HIGHER ORDER REFINED COMPUTATIONAL MODEL USING POST PROCESSING FOR THE TRANSVERSE SHEAR AND TRANSVERSE NORMAL STRESS IN COMPOSITE PLATES

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ABSTRACT

Many researchers presented failure analysis of multilayered plates. Some researchers used Equivalent Single Layer (ESL) for their analysis. Some researchers adopted layer wise theories for failure analysis. Accuracy in estimation of transverse stresses plays a pivotal role in such analysis. Transverse stresses at the layer interface calculated from the constitutive equation gives discontinuity of stresses. As such these stresses can be precisely forecasted by applying a technique. This Technique includes integration of the 3D equilibrium equations as post processing. The principal aim of the present work is to evaluate analytically the accuracy of a higher order refined computational model with 12 unknowns in predicting the transverse shear and normal stresses. The theoretical model presented herein considers the effects of both transverse shear and normal strain/stress. This model also considers the warping of transverse cross sections. The theoretical formulations and solution method for post processing is presented in the paper. Extensive numerical results are presented and discussed. Present model with post processing could predict the transverse shear and normal stresses more accurately than the models with lower variable.

Keywords: Composite Laminates, Transverse Stresses, Higher Order Shear Deformation Theory, Analytical Method.

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1. INTRODUCTION

Analysis of laminated of such covered composites gets troublesome because of various aspects. Material mismatch across the laminate interfaces is one of the aspect, Bending-stretching coupling and geometric non-linear effects are other aspects. Laminated composites have low transverse shear stiffness. Therefore laminated composites having lower side to

thickness ratio are significantly affected transverse shear deformation. Accuracy in predicting the transverse stresses are most important for predicting failure of composites. Various investigators used numerous theories for the accurate computation of the inplane and transverse stresses to be used in failure analysis. A complete review of literature up to the year 2019 is also found in the articles presented by many authors [1-7]. Some of the researchers are of the opinion that ESL can only predict global responses accurately. Deflection, natural frequency and buckling load comes under global response. These researchers are of the opinion that ESL theories fail to predict local response. Variation of stress distribution comes under local response. In their opinion layerwise theories which are computationally expensive as compared to ESL theories can only predict exact through the thickness variation of stresses. Most of the researchers used computational models with up to a maximum of five degrees of freedom in their analysis to arrive at these conclusions. Some researchers are of opposite school of thoughts [8-11]. In this paper one such higher order shear deformation theory with 12 DOF already available in the literature is considered. Analytical formulation and solution method to compute the transverse stresses by using post processing techniques not yet reported. Extensive numerical results using this model are provided. Analytical solutions for the in-plane stresses of plates using this computational model are reported in the articles by Swaminatan and Patil [13-15], Kant and Manjunatha [12]. It was shown that the results of linear stresses obtained using the model is very much closer to 3D elasticity solutions. Antisymmetric angle ply laminates are used to have more torsional rigidity viz. in fan blades. There is not much results published in the literature on the transverse stresses of antisymmetric angle ply composite plates.

2. THEORETICAL FORMULATION&ANALYTICAL SOLUTION

Equivalent Single layer theory is used for the analysis. The displacement field derived from the expanded Taylor's series in terms of the thickness coordinates z and having 12 DOF proposed by Kant and Manjunatha [12] is considered

$$\begin{aligned} u(x, y, z) &= u_o(x, y) + z\theta_x(x, y) + z^2u_o^*(x, y) + z^3\theta_x^*(x, y) \\ v(x, y, z) &= v_o(x, y) + z\theta_y(x, y) + z^2v_o^*(x, y) + z^3\theta_y^*(x, y) \\ w(x, y, z) &= w_o(x, y) + z\theta_z(x, y) + z^2w_o^*(x, y) + z^3\theta_z^*(x, y) \end{aligned} \quad (1)$$

Where

u_o, v_o and w_o are the displacements on the middle plane.

θ_x, θ_y and θ_z are rotations

Others are higher order terms. These terms are developed from the Taylor's series expansion.

For modeling each layer of laminate is assumed as homogeneous and orthotropic and the laminate throughout the thickness is assumed as anisotropic and heterogeneous.

The general linear strain-displacement relations are given as follows:

$$\epsilon_x = \frac{\partial u}{\partial x}; \epsilon_y = \frac{\partial v}{\partial y}; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \epsilon_z = \frac{\partial w}{\partial z}; \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}; \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (2)$$

Figure 1 Laminate Geometry with positive set of Lamina/laminate reference axes, displacement components

The stress-strain relations for L^{th} lamina is developed. Figure 1 gives laminate geometry. Thus,

$$\{\sigma\}^L = [Q_{ij}] \{\varepsilon\}^L \quad (3)$$

where $\{\sigma\}^L = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ are the stresses and

$\{\varepsilon\}^L = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ are the strains with respect to the laminate axes.

$[Q_{ij}]$ are the transformed elastic constants or the stiffness matrix with respect to the laminate axes x, y, z .

