NECESSARY OPTIMALITY CONDITIONS OF CONTROL OF STOCHASTIC COMPOUND DYNAMIC SYSTEM IN CASE OF FULL INFORMATION ABOUT STATE VECTOR

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Анотація. Стохастичні складені динамічні системи (СДС) відносяться до складних технічних систем, які створюються завдяки застосуванню методів прецизійної механіки в поєднанні з сучасними телекомунікаційними та комп'ютерними технологіями. Невизначеність у цих СДС виникає під впливом зовнішніх та внутрішніх стохастичних збурень. Складові елементи СДС поєднуються в єдину систему тому, що ці елементи виконують єдине комплексне завдання. Обмін інформацією відбувається безпроводово, механічний зв'язок між елементами СДС відсутній. Як різновиди таких складних технічних систем у статті розглядаються групи безпілотних літальних апаратів (БПЛА), які оснащені сенсорами або мультисенсорами і можуть утворювати мобільні сенсорні мережі. Траєкторії руху окремих елементів мобільної сенсорної мережі – це траєкторії, що формуються під впливом стохастичних збурень. Цей факт означає, що за фізичним змістом мобільна сенсорна мережа може бути класифікована як стохастична складена динамічна система і для математичного опису та оптимізації руху цієї системи адекватним є застосування моделей та методів оптимізації стохастичних СДС. Моделью руху СДС вважається траєкторія з розгалуженням або, іншими словами, розгалужена траєкторія. Стохастична математична модель руху мобільної сенсорної мережі, яка виконує поєднану єдиним задумом задачу обстеження зони надзвичайної ситуації, може бути класифікована як модель руху стохастичної складеної динамічної системи. Такий підхід до побудови математичної моделі мобільної сенсорної мережі є адекватним фізичним умовам її функціонування у зоні стихійного лиха природного або антропогенного походження. Дана стаття присвячена розв'язанню теоретичного завдання, яке пов'язано з формулюванням і доведенням необхідних умов оптимального керування стохастичною СДС, що пересувається по розгалуженій траєкторії із довільною схемою розгалужень.

Ключові слова: оптимальне керування, стохастичні системи, складені динамічні системи, траєкторія з розгалуженням, необхідні умови оптимальності.

Аннотация. Стохастические составные динамические системы относятся к сложным техническим системам, которые создаются благодаря использованию методов прецизионной механики в сочетании с современными телекоммуникационными и компьютерными технологиями. Неопределенность в этих СДС возникает под влиянием внешних и внутренних стохастических возмущений. Составляющие элементы СДС объединяются в единую систему потому, что эти элементы выполняют единую комплексную задачу. Обмен информацией происходит беспроводово, механическая связь между элементами СДС отсутствует. В качестве разновидностей таких сложных технических систем в статье рассматриваются группы беспилотных летательных аппаратов (БПЛА), которые оснащены сенсорами или мультисенсорами и могут образовывать мобильные сенсорные сети. Траектории движения отдельных элементов мобильной сенсорной сети – это траектории, которые формируются под влиянием стохастических возмущений. Этот факт означает, что по физическому смыслу мобильная сенсорная сеть может быть классифицирована как стохастическая составная динамическая система и для математического описания и оптимизации движения этой системы адекватным является применение моделей и методов оптимизации стохастических СДС. Моделью движения СДС считается траектория с разветвлением или, иными словами, разветвляющаяся траектория. Стохастическая математическая модель движения мобильной сенсорной сети, выполняющей соответствующую единому замыслу задачу обследования зоны чрезвычайной ситуации, может быть классифицирована как модель движения стохастической составной динамической системы. Такой подход к построению математической модели мобильной сенсорной сети является адекватным физическим условием ее функционирования в зоне стихийного бед-

ствия природного или антропогенного происхождения. Данная статья посвящена решению теоретического задания, которое связано с формулировкой и доказательством необходимых условий оптимального управления стохастической СДС, которая движется по разветвленной траектории с произвольной схемой разветвлений.

Ключевые слова: оптимальное управление, стохастические системы, составные динамические системы, разветвленная траектория, необходимые условия оптимальности

Abstract. *Stochastic Compound Dynamic Systems (CDS) are complex technical systems that are created through the use of precision mechanics in combination with modern telecommunications and computer technologies. Incertitude in these CDS shows up under the influence of external and internal stochastic perturbations. The constituent elements of CDS are combined into a single system because these elements perform a single complex mission. The information exchange is wireless, there is no mechanical connection between the elements of the CDS. The paper considers groups of unmanned aerial vehicles (UAVs), which are equipped with sensors or multisensors that are able to perform a mobile sensor network. The trajectories of individual elements of the mobile sensor network are trajectories formed under the influence of stochastic perturbations. This fact means that the nature of the mobile sensor network can be classified as a stochastic compound dynamic system and for the mathematical description and optimization of the movement of this system is adequate to use models and methods for optimizing stochastic CDS. The model of CDS motion is considered to be a branching trajectory or, as they say, a branched trajectory. A stochastic mathematical model of the motion of a mobile sensor network, which performs the combined mission of surveying an emergency zone, can be classified as a model of the motion of a stochastic compound dynamic system. This approach is an adequate for mathematical model creation to the mobile sensor network physical condition, for its operation in the zone of natural disaster by natural or anthropogenic origin. This paper is devoted to solving a theoretical problem related to the formulation and proof of the necessary conditions for stochastic CDS moving optimal control along a branched trajectory with an arbitrary branching scheme.*

Keywords: *optimal control, stochastic systems, compound dynamic systems, branching trajectory, necessary conditions for optimality.*

DOI: 10.34121/1028-9763-2020-4-136-147

1. Introduction

The efficiency of CDS depends on the operative optimal choice of spatial coordinates and time points at which structural transformations of CDS take place, as well as on optimal control of constituent elements during their movement on the branches of the trajectory in time intervals between successive structural transformations.

