flat-step seabed, for the case of $\alpha_1 = 12$, $C_{*s1} = 0.0098$ for $r = 1$, and as r increases, the reduction in the concentration limit increases, from about 38% for $r = 1.5$ to more than 72% for $r = 2.5$. However, the reduction in R_{*s1} is much smaller; of about 2.5% for $r = 2$ to about 3.6% for $r = 3$.

Fig. 9. The RMZ for discharges on a slope-step seabed when $\ell = 11$: (a) the concentration limit C_{*s1} and (b) the ratio of radius R_{*s1}/R_{*s}

IV. CONCLUDING REMARKS

The effect of the seabed depth profile upon mixing and spreading of effluent wastewater continuously being discharged into the sea is investigated analytically using simple models of a flat seabed with a constant water depth and a uniformly sloping seabed. Using the location of the maximum value of concentration at the shoreline, the radius of the RMZ and the concentration limit at the end of the RMZ are formulated. These model values are found to have agreed with the recommended size of a mixing zone imposed and implemented in practice by the regulatory authorities to control coastal effluent discharges from marine outfall systems [7], [10]. The radius of the RMZ obtained for a sloping seabed is less than a

tenth of that value for a flat seabed, and thus, it is suitable for the RMZ of acutely toxic coastal effluents discharge. To improve the models and to illustrate the effect of shallow water depth in the nearshore region, a step discontinuity in depth profile is then introduced.

The use of mathematical models has been a key strategy for assessing the potential environmental impacts of outfall effluent discharges. While the far field modelling in this paper involves drastic simplifications, key physical mixing and dispersion processes are represented, and thus the analytical solution remains useful in providing a qualitative understanding and in suggesting general behaviour of the effluent discharge plumes in coastal waters [19], [20]. The models can be applied to any type of pollutant continuously being discharged into the sea from a marine outfall system; for example, treated effluents from coastal sewage works [1], [2], toxic contaminants from industrial installations in coastal areas [2], hot water discharged from coastal power stations [21] and brine discharge from desalination plants [4], [7].

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