Authors have independently analysed in plane stresses. But due to space restraint are not provided here. Readers may refer papers of K. Swaminathan and Patil [12-15] for detailed formulation, analysis and solution to get in in plane stresses. Afterwards Transverse stresses are calculated by applying post processing technique. This technique includes integration of 3D elasticity equilibrium equations. The procedure given below:

The equilibrium equations are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0; \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0; \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (4)$$

Integrating the above set of equations with respect to the thickness coordinate 'z' we get

$$\begin{aligned}\tau_{xz}^{(L)} &= - \int_{z_1}^z \left(\frac{\partial \sigma_x^{(L)}}{\partial x} + \frac{\partial \sigma_{xy}^{(L)}}{\partial y} \right) dz + C_1^{(L)}(x, y) \\ \tau_{yz}^{(L)} &= - \int_{z_1}^z \left(\frac{\partial \sigma_{xy}^{(L)}}{\partial x} + \frac{\partial \sigma_{yz}^{(L)}}{\partial y} \right) dz + C_2^{(L)}(x, y) \\ \sigma_{zz}^{(L)} &= - \int_{z_1}^z \left(\frac{\partial \sigma_{xz}^{(L)}}{\partial x} + \frac{\partial \sigma_{yz}^{(L)}}{\partial y} \right) dz + C_3^{(L)}(x, y)\end{aligned}\quad (5)$$

τ_{xz} , τ_{yz} and σ_z are evaluated using the bounding plane criteria and the continuity of stresses.

From the bounding plane criteria we obtain

$$\tau_{xz}\left(x, y, -\frac{h}{2}\right) = 0 ; \quad \tau_{yz}\left(x, y, -\frac{h}{2}\right) = 0 ; \quad \sigma_z\left(x, y, -\frac{h}{2}\right) = 0 \quad (7)$$

Following conditions are used For Continuity of stress

$$\tau_{xz}^L(x, y, z_{L+1}) = \tau_{xz}^{L+1}(x, y, z_{L+1}); \quad \tau_{yz}^L(x, y, z_{L+1}) = \tau_{yz}^{L+1}(x, y, z_{L+1}) \quad (8)$$

$$\sigma_z^L(x, y, z_{L+1}) = \sigma_z^{L+1}(x, y, z_{L+1}) \quad (9)$$

2.1. Transverse Stress (τ_{xz})

From Equation (23) we can write the value of transverse stress τ_{xz} for any L^{th} layer as

$$\tau_{xz}^{(L)} = - \int_{z_L}^z \left(\frac{\partial \sigma_x^{(L)}}{\partial x} + \frac{\partial \tau_{xy}^{(L)}}{\partial y} \right) dz \quad (10)$$

Substituting the values of σ_x and τ_{xy} in terms of strains from equations (2) and (6) in to equation (10) we get

$$\tau_{xz}^{(L)} = - \int_{z_L}^z \left\{ \frac{\partial}{\partial x} [Q_{11} \varepsilon_x + Q_{12} \varepsilon_y + Q_{13} \varepsilon_z + Q_{14} \gamma_{xy}] + \frac{\partial}{\partial y} [Q_{41} \varepsilon_x + Q_{42} \varepsilon_y + Q_{43} \varepsilon_z + Q_{44} \gamma_{xy}] \right\} dz$$

And after solving above equation and using boundary conditions:

At the bounding plane we get

$$Z = Z_L = -\frac{h}{2} \quad \tau_{xz} = 0 \quad \text{i.e.} \quad \tau_{xz}\left(x, y, -\frac{h}{2}\right) = 0 \quad (11)$$

Condition for continuity of stress at layer interface are applied. The value of τ_{xz} is given by:

$$\begin{aligned}\tau_{xz}^{(L)}(x, y, z_L) = & - \left\{ Q_{11} H_1 \frac{\partial^2 u_0}{\partial x^2} + Q_{11} H_2 \frac{\partial^2 \theta_x}{\partial x^2} + Q_{11} H_3 \frac{\partial^2 u_0^*}{\partial x^2} + Q_{11} H_4 \frac{\partial^2 \theta_x^*}{\partial x^2} + Q_{12} H_1 \frac{\partial^2 v_0}{\partial x \partial y} \right. \\ & + Q_{12} H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{12} H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{12} H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} + Q_{13} H_1 \frac{\partial \theta_z}{\partial x} + Q_{13} H_2 \frac{\partial w_0^*}{\partial x} + Q_{13} H_3 \frac{\partial \theta_z^*}{\partial x} \\ & \left. + Q_{13} H_4 \frac{\partial w_0}{\partial x} \right\}\end{aligned}$$