From the general theoretical point of view, the control tasks of CDS are reduced to the control problems of discontinuous systems [1–8]. Mathematical models of discontinuous systems are described either by ordinary differential equations or by stochastic differential equations with piecewise continuous right-hand sides.

But in these works, the problem of control of such systems was formulated as the problem of control of discontinuous systems, when the priority element was allocated and the application of the theory of discontinuous systems was considered in relation to it. The trajectories of such systems were designed in advance. The methods of optimization of branched trajectories considered in these works are developed for a narrow range of deterministic problems, which does not allow to formulate conditions for optimality of optimality of branched CDS trajectories with different branching schemes. This article is aimed at developing universal conditions for the optimality of stochastic CDS motion. These conditions will form an applied theoretical basis for the construction of algorithms for operational trajectory synthesis and optimal control of CDS for each specific case of branches.

2. Problem Statement

We will use the concepts of the subsystem, block, branch, and instant of time of the structural transformation introduced in [1–5]. We extend these concepts to the stochastic CDS that move along an arbitrarily branching stochastic path.

We represent a stochastic CDS in the form of a stochastic discontinuous system with a variable size of the state vectors, control, noise exciting the system, and points of discontinuities that are optimized on average.

The movement of a subsystem (unit) along the path branches is described by the stochastic differential equation [6]

$$dx(t) = f(x, u; y, v; t)dt + \sigma(x, u; y, v; t)dw(t), \quad t \in [t_0, t_f], \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \Omega \subset \mathbb{R}^m$; and v – the phase coordinates and controls of other subsystems (units) of the stochastic CDS, affecting the system (1); f – size vector function $n \times 1$; σ – size matrix function $n \times l$; $w(t)$ – l -dimensional separable Wiener process [7]; t_0, t_f – moments of start and end of the subsystem (unit) motion along the considered path branch.

On the trajectory of subsystem (1), the scalar constraints of the form:

$$E[g_{oi}(x(t_0), y(t_0), t_0)] + E[g_{fi}(x(t_f), y(t_f), t_f)] \begin{cases} \leq 0, i = \overline{1, k_g}, \\ = 0, i = \overline{k_g + 1, n_g} \end{cases}, \quad (2)$$

$$E[q_i(x(t), u(t), y(t), v(t), t)] \begin{cases} \leq 0, i = \overline{1, k_q}, \\ = 0, i = \overline{k_q + 1, n_q} \end{cases}, \quad t \in [t_0, t_f], \quad (3)$$

where $E[\dots]$ – symbol of an operation that requires calculating the expectation of bracketed expression.

The control efficiency of stochastic CDS is estimated by the minimum of functionality [8]

$$P = E[\Pi + \rho_\Sigma] \rightarrow \min, \quad (4)$$

where Π – the terminal component of the criterion, which depends on the phase coordinates of subsystems (units) at time instants of structural transformations of CDS and time instants themselves; ρ_Σ – the integral component of the criterion, consisting of a sum of partial integral components similar to those used in [1]. The statement of the problem of optimizing the stochastic CDS takes the form (1)–(4).

According to this statement, a search is made for the average optimal controls for the stochastic subsystems (units), as well for the optimal average phase coordinates at the time instants of the structural transformations of the CDS and the time instants themselves.

3. Problem Solution

The problem (1)–(4) is solved in three stages in the same manner as [1–5].

At the first stage, a transition from the stochastic CDS to the discontinuous stochastic system with a variable size of control vectors, state, and noise exciting the system is made.

Then, at the second stage, the discontinuous stochastic system is optimized with optimized discontinuity points. And finally, at the third stage, there is a return to the terms of the original statement of the problem.

This means that the result obtained at the second stage as applied to a discontinuous stochastic system should be reformulated to the form corresponding to the optimization problem of the stochastic CDS that moves along a branching path.

Having saved for the stochastic CDS the method of converting CDS to a discontinuous dynamical system described in [1], we take into account that in the stochastic problem (1)–(4), the sizes of vectors are changing at the points of discontinuity, not only of the state and control vectors, but also of the vector of noise exciting the system.

As a result, we arrive at the following statement of the problem of optimizing the discontinuous stochastic system in case of full state information:

$$J(U, T) = \sum_{i=1}^N \left\{ E[\mathfrak{S}_i(\cdot, X, t_{i-1})] + E[S_i(\cdot, X, t_i)] + \int_{t_{i-1}}^{t_i} E[\Phi_i(\cdot, X, U, t)] dt \right\} \rightarrow \min_{U, T}, \quad (5)$$

$$J_j^G = \sum_{i=1}^N E[\mathfrak{G}_j(\cdot, X, t_{i-1})] + E[G_j(\cdot, X, t_i)] \begin{cases} \leq 0, j = \overline{1, K_G}, \\ = 0, j = \overline{K_G + 1, N_G} \end{cases}, \quad (6)$$

$${}_i J_j^\Psi = {}_i \Psi_j(\cdot, X(t_i), t_i) - {}_{i+1} X_j(t_i) = 0 \quad (j = \overline{1, n_{\Sigma i+1}}; i = \overline{1, N-1}), \quad (7)$$

$${}_i J_j^Q = E[{}_i Q_j(\cdot, X(t), {}_i U(t), t)] \begin{cases} \leq 0, j = \overline{1, K_{Q_i}}, \\ = 0, j = \overline{K_{Q_i} + 1, N_{Q_i}} \end{cases}, \quad (8)$$

$$\begin{aligned} d_i X(t) &= {}_i F(\cdot, X, {}_i U, t) dt + {}_i \mathfrak{D}(\cdot, X, {}_i U, t) d_i W(t), \\ t &\in (t_{i-1}, t_i), {}_1 X(t_0) = {}_1 X_0, {}_i X(t) \in \mathbb{R}^{n_{\Sigma i}}, \end{aligned} \quad (9)$$

$${}_i U(t) \in \Omega_i \subset \mathbb{R}^{m_{\Sigma i}}, \quad (10)$$

here ${}_i U(\cdot)$ – piecewise continuous, $i = \overline{1, N}$.

