Note on History of Age Replacement Policies

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Abstract

This paper tries to trace our research history briefly from Barlow and Proschan to attain general replacement models. We begin with a random age replacement policy that is planned at a random time Y and call it as random replacement. When the distribution of Y becomes a degenerate distribution placing unit mass at T , age replacement is formulated. We obtain the general formulas for optimum replacement times. We next suppose the unit works for a job with random works, and replacement policies with N cycles are discussed. As follows, we combine age and random replacement models and discuss replacement first, replacement last, replacement overtime, replacement overtime first and replacement overtime last. By formulating the distributions of replacement times with n variables, general replacement models with n replacement times are obtained.

Keywords- Age replacement, Replacement first, Replacement last, Replacement overtime, General replacement.

1. Introduction

Most basic model in maintenance theory is age replacement, in which we plan to replace an operating unit before failure at an optimum time T^* to minimize the expected replacement cost rate. Since Barlow and Proschan (Barlow and Proschan, 1996) have introduced age replacement model in 1965, a great number of age replacement models have been discussed by researchers for a half century, which have been summarized (Barlow, and Proschan, 1996; Nakagawa, 2005; Sarkar et al., 2011). Recently, we have proposed several new notions of age replacement such as random replacement, replacement first, replacement last, replacement overtime, and replacement middle (Nakagawa, 2014; Zhao and Nakagawa, 2012; Nakagawa and Zhao, 2015; Zhao et al., 2015). General replacement models combining age and random replacement policies with n replacement times have been studied (Chen et al., 2016).

This paper has an object of tracing our research history briefly from Barlow and Proschan (Barlow and Proschan, 1996) to attain general replacement models, which is shown in Fig. 1.

Fig. 1. The evolution of age replacement policies

In Section 2, general assumptions are given for modelings. In Section 3, we introduce age and random replacement models (Barlow and Proschan, 1996). In Sections 4-6, we give models of replacement first, replacement last, replacement overtime and replacement middle, respectively, and compare their optimum policies analytically. When the costs of preventive replacement are the same, age replacement is most economical than others. However, when a unit works for a job with random works, it is not wise in practice to stop its operation and to replace it at the optimum time, e.g., planned time T . In this situation, replacement overtime would be better than others. In Section 7, we propose general replacement policies and call them as redundant maintenances in general forms, as their distributions of replacement times agree with the failure distributions of a series system and a parallel system with n units.

2. General Assumptions

We suppose that an operating unit has a failure distribution $F(t)$ with finite mean $\mu \equiv \int_0^\infty \bar{F}(t)dt$, where $\bar{\varphi}(t) \equiv 1 - \varphi(t)$ for any distribution $\varphi(t)$. Denote that $F(t)$ has a density function $f(t) \equiv dF(t)/dt$, i.e., $F(t) = \int_0^t f(u)du$, and suppose the failure rate $h(t) \equiv f(t)/\bar{F}(t)$ for $\bar{F}(t) < 1$ increases strictly from $h(0) \equiv 0$ to $h(\infty) \equiv \lim_{t \rightarrow \infty} h(t) = \infty$.

In addition, we suppose the unit operates for an infinite time horizon by means of replacement. All times for corrective replacement and preventive replacement are supposed to be neglected and their costs are constantly given respectively. Our objective is to find optimum preventive replacement times to minimize the expected cost rates.

3. Age and Random Replacement Models

Suppose that an operating unit is replaced at a random time $Y(0 < Y \leq \infty)$ or at failure, whichever occurs first. The random variable Y has a general distribution $G(t)$ and is independent with the failure time X . We call this policy as random replacement. The expected cost rate is (Barlow and Proschan, 1996; Nakagawa, 2014)

$$C(G) = \frac{c_R + (c_F - c_R) \int_0^\infty \bar{G}(t) dF(t)}{\int_0^\infty \bar{G}(t) F(t) dt}, \quad (1)$$

where c_R = replacement cost at time Y , c_F = replacement cost at failure and $c_F > c_R$.