$$\begin{aligned}
 & + Q_{14}H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{14}H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{14}H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{14}H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{14}H_1 \frac{\partial^2 v_0}{\partial x^2} \\
 & + Q_{14}H_2 \frac{\partial^2 \theta_y}{\partial x^2} + Q_{14}H_3 \frac{\partial^2 v_0^*}{\partial x^2} + Q_{14}H_4 \frac{\partial^2 \theta_y^*}{\partial x^2} + Q_{41}H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{41}H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{41}H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} \\
 & + Q_{41}H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{42}H_1 \frac{\partial^2 v_0}{\partial y^2} + Q_{42}H_2 \frac{\partial^2 \theta_y}{\partial y^2} + Q_{42}H_3 \frac{\partial^2 v_0^*}{\partial y^2} + Q_{42}H_4 \frac{\partial^2 \theta_y^*}{\partial y^2} + Q_{43}H_1 \frac{\partial \theta_z}{\partial y} \\
 & + Q_{43}H_2 \frac{\partial^2 w_0^*}{\partial y} + Q_{43}H_3 \frac{\partial \theta_z^*}{\partial y} + Q_{44}H_1 \frac{\partial^2 u_0}{\partial y^2} + Q_{44}H_2 \frac{\partial^2 \theta_x}{\partial y^2} + Q_{44}H_3 \frac{\partial^2 u_0^*}{\partial y^2} \\
 & + Q_{44}H_4 \frac{\partial^2 \theta_x^*}{\partial y^2} + Q_{44}H_1 \frac{\partial^2 v_0}{\partial x \partial y} + Q_{44}H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{44}H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{44}H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} \} \\
 & - \{ Q_{11}H_1 \frac{\partial^2 u_0}{\partial x^2} + Q_{11}H_2 \frac{\partial^2 \theta_x}{\partial x^2} + Q_{11}H_3 \frac{\partial^2 u_0^*}{\partial x^2} + Q_{11}H_4 \frac{\partial^2 \theta_x^*}{\partial x^2} + Q_{12}H_1 \frac{\partial^2 v_0}{\partial x \partial y} + Q_{12}H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} \\
 & + Q_{12}H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{12}H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} + Q_{13}H_1 \frac{\partial \theta_z}{\partial x} + Q_{13}H_2 \frac{\partial w_0^*}{\partial x} + Q_{13}H_3 \frac{\partial \theta_z^*}{\partial x} + Q_{14}H_1 \frac{\partial^2 u_0}{\partial x \partial y} \\
 & + Q_{14}H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{14}H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{14}H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{14}H_1 \frac{\partial^2 v_0}{\partial x^2} + Q_{14}H_2 \frac{\partial^2 \theta_y}{\partial x^2} + Q_{14}H_3 \frac{\partial^2 v_0^*}{\partial x^2} \\
 & + Q_{14}H_4 \frac{\partial^2 \theta_y^*}{\partial x^2} + Q_{41}H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{41}H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{41}H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{41}H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} \\
 & + Q_{42}H_1 \frac{\partial^2 v_0}{\partial y^2} + Q_{42}H_2 \frac{\partial^2 \theta_y}{\partial y^2} + Q_{42}H_3 \frac{\partial^2 v_0^*}{\partial y^2} + Q_{42}H_4 \frac{\partial^2 \theta_y^*}{\partial y^2} + Q_{43}H_1 \frac{\partial \theta_z}{\partial y} + Q_{43}H_2 \frac{\partial^2 w_0^*}{\partial y} \\
 & + Q_{43}H_3 \frac{\partial \theta_z^*}{\partial y} + Q_{44}H_1 \frac{\partial^2 u_0}{\partial y^2} + Q_{44}H_2 \frac{\partial^2 \theta_x}{\partial y^2} + Q_{44}H_3 \frac{\partial^2 u_0^*}{\partial y^2} + Q_{44}H_4 \frac{\partial^2 \theta_x^*}{\partial y^2} \\
 & + Q_{44}H_1 \frac{\partial^2 v_0}{\partial x \partial y} + Q_{44}H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{44}H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{44}H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} \} \quad (12)
 \end{aligned}$$

Where $H_i = \frac{1}{i} (z_{L+1}^i - z_L^i) \dots \dots \dots (i = 1 \text{ to } 7)$

Similarly the final expressions for the computation of τ_{yz} , σ_z are given as below

2.2. Transverse Stress (τ_{yz})

The general expression for shear stress $\tau_{yz}^{(L)}$ for any L^{th} layer is given by

$$\begin{aligned}
 \tau_{yz}^{(L)}(x, y, z_L) = - \{ & Q_{41}H_1 \frac{\partial^2 u_0}{\partial x^2} + Q_{41}H_2 \frac{\partial^2 \theta_x}{\partial x^2} + Q_{41}H_3 \frac{\partial^2 u_0^*}{\partial x^2} + Q_{41}H_4 \frac{\partial^2 \theta_x^*}{\partial x^2} + Q_{42}H_1 \frac{\partial^2 v_0}{\partial x \partial y} \\
 & + Q_{42}H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{42}H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{42}H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} + Q_{43}H_1 \frac{\partial \theta_z}{\partial x} + Q_{43}H_2 \frac{\partial w_0^*}{\partial x} + Q_{43}H_3 \frac{\partial \theta_z^*}{\partial x} \\
 & + Q_{44}H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{44}H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{44}H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{44}H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{44}H_1 \frac{\partial^2 v_0}{\partial x^2} + Q_{44}H_2 \frac{\partial^2 \theta_y}{\partial x^2} \\
 & + Q_{44}H_3 \frac{\partial^2 v_0^*}{\partial x^2} + Q_{44}H_4 \frac{\partial^2 \theta_y^*}{\partial x^2} \}
 \end{aligned}$$