Here t – time, $T = (t_0, t_1, \dots, t_N)$, $t_0, t_1, \dots, t_N$ – time moments satisfying the inequalities $t_0 < t_1 < \dots < t_N$; ${}_i X$ – $n_{\Sigma i}$ –dimensional extended vector-function of phase state of a discontinuous stochastic system, continuous along t at each time interval $(t_{i-1}, t_i]$ and taking at every $t \in (t_{i-1}, t_i]$ values ${}_i X(t) \in \mathbb{R}^{n_{\Sigma i}} = \{X : -\infty < X < \infty\}$; ${}_i U$ – extended vector of control actions corresponding to the i -th time interval between structural transformations of stochastic CDS, dimensions $m_{\Sigma i}$; $U = ({}_1 U, \dots, {}_N U)$; ${}_i W(t)$ – $\nu_{\Sigma i}$ – dimensional extended vector of the standard Wiener process (${}_i W(t_{i-1}) = 0$, $E[{}_i W(t)] = 0$, $E[{}_i W(t) {}_i W^T(t)] = It$, I – identity matrix, $t \in (t_{i-1}, t_i]$, $i = \overline{1, N}$, [10–12]; ${}_i F(\cdot, X, {}_i U, t)$ – $n_{\Sigma i}$ – dimensional vector-function; ${}_i \mathfrak{D}(\cdot, X, {}_i U(\cdot, X, t), t)$ – matrix size function $n_{\Sigma i} \times \nu_{\Sigma i}$; ${}_i F(\cdot, X, {}_i U(\cdot, X, t), t)$ and ${}_i \mathfrak{D}(\cdot, X, {}_i U(\cdot, X, t), t)$ – are measured by t and ${}_i X$ at $t \in (t_{i-1}, t_i]$ and satisfy the conditions

$$\begin{aligned} &\|{}_i F(\cdot, X, {}_i U(\cdot, X, t), t) - {}_i F(\cdot, Y, {}_i U(\cdot, Y, t), t)\| + \\ &+ \|{}_i \mathfrak{D}(\cdot, X, {}_i U(\cdot, X, t), t) - {}_i \mathfrak{D}(\cdot, Y, {}_i U(\cdot, Y, t), t)\| \leq K \|{}_i X - {}_i Y\|, t \in (t_{i-1}, t_i], \end{aligned}$$

where K – corresponding constant, ${}_iX(t_{i-1})$ – random variables independent of $d_iW(t), t \in (t_{i-1}, t_i]$.

Under the above conditions, equation (9) has a unique solution with probability 1 continuous on the interval $(t_{i-1}, t_i]$ ($i = \overline{1, N}$) decision, the stochastic integral of the equation will be a diffusion Markov random process [9–14].

One-dimensional probability density $p_i({}_iX, t)$ of vector process ${}_iX(t)$ satisfies the Fokker-Planck-Kolmogorov equation [12]

$$\begin{aligned} \frac{\partial p_i({}_iX, t)}{\partial t} &= -\frac{\partial^T}{\partial {}_iX} [{}_iF({}_iX, {}_iU, t)p_i({}_iX, t)] + \\ &+ \frac{1}{2} tr \left\{ \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} [{}_i\mathfrak{D}({}_iX, {}_iU, t){}_i\mathfrak{D}^T({}_iX, {}_iU, t)p_i({}_iX, t)] \right\} = \\ &= \mathfrak{B}(p_i, {}_iU, t, t \in (t_{i-1}, t_i], i = \overline{1, N}) \end{aligned} \quad (11)$$

with a given distribution density at the time moment t_0 of the start of interval $[t_0, t_1]$

$$p_1({}_1X, t)|_{t=t_0} = p_1({}_1X, t_0) \geq 0, \int_{\Xi_1} p_1({}_1X, t_0) d_1X = 1$$

and with subsequent holding the condition

$${}_{i+1}X_j(t_i) = {}_i\Psi_j({}_iX, t_i) \quad (i = \overline{1, N-1}; j = \overline{1, n_{\Sigma i+1}}) \quad (12)$$

at discontinuity points. In equation (11) the following designations are used:

$\frac{\partial}{\partial {}_iX} = \left(\frac{\partial}{\partial {}_iX_1}, \dots, \frac{\partial}{\partial {}_iX_{n_{\Sigma i}}} \right)$ – gradient vector in $n_{\Sigma i}$ – dimensional space and $\frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX}$ – square matrix of double differentiation operators with respect to the components of the vector ${}_iX$. Note that J_j^G ($j = \overline{1, N_G}$), ${}_iJ_j^Q$ ($i = \overline{1, N}; j = \overline{1, N_{Q_i}}$) – continuously differentiable bounded functional; ${}_i\Psi_j({}_iX, t_i)$ ($i = \overline{1, N-1}; j = \overline{1, n_{\Sigma i+1}}$) – continuously differentiable functions; $J(U, T)$, J_j^G ($j = \overline{1, K_G}$), ${}_iJ_j^Q$ ($i = \overline{1, N}; j = \overline{1, K_{Q_i}}$) – convex bounded functionals.