$$\begin{aligned}
 & + Q_{44} H_3 \frac{\partial^2 v_0^*}{\partial x^2} + Q_{44} H_4 \frac{\partial^2 \theta_y^*}{\partial x^2} + Q_{21} H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{21} H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{21} H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{21} H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} \\
 & + Q_{22} H_1 \frac{\partial^2 v_0}{\partial y^2} + Q_{22} H_2 \frac{\partial^2 \theta_y}{\partial y^2} + Q_{22} H_3 \frac{\partial^2 v_0^*}{\partial y^2} + Q_{22} H_4 \frac{\partial^2 \theta_y^*}{\partial y^2} + Q_{23} H_1 \frac{\partial \theta_z}{\partial y} \\
 & + Q_{23} H_2 \frac{\partial^2 w_0^*}{\partial y} + Q_{23} H_3 \frac{\partial \theta_z^*}{\partial y} + Q_{24} H_1 \frac{\partial^2 u_0}{\partial y^2} + Q_{24} H_2 \frac{\partial^2 \theta_x}{\partial y^2} + Q_{24} H_3 \frac{\partial^2 u_0^*}{\partial y^2} \\
 & + Q_{24} H_4 \frac{\partial^2 \theta_x^*}{\partial y^2} + Q_{24} H_1 \frac{\partial^2 v_0}{\partial x \partial y} + Q_{24} H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{24} H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{24} H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} \Bigg\} \\
 & - \Bigg\{ Q_{41} H_1 \frac{\partial^2 u_0}{\partial x^2} + Q_{41} H_2 \frac{\partial^2 \theta_x}{\partial x^2} + Q_{41} H_3 \frac{\partial^2 u_0^*}{\partial x^2} + Q_{41} H_4 \frac{\partial^2 \theta_x^*}{\partial x^2} + Q_{42} H_1 \frac{\partial^2 v_0}{\partial x \partial y} + Q_{42} H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} \\
 & + Q_{42} H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{42} H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} + Q_{43} H_1 \frac{\partial \theta_z}{\partial x} + Q_{43} H_2 \frac{\partial w_0^*}{\partial x} + Q_{43} H_3 \frac{\partial \theta_z^*}{\partial x} + Q_{44} H_1 \frac{\partial^2 u_0}{\partial x \partial y} \\
 & + Q_{44} H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{44} H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{44} H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{44} H_1 \frac{\partial^2 v_0}{\partial x^2} + Q_{44} H_2 \frac{\partial^2 \theta_y}{\partial x^2} + Q_{44} H_3 \frac{\partial^2 v_0^*}{\partial x^2} \\
 & + Q_{44} H_4 \frac{\partial^2 \theta_y^*}{\partial x^2} + Q_{21} H_1 \frac{\partial^2 u_0}{\partial x \partial y} + Q_{21} H_2 \frac{\partial^2 \theta_x}{\partial x \partial y} + Q_{21} H_3 \frac{\partial^2 u_0^*}{\partial x \partial y} + Q_{21} H_4 \frac{\partial^2 \theta_x^*}{\partial x \partial y} + Q_{22} H_1 \frac{\partial^2 v_0}{\partial y^2} \\
 & + Q_{22} H_2 \frac{\partial^2 \theta_y}{\partial y^2} + Q_{22} H_3 \frac{\partial^2 v_0^*}{\partial y^2} + Q_{22} H_4 \frac{\partial^2 \theta_y^*}{\partial y^2} + Q_{23} H_1 \frac{\partial \theta_z}{\partial y} + Q_{23} H_2 \frac{\partial^2 w_0^*}{\partial y} + Q_{23} H_3 \frac{\partial \theta_z^*}{\partial y} \\
 & + Q_{24} H_1 \frac{\partial^2 u_0}{\partial y^2} + Q_{24} H_2 \frac{\partial^2 \theta_x}{\partial y^2} + Q_{24} H_3 \frac{\partial^2 u_0^*}{\partial y^2} + Q_{24} H_4 \frac{\partial^2 \theta_x^*}{\partial y^2} + Q_{24} H_1 \frac{\partial^2 v_0}{\partial x \partial y} \\
 & + Q_{24} H_2 \frac{\partial^2 \theta_y}{\partial x \partial y} + Q_{24} H_3 \frac{\partial^2 v_0^*}{\partial x \partial y} + Q_{24} H_4 \frac{\partial^2 \theta_y^*}{\partial x \partial y} \Bigg\}
 \end{aligned} \tag{13}$$

2.3. Transverse Stress (σ_z)

The general expression for transverse stress σ_z for L^{th} layer becomes

$$\begin{aligned}
 \sigma_z(x, y, z) = & - \Bigg\{ Q_{65} \Bigg\{ H_1 \left(\frac{\partial \theta_y}{\partial x} + \frac{\partial^2 w_0}{\partial x \partial y} \right) + H_2 \left(2 \frac{\partial v_0^*}{\partial x} + \frac{\partial^2 \theta_z}{\partial x \partial y} \right) \\
 & + H_3 \left(3 \frac{\partial \theta_y^*}{\partial x} + \frac{\partial^2 w_0^*}{\partial x \partial y} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x \partial y} \right) \Bigg\} \\
 & + Q_{66} \Bigg\{ H_1 \left(\frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) + H_2 \left(2 \frac{\partial u_0^*}{\partial x} + \frac{\partial^2 \theta_z}{\partial x^2} \right) \\
 & + H_3 \left(3 \frac{\partial \theta_x^*}{\partial x} + \frac{\partial^2 w_0^*}{\partial x^2} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x^2} \right) \Bigg\} \\
 & + Q_{55} \Bigg\{ H_1 \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \right) + H_2 \left(2 \frac{\partial v_0^*}{\partial y} + \frac{\partial^2 \theta_z}{\partial y^2} \right) \\
 & + H_3 \left(3 \frac{\partial \theta_y^*}{\partial y} + \frac{\partial^2 w_0^*}{\partial y^2} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial y^2} \right) \Bigg\}
 \end{aligned}$$

$$\begin{aligned}
& + Q_{56} \left\{ H_1 \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial^2 w_0}{\partial x \partial y} \right) + H_2 \left(2 \frac{\partial u_0^*}{\partial y} + \frac{\partial^2 \theta_z}{\partial x \partial y} \right) \right. \\
& \quad \left. + H_3 \left(3 \frac{\partial \theta_x^*}{\partial y} + \frac{\partial^2 w_0^*}{\partial x \partial y} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x \partial y} \right) \right\} \Bigg\} \\
& - \left\{ Q_{65} \left\{ H_1 \left(\frac{\partial \theta_y}{\partial x} + \frac{\partial^2 w_0}{\partial x \partial y} \right) + H_2 \left(2 \frac{\partial v_0^*}{\partial x} + \frac{\partial^2 \theta_z}{\partial x \partial y} \right) \right. \right. \\
& \quad \left. + H_3 \left(3 \frac{\partial \theta_y^*}{\partial x} + \frac{\partial^2 w_0^*}{\partial x \partial y} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x \partial y} \right) \right\} \Bigg\} \\
& + Q_{66} \left\{ H_1 \left(\frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2} \right) + H_2 \left(2 \frac{\partial u_0^*}{\partial x} + \frac{\partial^2 \theta_z}{\partial x^2} \right) \right. \\
& \quad \left. + H_3 \left(3 \frac{\partial \theta_x^*}{\partial x} + \frac{\partial^2 w_0^*}{\partial x^2} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x^2} \right) \right\} \Bigg\} \\
& + Q_{55} \left\{ H_1 \left(\frac{\partial \theta_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2} \right) + H_2 \left(2 \frac{\partial v_0^*}{\partial y} + \frac{\partial^2 \theta_z}{\partial y^2} \right) \right. \\
& \quad \left. + H_3 \left(3 \frac{\partial \theta_y^*}{\partial y} + \frac{\partial^2 w_0^*}{\partial y^2} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial y^2} \right) \right\} \Bigg\} \\
& + Q_{56} \left\{ H_1 \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial^2 w_0}{\partial x \partial y} \right) + H_2 \left(2 \frac{\partial u_0^*}{\partial y} + \frac{\partial^2 \theta_z}{\partial x \partial y} \right) \right. \\
& \quad \left. + H_3 \left(3 \frac{\partial \theta_x^*}{\partial y} + \frac{\partial^2 w_0^*}{\partial x \partial y} \right) + H_4 \left(\frac{\partial^2 \theta_z^*}{\partial x \partial y} \right) \right\} \Bigg\} \Bigg\} \tag{14}
\end{aligned}$$

3. NUMERICAL RESULTS AND DISCUSSIONS

Two numerical examples are solved and presented.

3.1. Material Properties

The following sets of material constants are used.

Material 1

$E_1 = 25 \times 10^6$ ps (276 GPa) $E_2 = E_3 = 1 \times 10^6$ ps (6.895 Gpa) $G_{12} = G_{13} = 0.5 \times 10^6$ ps (3.45 GPa) $G_{23} = 0.2 \times 10^6$ ps (4.12 GPa)

$$\nu_{12} = \nu_{23} = \nu_{13} = 0.25$$

Following non-dimensional forms are used to present the results.

$$\begin{aligned}
\bar{\sigma}_x &= \sigma_x \left(\frac{h^2}{P_0 a^2} \right) \bar{\sigma}_y = \sigma_y \left(\frac{h^2}{P_0 a^2} \right) \bar{\tau}_{xy} = \tau_{xy} \left(\frac{h^2}{P_0 a^2} \right) \\
\bar{\tau}_{xz} &= \tau_{xz} \left(\frac{h}{P_0 a} \right) \bar{\tau}_{yz} = \tau_{yz} \left(\frac{h}{P_0 a} \right) \bar{\sigma}_z = \sigma_z \left(\frac{h^2}{P_0 a^2} \right)
\end{aligned}$$