The expectation in the relations (5), (6), (8) are calculated by the formula

$$E[\varphi] = \int_{\Xi_i} \varphi p_i({}_iX, \tau) d_iX, \quad (13)$$

where by φ is meant any of the functions, standing in square brackets under the sign of the operator, the mathematical expectation in the specified expressions, and τ , according to the expression for φ , understood either the current point in time t , either t_i ; $\Xi_i \subset \mathbb{R}^{n_{\Sigma i}}$ – the implementation of the phase vector states ${}_iX$.

4. The Principle of Optimality on Average: Necessary Conditions for the Optimal Control of the Stochastic CDS with an Arbitrary Branching Scheme of the Trajectory

The stochastic problem (5)–(10) can be considered as the deterministic task of optimizing the functional (5) taking into account the intermediate (6) and continuously acting (8) constraints in the presence of differential coupling (11), (12) [13].

Theorem. For optimality of allowed control $V = (U, T)$ of problem (5)–(10) it is necessary the existence of vectors $\alpha = [\alpha_1, \dots, \alpha_{N_G}]^T$, ${}_i\beta(t) = [{}_i\beta_1(t), \dots, {}_i\beta_{N_{Qi}}(t)]^T$ ($i = \overline{1, N}$), ${}_i\nu = [{}_i\nu_1, \dots, {}_i\nu_{n_{\Sigma i}}]^T$ ($i = \overline{1, N-1}$) and functions of random argument and time $\lambda_i({}_iX, t)$ ($i = \overline{1, N}$), determined from the equations.

$$-\lambda_{it} = -\frac{\partial \lambda_i({}_iX, t)}{\partial t} = \Phi_i({}_iX, {}_i\hat{U}, t) + {}_i\beta^T(t) {}_iQ({}_iX, {}_i\hat{U}, t) + \frac{\partial^T \lambda_i({}_iX, t)}{\partial t} {}_iF({}_iX, {}_i\hat{U}, t) + \frac{1}{2} \text{tr} [{}_i\mathfrak{D}({}_iX, {}_i\hat{U}, t) {}_i\mathfrak{D}^T({}_iX, {}_i\hat{U}, t) \times \times \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} \lambda_i({}_iX, t)], \quad t \in (\hat{t}_{i-1}, \hat{t}_i] (i = \overline{1, N}), \quad (14)$$

$$S_i({}_iX(\hat{t}_i), \hat{t}_i) + \alpha^T {}_iG({}_iX(\hat{t}_i), \hat{t}_i) + {}_i\nu^T {}_i\Psi({}_iX(\hat{t}_i), \hat{t}_i) - \lambda_i({}_iX(\hat{t}_i), \hat{t}_i) = 0 (i = \overline{1, N-1}), \quad (15)$$

$$\lambda_{i+1}({}_{i+1}X(\hat{t}_i), \hat{t}_i) + \mathfrak{S}_{i+1}({}_{i+1}X(\hat{t}_i), \hat{t}_i) + \alpha^T {}_{i+1}\mathfrak{G}({}_{i+1}X(\hat{t}_i), \hat{t}_i) - {}_i\nu^T {}_{i+1}X(\hat{t}_i) = 0 (i = \overline{1, N-1}), \quad (16)$$

$$\mathfrak{S}_1({}_1X(\hat{t}_0), \hat{t}_0) + \alpha^T {}_1\mathfrak{G}({}_1X(\hat{t}_0), \hat{t}_0) + \lambda_1({}_1X(\hat{t}_0), \hat{t}_0) = 0, \quad (17)$$

$$S_N({}_NX(\hat{t}_N), \hat{t}_N) + \alpha^T {}_NG({}_NX(\hat{t}_N), \hat{t}_N) - \lambda_N({}_NX(\hat{t}_N), \hat{t}_N) = 0. \quad (18)$$

Such that the following conditions are true: non-triviality, non-negativity, complementing nonrigidity for vectors α , ${}_i\beta(t)$ ($i = \overline{1, N}$), ${}_i\nu$ ($i = \overline{1, N-1}$)

$$\sum_{j=1}^{N_G} \alpha_j + \sum_{i=1}^{N-1} \sum_{j=1}^{n_{\Sigma i}} {}_i\nu_j + \sum_{i=1}^N \sum_{j=1}^{N_{Qi}} \int_{\hat{t}_{i-1}}^{\hat{t}_i} {}_i\beta_j(t) dt = 1; \quad (19)$$

$$\alpha_j \geq 0 (j = \overline{1, N_G}), {}_i\nu_j \geq 0 (i = \overline{1, N-1}, j = \overline{1, n_{\Sigma i}}), {}_i\beta_j \geq 0 (i = \overline{1, N}, j = \overline{1, N_{Qi}}), \quad (20)$$

$$\alpha_j J_j^G = 0 (j = \overline{1, K_G}), {}_i\beta_j(t) J_{ij}^Q = 0, t \in (\hat{t}_{i-1}, \hat{t}_i] (i = \overline{1, N}, j = \overline{1, K_{Qi}}); \quad (21)$$

vector optimality T

$$\frac{\partial}{\partial \hat{t}_0} E[\mathfrak{S}_1({}_1X(\hat{t}_0), \hat{t}_0)] \Big|_{\wedge} + \alpha^T \frac{\partial}{\partial \hat{t}_0} E[{}_1\mathfrak{G}({}_1X(\hat{t}_0), \hat{t}_0)] \Big|_{\wedge} + E[\lambda_{1t}] \Big|_{\wedge, t=\hat{t}_0} = 0, \quad (22)$$