3.2. Numerical Examples

Example 1: A simply supported, square composite plate is considered. Plate has two layers. Plate is made up of antisymmetric angle ply ($\theta / -\theta$). Plate is subjected to sinusoidal transverse load. The layers are of equal thickness. Material Set 1 is used. The nondimensionalized maximum in plane stresses ($\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$) and transverse stresses ($\bar{\tau}_{xz}, \bar{\tau}_{yz}$) for plates with varying a/h ratios and fibre orientations are given in Table 1 to Table 3. The solutions using 3D elasticity theory [14] and other shear deformation theories [15,16] already reported in literature are used for comparison. In thick plates, it has been found that for all the fibre orientations considered the inplane and transverse stress values obtained using the present model are very close to 3D elasticity values. For plate with a/h ratio equal to 4 and fibre orientation equal to 30° the values of $\bar{\tau}_{xz}(a/2, 0)$ predicted by the present model, RSDT and HSDT-5 are respectively 10.34%, 19.08% and 35.73% higher as compared to 3D elasticity solutions. Similarly for the same plate the values of $\bar{\tau}_{yz}(0, a/2)$ predicted by the present and other models are respectively 8.06%, 14.98% and 28.34% higher as compared to the solutions reported using 3D elasticity theory. Similar patterns in the percentage difference between the models can be observed in all other stress values also for all the fibre orientations considered. Figures 2 show the through the thickness variation of nondimensionalized transverse stress $\bar{\tau}_{xz}, \bar{\tau}_{yz}$ at $(a/2, 0)$ for plates with a/h ratios 4 and fibre orientation equal to 15° . Figures 3 show the through the thickness variation of nondimensionalized transverse stress at $(a/2, 0)$ for plates with a/h ratios 10 and fibre orientation equal to 15° . The plot of through the thickness variation of transverse normal stress ($\bar{\sigma}_z$) for plates with a/h ratios 4 and 10 is given in Figure 4. Table 5 gives comparison of transverse stresses with the present model for 4 layer and 8 layer composite plate.

Figure 2 Variation of transverse shear $\bar{\tau}_{xz}$ stress ($\bar{\tau}_{xz}$) **Figure 3** Variation of transverse shear stress ($\bar{\tau}_{xz}$)

Example 2: A simply supported, rectangular plate ($b=3a$) composite plate is used. Type of plate, material of plate, Number of layers and their thickness along with loading on the plate is same as given in Example 1. The variation of nondimensionalized maximum stresses for plates with varying a/h ratios is considered. a/h ratios 4 and 10 with fibre angles $\theta = 15^\circ, 30^\circ$

and 45° are tabulated in Table 4. Comparison of results obtained using the present model with the 3D elasticity and other models solutions already available [14, 15, 16] clearly indicates that for all the parameters considered in the present investigation, the present model predicts all the stresses very much closer to 3D elasticity solutions. For a particular value of fibre orientation $\theta=30^\circ$ and a/h value equal to 4, the value of $\bar{\tau}_{xz}(a/2,0)$ obtained using the present model, RSDT [15] and HSDT-5 [16] are respectively 14.0%, 25.82% and 56.89% higher and the values of $\bar{\tau}_{yz}(0,b/2)$ are respectively 4.5%, 6.0% and 9.98% higher than the 3D elasticity solutions.

Figure 4 Variation of transverse normal stress ($\bar{\sigma}_z$)

Table 1 Non-dimensionalised maximum stresses in simply supported, two layered (15/-15) angle ply, square laminate

a/h	Theory	$\bar{\sigma}_x(a/2,a/2)$	$\bar{\sigma}_y(a/2,a/2)$	$\bar{\tau}_{xy}(a/2,a/2)$	$\bar{\tau}_{xz}(a/2,0)$	$\bar{\tau}_{yz}(a/2,0)$	$\bar{\tau}_{xz}(0,a/2)$	$\bar{\tau}_{yz}(0,a/2)$
4	Present	0.6559	0.1201	0.1451	0.0907	0.0834	0.2991	0.0403
	3 D-Elasticity [16]	0.6645	0.1135	0.1489	0.0841	0.0960	0.3145	0.0399
	RSDT* [17]	0.5884	0.0980	0.1305	0.1034	0.0929	0.3186	0.0428
	HSDT-5# [17]	0.7032	0.1064	0.1633	0.1282	0.0899	0.3061	0.0465
10	Present	0.5606	0.0810	0.1243	0.1001	0.0804	0.3419	0.0412
	3 D-Elasticity [16]	0.5633	0.0806	0.1244	0.0991	0.0814	0.3421	0.0412
	RSDT* [17]	0.5514	0.0778	0.1216	0.1039	0.0807	0.3427	0.0418
	HSDT-5# [17]	0.5680	0.0792	0.1264	0.1079	0.0800	0.3133	0.0424
20	Present	0.5463	0.0748	0.1208	0.1015	0.0767	0.3477	0.0411
	RSDT* [17]	0.5456	0.0741	0.1201	0.1033	0.0780	0.3478	0.0417
	HSDT-5# [17]	0.5497	0.0744	0.1213	0.1043	0.0725	0.3152	0.0418
100	Present	0.5415	0.0724	0.1196	0.1019	0.0762	0.3500	0.0412
	RSDT* [17]	0.5437	0.0728	0.1196	0.1030	0.0771	0.3496	0.0417
	HSDT-5# [17]	0.5439	0.0728	0.1196	0.1030	0.0714	0.3159	0.0417