$$\begin{aligned} & \frac{\partial}{\partial \hat{t}_i} E[S_i({}_iX, \hat{t}_i)] \Big|_{\wedge} + E[\mathfrak{S}_{i+1}({}_{i+1}X, \hat{t}_i)] \Big|_{\wedge} + \alpha^T E[{}_{i+1}\mathfrak{G}({}_{i+1}X, \hat{t}_i)] \Big|_{\wedge} + \\ & + \alpha^T E[{}_iG({}_iX, \hat{t}_i)] \Big|_{\wedge} + {}_i\nu^T E[{}_i\Psi({}_iX, \hat{t}_i) - {}_{i+1}X(\hat{t}_i)] \Big|_{\wedge} + \\ & + E[\lambda_{i+1,t} - \lambda_{i,t}] \Big|_{\wedge, t=\hat{t}_i} = 0 \quad (i = \overline{1, N-1}), \end{aligned} \quad (23)$$

$$\frac{\partial}{\partial \hat{t}_N} E[S_N({}_NX, \hat{t}_N)] \Big|_{\wedge} + \alpha^T E[{}_N\mathfrak{G}({}_NX, \hat{t}_N)] \Big|_{\wedge} - E[\lambda_{N,t}] \Big|_{\wedge, t=\hat{t}_N} = 0, \quad (24)$$

function minimum:

$$\min_{U \in \Omega_i} E[\mathfrak{N}_i(t, {}_iX, {}_iU, \lambda_i)] \Big|_{\wedge} = \min_{U \in \Omega_i} \int_{\Xi_i} \mathfrak{N}_i(t, {}_iX, {}_iU, \lambda_i) \hat{p}_i({}_iX, t) d_iX. \quad (25)$$

Here

$$\begin{aligned} \mathfrak{N}_i(t, {}_iX, {}_iU, \lambda_i) = & \Phi_i({}_iX, {}_iU, t) + {}_i\beta^T(t) {}_iQ({}_iX, {}_iU, t) + \\ & + \frac{\partial^T \lambda_i({}_iX, t)}{\partial {}_iX} {}_iF({}_iX, {}_iU, t) + \frac{1}{2} tr \left[{}_i\mathfrak{D}({}_iX, {}_iU, t) {}_i\mathfrak{D}^T({}_iX, {}_iU, t) \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} \lambda_i({}_iX, t) \right], \end{aligned} \quad (26)$$

$${}_iQ({}_iX, {}_iU, t) = \left[{}_iQ_1({}_iX, {}_iU, t), \dots, {}_iQ_{N_{Q_i}}({}_iX, {}_iU, t) \right]^T, \quad (27)$$

$${}_i\mathfrak{G}({}_iX, t_{i-1}) = \left[{}_i\mathfrak{G}_1({}_iX, t_{i-1}), \dots, {}_i\mathfrak{G}_{N_G}({}_iX, t_{i-1}) \right]^T, \quad (28)$$

$${}_iG({}_iX, t_i) = \left[{}_iG_1({}_iX, t_i), \dots, {}_iG_{N_G}({}_iX, t_i) \right]^T, \quad i = \overline{1, N}, \quad (29)$$

$$\begin{aligned} {}_i\Psi({}_iX, t_i) - {}_{i+1}X_1(t_i) = & \left[{}_i\Psi_1({}_iX, t_i) - {}_{i+1}X_1(t_i), \dots, \right. \\ & \left. {}_i\Psi_{n_{\Sigma i+1}}({}_iX, t_i) - {}_{i+1}X_{n_{\Sigma i+1}}(t_i) \right]^T \quad (i = \overline{1, N-1}), \end{aligned} \quad (30)$$

$$E[\varphi({}_iX, t)] \Big|_{\wedge} = \int_{\Xi_i} \varphi({}_iX, t) \hat{p}_i({}_iX, t) d_iX.$$

The theorem is proved according to the technique proposed in [13]. Let's make extended functional

$$\begin{aligned} J_R(U, T, \gamma, \lambda_i({}_iX, t) \quad (i = \overline{1, N})) = & J(U, T) + \alpha^T J^G + \\ & + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} {}_i\beta^T(t) {}_iJ^Q dt + \sum_{i=1}^{N-1} {}_i\nu^T E[{}_iJ^\Psi] + \\ & + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Xi_i} \lambda_i({}_iX, t) \left[\mathfrak{B}_i(p_i, {}_iu, t) - \frac{\partial p_i}{\partial t} \right] p_i({}_iX, t) d_iX dt, \end{aligned} \quad (31)$$

where

$$J^G = [J_1^G, \dots, J_{NG}^G]^T, {}_i J^G = [{}_i J_1^G, \dots, {}_i J_{N_{Qi}}^G]^T, {}_i J^\Psi = [{}_i J_1^\Psi, \dots, {}_i J_{n_{\Sigma i}}^\Psi]^T, \quad (32)$$

$\gamma = [\alpha^T; {}_1 \beta^T(t), t \in [t_0, t_1]; \dots; {}_N \beta^T(t), t \in [t_{N-1}, t_N]; {}_1 \nu^T, \dots, {}_{n-1} \nu^T]^T$ – vector of deterministic Lagrange multipliers; $\lambda_i({}_i X, t)$ ($i = \overline{1, N}$) – Lagrange multipliers representing scalar random functions.