*Refined Shear Deformation Theory (RSDT)

#Higher Order Shear Deformation Theory with 5 degrees of freedom (HSDT-5)

Higher Order Refined Computational Model using Post Processing for the Transverse Shear and Transverse Normal Stress in Composite Plates

Table 2 Non-dimensionalised maximum stresses in simply supported, two layered (30/-30) square laminate

a/h	Theory	$\bar{\sigma}_x(a/2,a/2)$	$\bar{\sigma}_y(a/2,a/2)$	$\bar{\tau}_{xy}(a/2,a/2)$	$\bar{\tau}_{xz}(a/2,0)$	$\bar{\tau}_{yz}(a/2,0)$	$\bar{\tau}_{xz}(0,a/2)$	$\bar{\tau}_{yz}(0,a/2)$
4	Present	0.4747	0.2067	0.2181	0.1504	0.1481	0.2509	0.0938
	3 D-Elasticity [16]	0.4945	0.2089	0.2335	0.1363	0.1546	0.2561	0.0868
	RSDT* [17]	0.4239	0.1763	0.1999	0.1623	0.1584	0.2673	0.0998
	HSDT-5# [17]	0.5321	0.2104	0.2684	0.1850	0.1581	0.2666	0.1114
10	Present	0.4154	0.1682	0.1989	0.1666	0.1526	0.2634	0.1015
	3 D-Elasticity [16]	0.4204	0.1696	0.2018	0.1639	0.1570	0.2699	0.1002
	RSDT* [17]	0.4090	0.1639	0.1968	0.1703	0.1578	0.2723	0.1035
	HSDT-5# [17]	0.4267	0.1695	0.2081	0.1741	0.1547	0.2664	0.1055
20	Present	0.4074	0.1626	0.1966	0.1694	0.1532	0.2654	0.1029
	RSDT* [17]	0.4070	0.1620	0.1964	0.1715	0.1576	0.2732	0.1041
	HSDT-5# [17]	0.4114	0.1634	0.1993	0.1725	0.1541	0.2663	0.1046
100	Present	0.4049	0.1608	0.1959	0.1703	0.1534	0.2661	0.1034
	RSDT* [17]	0.4063	0.1614	0.1963	0.1719	0.1576	0.2735	0.1043
	HSDT-5# [17]	0.4065	0.1615	0.1964	0.1720	0.1540	0.2663	0.1043

*Refined Shear Deformation Theory (RSDT)

#Higher Order Shear Deformation Theory with 5 degrees of freedom (HSDT-5)

Table 3 Non-dimensionalised maximum stresses in simply supported, two layered (45/-45) angle ply, square laminate

a/h	Theory	$\bar{\sigma}_x(a/2,a/2)$	$\bar{\sigma}_y(a/2,a/2)$	$\bar{\tau}_{xy}(a/2,a/2)$	$\bar{\tau}_{xz}(a/2,0)$	$\bar{\tau}_{yz}(a/2,0)$	$\bar{\tau}_{xz}(0,a/2)$	$\bar{\tau}_{yz}(0,a/2)$
4	Present	0.2976	0.2976	0.2144	0.1357	0.2029	0.2029	0.1357
	3 D-Elasticity [16]	0.3092	0.3092	0.2327	0.1239	0.2062	0.2062	0.1239
	RSDT* [17]	0.2588	0.2588	0.1956	0.1452	0.2155	0.2155	0.1452
	HSDT-5# [17]	0.3392	0.3392	0.2839	0.1618	0.2160	0.2160	0.1618
10	Present	0.2562	0.2562	0.1994	0.1483	0.2119	0.2119	0.1483
	3 D-Elasticity [16]	0.2594	0.2594	0.2029	0.1461	0.2154	0.2154	0.1461
	RSDT* [17]	0.2512	0.2512	0.1975	0.1513	0.2175	0.2175	0.1513
	HSDT-5# [17]	0.2646	0.2646	0.2123	0.1542	0.2145	0.2145	0.1542
20	Present	0.2505	0.2505	0.1978	0.1506	0.2133	0.2133	0.1506
	RSDT* [17]	0.2501	0.2498	0.1981	0.1523	0.2178	0.2178	0.1523
	HSDT-5# [17]	0.2535	0.2535	0.2016	0.1530	0.2144	0.2144	0.1530
100	Present	0.2487	0.2487	0.1974	0.1513	0.2138	0.2138	0.1513
	RSDT* [17]	0.2498	0.2498	0.1981	0.1526	0.2179	0.2179	0.1526
	HSDT-5# [17]	0.2499	0.2499	0.1982	0.1527	0.2143	0.2143	0.1527