Let $\hat{V} = (\hat{U}, \hat{T})$ be an optimal control. Consider the control $V = (U, T)$:

$$U(t) = \hat{U}(t) + \Delta U(t), \quad T = \hat{T} + \varepsilon \delta T, \quad (33)$$

$$\Delta U(t) = \begin{cases} 0, & t \in [t_{i-1}, t_i] \setminus \mathfrak{J}_{i\bar{\varepsilon}}, \\ \omega - {}_i \hat{U}(t), & t \in \mathfrak{J}_{i\bar{\varepsilon}}, \end{cases} \quad i = \overline{1, N}, \bar{\varepsilon} = \overline{1, l_i}; \quad \delta T = (\delta t_0, \dots, \delta t_N).$$

Here ε – small positive value, $\mathfrak{J}_{i\bar{\varepsilon}}$ ($i = \overline{1, N}, \bar{\varepsilon} = \overline{1, l_i}$) – half-intervals on which "needle" variation occurs $\Delta U(t) = \omega - {}_i \hat{U}(t)$, $t \in \mathfrak{J}_{i\bar{\varepsilon}}$, defined on $t \in [t_{i-1}, t_i]$ inequalities $\tau_{i\bar{\varepsilon}} \leq t < \tau_{i\bar{\varepsilon}} + \varepsilon \rho_{i\bar{\varepsilon}}$, $\tau_{i\bar{\varepsilon}}$ – continuity points ${}_i U$ t ; ${}_{i\bar{\varepsilon}} \omega = [{}_{i\bar{\varepsilon}} \omega_1, \dots, {}_{i\bar{\varepsilon}} \omega_{mi}]^T$ – arbitrary points of a allowed control area Ω_i ; $\rho_{i\bar{\varepsilon}}$ $i = \overline{1, N}, \bar{\varepsilon} = \overline{1, l_i}$ – some non-negative numbers.

For sufficiently small $\varepsilon > 0$ half intervals $\mathfrak{J}_{i\bar{\varepsilon}}$ ($\bar{\varepsilon} = \overline{1, l_i}$) do not intersect and are located entirely on $[t_{i-1}, t_i]$. Then it is necessary that first functional variation (34) was non-negative at the point $\hat{V} = (\hat{U}, \hat{T})$, i.e.

$$\delta J_R(U, T, \gamma, \lambda_i({}_i X, t) (i = \overline{1, N})) \geq 0. \quad (34)$$

To calculate δJ_R , we use the formula

$$\delta J_R = \lim_{\varepsilon \rightarrow 0} J_R(\hat{U} + \Delta U, \hat{T} + \varepsilon \delta T, \gamma, \lambda_i (i = \overline{1, N})) - J_R(\hat{U}, \hat{T}, \gamma, \lambda_i (i = \overline{1, N})). \quad (35)$$

We take into account that control variation $\Delta V = (\Delta U, \varepsilon \delta T)$ leads to variations in probability density

$$\Delta p_i({}_i X(t), t) = \begin{cases} \delta p_i({}_i X(t), t), & t \in [t_{i-1}, t_i], \\ \delta p_i({}_i X, \hat{t}_{i-1}) + \frac{\partial p_i}{\partial t_{i-1}} \Big|_{t_i = \hat{t}_{i-1}} \delta t_{i-1}, & t = \hat{t}_{i-1}, \\ \delta p_i({}_i X, \hat{t}_i) + \frac{\partial p_i}{\partial t_i} \Big|_{t_i = \hat{t}_i} \delta t_i, & t = \hat{t}_i, (i = \overline{1, N}). \end{cases} \quad (36)$$

Using the transformation formulas of the integrals [14], we write the last term of expression (31) in the form

$$\begin{aligned}
& \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \int_{\Xi_i} \lambda_i(\cdot, X, t) \left[\mathfrak{B}(p_i, \cdot, U, t) - \frac{\partial p_i}{\partial t} \right] p_i(\cdot, X, t) d_i X dt = \\
& = \sum_{i=1}^N \left\{ \int_{t_{i-1}}^{t_i} \int_{\Xi_i} \left[\frac{\partial \lambda_i(\cdot, X, t)}{\partial t} + \frac{\partial^T \lambda_i}{\partial_i X} F(\cdot, X, \cdot, U, t) + \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \text{tr} \left[{}_i \mathfrak{D}(\cdot, X, \cdot, U, t) {}_i \mathfrak{D}^T(\cdot, X, \cdot, U, t) \frac{\partial}{\partial_i X} \frac{\partial^T}{\partial_i X} (\lambda_i(\cdot, X, t)) \right] \right\} p_i(\cdot, X, t) d_i X dt - \right. \\
& \quad \left. - \int_{\Xi_i} \left[\lambda_i(\cdot, X, t_i) p_i(\cdot, X, t_i) d_i X(t_i) - \lambda_i(\cdot, X, t_{i-1}) p_i(\cdot, X, t_{i-1}) d_i X(t_{i-1}) \right]. \right.
\end{aligned} \tag{37}$$

Then the expression to calculate δJ_R , taking into account (33)–(37) and the theorem on finite increments, takes the form