*Refined Shear Deformation Theory (RSDT)

#Higher Order Shear Deformation Theory with 5 degrees of freedom (HSDT-5)

Table 4 Non-dimensionalised maximum stresses in simply supported, two layered (θ / $-\theta$) angle ply, Rectangular laminate

θ	a/h	Theory	$\bar{\sigma}_x(a/2, a/2)$	$\bar{\sigma}_y(a/2, a/2)$	$\bar{\tau}_{xy}(a/2, a/2)$	$\bar{\tau}_{xz}(a/2, 0)$	$\bar{\tau}_{yz}(a/2, 0)$	$\bar{\tau}_{xz}(0, a/2)$	$\bar{\tau}_{yz}(0, a/2)$
15°	4	Present	0.9562	0.0969	0.1980	0.0511	0.0337	0.3920	0.0609
		3 D-Elasticity [16]	0.9581	0.0911	0.2018	0.0457	0.0365	0.3988	0.0591
		RSDT* [17]	0.8678	0.0762	0.1805	0.0575	0.0366	0.4021	0.0627
		HSDT-5# [17]	1.0111	0.0873	0.2202	0.0717	0.0375	0.3922	0.0650
	10	Present	0.8085	0.0713	0.1641	0.0570	0.0328	0.4005	0.0598
		RSDT* [17]	0.7973	0.0676	0.1615	0.0583	0.0338	0.4091	0.0602
30°	4	Present	0.9067	0.2994	0.4113	0.1141	0.0967	0.3806	0.1496
		3 D-Elasticity [16]	0.9282	0.3000	0.4260	0.1029	0.0960	0.3885	0.1427
		RSDT* [17]	0.8514	0.2689	0.3906	0.1225	0.1022	0.3956	0.1532
		HSDT-5# [17]	0.9720	0.3087	0.4642	0.1389	0.1060	0.3905	0.1592
	10	Present	0.8191	0.2614	0.3761	0.1336	0.1069	0.3858	0.1519
		RSDT* [17]	0.8120	0.2569	0.3737	0.1358	0.1095	0.3958	0.1530
45°	4	Present	0.7314	0.5862	0.5020	0.1461	0.1976	0.3445	0.2097
		3 D-Elasticity [16]	0.7427	0.5943	0.5182	0.1369	0.1956	0.3564	0.2009
		RSDT* [17]	0.6949	0.5512	0.4861	0.1531	0.2050	0.3609	0.2160
		HSDT-5# [17]	0.7731	0.6297	0.5702	0.1646	0.2076	0.3541	0.2259
	10	Present	0.6624	0.5352	0.4695	0.1650	0.2200	0.3465	0.2164
		RSDT* [17]	0.6586	0.5313	0.4687	0.1668	0.2230	0.3573	0.2182
		HSDT-5# [17]	0.6719	0.5441	0.4824	0.1688	0.2211	0.3487	0.2199

Table 5 Non-dimensionalised maximum stresses in a four layered (θ / $-\theta$)₂ and six (θ / $-\theta$)₃ layered square laminate plate

θ	Theory	$\bar{w}$	$\bar{\sigma}_x(a/2, a/2)$	$\bar{\sigma}_y(a/2, a/2)$	$\bar{\tau}_{xy}(0, 0)$	$\bar{\tau}_{xz}(0, a/2)$	$\bar{\tau}_{yz}(a/2, 0)$
Four Layered							
15	Present	0.5957	0.4396	0.0621	0.0658	0.4124	0.1069
	3 D-Elasticity [16]	0.6150	0.4453	0.0615	0.0674	0.4116	0.1074
30	Present	0.5136	0.2710	0.1078	0.1262	0.3447	0.2027
	3 D-Elasticity [16]	0.5619	0.2833	0.1115	0.1331	0.3381	0.1995
45	Present	0.4826	0.1655	0.1655	0.1533	0.2756	0.2756
	3 D-Elasticity [16]	0.5430	0.1749	0.1749	0.1642	0.2684	0.2684
Six Layered							
15	Present	0.5649	0.4277	0.0595	0.0680	0.3669	0.0961
	3 D-Elasticity [16]	0.5781	0.4295	0.0584	0.0689	0.3696	0.0981
30	Present	0.4726	0.2605	0.1025	0.1252	0.2924	0.1723
	3 D-Elasticity [16]	0.5087	0.2662	0.1037	0.1290	0.2957	0.1749
45	Present	0.4420	0.1596	0.1596	0.1487	0.2323	0.2323
	3 D-Elasticity [16]	0.4890	0.1640	0.1640	0.1548	0.2355	0.2355

4. CONCLUSION

The results obtained are in very good agreement with 3D elasticity solutions. Hence ESL theory with higher degrees of freedom with post processing technique can predict the local behavior i.e. transverse stresses accurately.

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