$$\begin{aligned}
\delta J_R = & \int_{\Xi_1} \left[\mathfrak{S}_1(\cdot, X, \hat{t}_0) + \alpha^T {}_1 \mathfrak{G}(\cdot, X, \hat{t}_0) + \lambda_1(\cdot, X, \hat{t}_0) \right] \cdot \Delta p_1(\cdot, X, \hat{t}_0) p_1(\cdot, X, \hat{t}_0) d_1 X + \\
& + \left\{ \frac{\partial}{\partial \hat{t}_0} E \left[\mathfrak{S}_1(\cdot, X, \hat{t}_0) \right] + \alpha^T \frac{\partial}{\partial \hat{t}_0} E \left[{}_1 \mathfrak{G}(\cdot, X, \hat{t}_0) \right] \right\} \Big|_{\wedge} \cdot \delta t_0 + \\
& + E \left[\Phi_1(\cdot, X, {}_1 \hat{U}(\hat{t}_0), \hat{t}_0) + {}_1 \beta^T(\hat{t}_0) {}_1 Q(\cdot, X, {}_1 \hat{U}(\hat{t}_0), \hat{t}_0) \right] \Big|_{\wedge} \cdot \delta t_0 + \\
& + E \left\{ \frac{\partial^T \lambda_1(\cdot, X, \hat{t}_0)}{\partial_1 X} {}_1 F(\cdot, X, {}_1 \hat{U}(\hat{t}_0), \hat{t}_0) \right\} \Big|_{\wedge} \cdot \delta t_0 + \\
& + E \left\{ \frac{1}{2} \text{tr} \left[{}_1 \mathfrak{D}(\cdot, X, {}_1 \hat{U}(\hat{t}_0), \hat{t}_0) {}_1 \mathfrak{D}^T(\cdot, X, {}_1 \hat{U}(\hat{t}_0), \hat{t}_0) \frac{\partial}{\partial_1 X} \frac{\partial^T}{\partial_1 X} (\lambda_1(\cdot, X, \hat{t}_0)) \right] \right\} \Big|_{\wedge} \cdot \delta t_0 + \\
& + \dots + \int_{\Xi_i} \left[S_i(\cdot, X, \hat{t}_i) + \alpha^T {}_i G(\cdot, X, \hat{t}_i) \right] \cdot \Delta p_i(\cdot, X, \hat{t}_i) d_i X + \\
& + \int_{\Xi_i} \left[\nu^T {}_i \psi(\cdot, X, \hat{t}_i) - \lambda_i(\cdot, X, \hat{t}_i) \right] \cdot \Delta p_i(\cdot, X, \hat{t}_i) d_i X + \\
& + \int_{\Xi_{i+1}} \left[\lambda_{i+1}(\cdot, X, \hat{t}_i) - \nu^T {}_{i+1} X(\hat{t}_i) + \mathfrak{S}_{i+1}(\cdot, X, \hat{t}_i) + \right. \\
& \quad \left. + \alpha^T {}_{i+1} \mathfrak{G}(\cdot, X, \hat{t}_i) \right] \Delta p_{i+1}(\cdot, X, \hat{t}_i) d_{i+1} X + \left\{ \frac{\partial}{\partial \hat{t}_i} \left[E(S_i(\cdot, X, \hat{t}_i)) \right] \right\} \Big|_{\wedge} + \\
& + E(\mathfrak{S}_{i+1}(\cdot, X, \hat{t}_i)) \Big|_{\wedge} + \alpha^T E({}_{i+1} \mathfrak{G}(\cdot, X, \hat{t}_i)) \Big|_{\wedge} + \alpha^T E({}_i G(\cdot, X, \hat{t}_i)) +
\end{aligned}$$

$$\begin{aligned}
& + {}_i\nu^T E({}_i\Psi({}_iX, \hat{t}_i) - {}_{i+1}X(\hat{t}_i)) \Big|_{\wedge} \Big] + E\left[\Phi_i({}_iX, {}_i\hat{U}(\hat{t}_i), \hat{t}_i) + \right. \\
& \left. + {}_i\beta^T(\hat{t}_i) {}_iQ({}_iX, {}_i\hat{U}(\hat{t}_i), \hat{t}_i) + \frac{\partial^T \lambda_i({}_iX, \hat{t}_i)}{\partial {}_iX} {}_iF({}_iX, {}_i\hat{U}(\hat{t}_i), \hat{t}_i) + \right. \\
& \left. + \frac{1}{2} tr \left({}_i\mathfrak{D}({}_iX, {}_i\hat{U}(\hat{t}_i), \hat{t}_i) {}_i\mathfrak{D}^T({}_iX, {}_i\hat{U}(\hat{t}_i), \hat{t}_i) \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} (\lambda_i({}_iX, \hat{t}_i)) \right) \right] \Big|_{\wedge} - \\
& - E\left[\Phi_{i+1}({}_{i+1}X, {}_{i+1}\hat{U}(\hat{t}_i), \hat{t}_i) + {}_{i+1}\beta^T {}_{i+1}Q({}_{i+1}X, {}_{i+1}\hat{U}(\hat{t}_i), \hat{t}_i) + \right. \\
& \left. + \frac{\partial^T \lambda_{i+1}({}_{i+1}X, \hat{t}_i)}{\partial {}_{i+1}X} {}_{i+1}F({}_{i+1}X, {}_{i+1}\hat{U}(\hat{t}_i), \hat{t}_i) + \right. \\
& \left. + \frac{1}{2} tr ({}_{i+1}\mathfrak{D}({}_{i+1}X, {}_{i+1}\hat{U}(\hat{t}_i), \hat{t}_i) {}_{i+1}\mathfrak{D}^T({}_{i+1}X, {}_{i+1}\hat{U}(\hat{t}_i), \hat{t}_i) \times \right. \\
& \left. \times \frac{\partial}{\partial {}_{i+1}X} \frac{\partial^T}{\partial {}_{i+1}X} (\lambda_{i+1}({}_{i+1}X, \hat{t}_i)) \right) \Big|_{\wedge} \Big\} \delta t_i + \dots \\
& \dots + \int_{\Xi_N} \left[S_N({}_NX, \hat{t}_N) + \alpha^T {}_NG({}_NX, \hat{t}_N) - \lambda_N({}_NX, \hat{t}_N) \right] \Delta p_N({}_NX, \hat{t}_N) d_NX + \\
& + \left\{ \frac{\partial}{\partial \hat{t}_N} \left[E(S_N({}_NX, \hat{t}_N)) \Big|_{\wedge} + \alpha^T E({}_NG({}_NX, \hat{t}_N)) \Big|_{\wedge} \right] + E\left[\Phi_N({}_NX, {}_N\hat{U}(\hat{t}_N), \hat{t}_N) + \right. \\
& \left. + {}_N\beta^T {}_NQ({}_NX, {}_N\hat{U}(\hat{t}_N), \hat{t}_N) + \frac{\partial^T \lambda_N({}_NX, \hat{t}_N)}{\partial {}_NX} {}_NF({}_NX, {}_N\hat{U}(\hat{t}_N), \hat{t}_N) + \right. \\
& \left. + \frac{1}{2} tr({}_N\mathfrak{D}({}_NX, {}_N\hat{U}(\hat{t}_N), \hat{t}_N) {}_N\mathfrak{D}^T({}_NX, {}_N\hat{U}(\hat{t}_N), \hat{t}_N) \times \right. \\
& \left. \times \frac{\partial}{\partial {}_NX} \frac{\partial^T}{\partial {}_NX} \lambda_N({}_NX, \hat{t}_N)) \right] \Big|_{\wedge} \Big\} \delta t_N + \sum_{i=1}^N \int_{\wedge}^{\hat{t}_i} \int_{\Xi_i} [\Phi_i({}_iX, {}_i\hat{U}, t) + \\
& + {}_i\beta^T {}_iQ({}_iX, {}_i\hat{U}, t) + \frac{\partial \lambda_i({}_iX, t)}{\partial t} + \frac{\partial^T \lambda_i({}_iX, t)}{\partial {}_iX} {}_iF({}_iX, {}_i\hat{U}, t) + \\
& + \frac{1}{2} tr \left({}_i\mathfrak{D}({}_iX, {}_i\hat{U}, t) {}_i\mathfrak{D}^T({}_iX, {}_i\hat{U}, t) \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} \lambda_i({}_iX, t) \right) \Big] \delta p_i({}_iX, t) d_iX dt +
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \int_{\hat{t}_{i-1}}^{\hat{t}_i} E \left[\Phi_i(\cdot, X, \hat{U}(\tau_i), \tau_i) + {}_i\beta^T(\tau_i) {}_iQ(\cdot, X, \hat{U}(\tau_i), \tau_i) + \right. \\
& + \frac{\partial \lambda_i(\cdot, X, \tau_i)}{\partial {}_iX} {}_iF(\cdot, X, \hat{U}(\tau_i), \tau_i) + \frac{1}{2} \text{tr}({}_i\mathfrak{D}(\cdot, X, \hat{U}(\tau_i), \tau_i)) \times \\
& \times {}_i\mathfrak{D}^T(\cdot, X, \hat{U}(\tau_i), \tau_i) \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} \lambda_i(\cdot, X, \tau_i) - \Phi_i(\cdot, X, \hat{U}(\tau_i), \tau_i) - \\
& - {}_i\beta^T(\tau_i) {}_iQ(\cdot, X, \hat{U}(\tau_i), \tau_i) - \frac{\partial \lambda_i(\cdot, X, \tau_i)}{\partial {}_iX} {}_iF(\cdot, X, \hat{U}(\tau_i), \tau_i) - \\
& \left. - \frac{1}{2} \text{tr}({}_i\mathfrak{D}(\cdot, X, \hat{U}(\tau_i), \tau_i) {}_i\mathfrak{D}^T(\cdot, X, \hat{U}(\tau_i), \tau_i) \frac{\partial}{\partial {}_iX} \frac{\partial^T}{\partial {}_iX} \lambda_i(\cdot, X, \tau_i)) \right] d\tau_i \geq 0.
\end{aligned} \tag{38}$$

Note that to simplify the presentation one needle-like variation was considered on each interval $[t_{i-1}, t_i]$ ($i = \overline{1, N}$) in (38), when varying the control $U_i(t)$, $i = \overline{1, N}$,

This approach does not limit the generality of the results due to the effect of several needle-shaped variations between time instants t_{i-1} and t_i occurring in various infinitesimal time intervals, on account of the additivity inherent in (38), can be considered independently of each other.

Condition (38) should be satisfied for any choice of variations ΔU and δT at a point $\hat{V} = (\hat{U}, \hat{T})$, which is possible only if the meeting relations coincide with the necessary optimality conditions formulated in the theorem, as we wished to prove.

Comment. By summing up equations (15) and (16), with consideration for relation (7), we obtain the jump condition for the function of random argument and time

$$\begin{aligned}
& \lambda_{i+1}({}_{i+1}X, \hat{t}_i) - \lambda_i({}_iX, \hat{t}_i) + S_i({}_iX, \hat{t}_i) + \mathfrak{S}_{i+1}({}_{i+1}X, \hat{t}_i) + \\
& + \alpha^T {}_iG({}_iX, \hat{t}_i) + \alpha^T {}_{i+1}\mathfrak{G}({}_{i+1}X, \hat{t}_i) = 0 \quad (i = \overline{1, N-1}).
\end{aligned}$$

5. Conclusions

1. The problem of optimizing control of a stochastic composite dynamic system in case of full information about the state vector and with an arbitrary branching scheme of the trajectory can be solved using the theorems formulated in the article and proved necessary optimality conditions.

Before applying the above theorems, it is necessary, according to the transformation rule, to turn from a stochastic CDS to a discontinuous stochastic system with a variable size of state vectors, control, and exciting noise, and then, using the inverse transition rule, return to the terms of the original formulation of the problem for a stochastic CDS.

2. The result of solving the problem of optimizing the control of a stochastic CDS in case of full information about the state vector is the analytical or algorithmic dependence of the control vector of each subsystem (unit) on time or the current value of the phase coordinate and time, as well as the optimal values of the time instants of structural transformations.

3. The statement of the control optimization problem for a stochastic composite dynamic system suggests that the evolution of the state vector of a subsystem (unit) is described by a stochastic differential equation excited by a standard Wiener process, from which it is possible to turn to the Fokker-Planck-Kolmogorov equation and replace the stochastic CDS control optimization problem of by the problem of deterministic CDS control optimization, which subsystems (units) dynamics is described by equations with distributed parameters.

4. The practical use of the optimality conditions developed in this section is preferable to search for optimal control of compound aircraft that are subject to noise (for example, flying in a turbulent atmosphere) under conditions where the errors of real on-board measurement systems can be neglected.

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Стаття надійшла до редакції 03.11.2020