It has been shown (Barlow and Proschan, 1996; Nakagawa, 2014) that $C(G)$ can be rewritten as

$$C(G) = \frac{\int_0^\infty Q(t) dG(t)}{\int_0^\infty S(t) dG(t)},$$

where

$$Q(t) \equiv c_R + (c_F - c_R)F(t) \quad \text{and} \quad S(t) \equiv \int_0^t \bar{F}(u) du.$$

Suppose that there exists a minimum value T ($0 < T \leq \infty$) for $Q(t)/S(t) \geq Q(T)/S(T)$, i.e.,

$$\int_0^\infty Q(t) dG(t) \geq \frac{Q(T)}{S(T)} \int_0^\infty S(t) dG(t).$$

Thus,

$$C(G) \geq \frac{Q(T)}{S(T)} \geq C(G_T),$$

where $G_T(t)$ is the degenerate distribution placing unit mass at T , i.e., $G_T(t) \equiv 1$ for $t \geq T$ and $G_T(t) \equiv 0$ for $t < T$. If $T = \infty$, then the unit is replaced at failure.

It was written in (Barlow and Proschan, 1996) that the elegant proof was due to S. Karlin, who was the author of (Karlin and Taylor, 1975). This is very easy and simple, however, it could not be occurred to anybody from (1). In other words, if preventive replacement costs are equal, the policy at time T is superior to other ones, which will be shown theoretically in the following sections.

Suppose the unit is replaced at time T or at failure, whichever occurs first. We call this policy as age replacement, and the expected cost rate is

$$C(T) = \frac{c_T + (c_F - c_T)F(T)}{\int_0^T \bar{F}(t) dt}, \quad (2)$$

where c_T = replacement cost at time T with $c_T < c_F$. Optimum T^* to minimize $C(T)$ satisfies

$$h(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_T}{c_F - c_T},$$

or

$$\int_0^T \bar{F}(t)[h(T) - h(t)]dt = \frac{c_T}{c_F - c_T}, \quad (3)$$

which is the standard formula of deriving optimum age replacement time, and the resulting cost rate is

$$C(T^*) = (c_F - c_T)h(T^*). \quad (4)$$

Equation (3) means physically that optimum replacement time satisfies: *Failure rate* $\times$ *Mean replacement time* – *Failure probability* = *Replacement cost ratio*. This general formula will be shown in all sections.

Next, the unit works for a job with working cycles Y_j ($j = 1, 2, \dots$), where Y_j are independent and have an identical distribution $G(t) \equiv \Pr\{Y_j \leq t\}$ with finite mean $1/\theta \equiv \int_0^\infty \bar{G}(t) dt$. It is assumed that $G^{(j)}(t) \equiv \Pr\{Y_1 + Y_2 + \dots + Y_j \leq t\}$ ($j = 1, 2, \dots$) is the j -fold Stieltjes convolution of $G(t)$ with itself and $G^{(0)}(t) \equiv 1$ for $t \geq 0$.

Suppose that the unit is replaced at cycle N or at failure, whichever occurs first. We also call this policy as random replacement, and the expected cost rate is (Nakagawa, 2014).

$$C(N) = \frac{c_N + (c_F - c_N) \int_0^\infty [1 - G^{(N)}(t)] dF(t)}{\int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) dt} \quad (5)$$

where c_N = replacement cost at cycle N with $c_N < c_F$. The expected cost rate $C(N)$ is easily obtained by replacing $G(t)$ and c_R in (1) with $G^{(N)}(t)$ and c_N , respectively.

Optimum N^* to minimize $C(N)$ satisfies

$$\int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) [Q(N) - h(t)] dt \geq \frac{c_N}{c_F - c_N}, \quad (6)$$

where

$$Q(T, N) = \frac{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] dF(t)}{\int_0^T [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt} < h(T),$$

and $Q(N) \equiv \lim_{T \rightarrow \infty} Q(T, N)$.

In particular, when $G(t) = 1 - e^{-\theta t}$,

$$Q(T, N) = \frac{\int_0^T (\theta t)^N e^{-\theta t} dF(t)}{\int_0^T (\theta t)^N e^{-\theta t} \bar{F}(t) dt}$$

increases strictly with N to $h(T)$ and increases with T to $Q(\infty, N)$ [4]. When $h(\infty) = \infty$ and $G(t) = 1 - e^{-\theta t}$, both optimum T^* and N^* to satisfy (3) and (6) exist.

4. Replacement First and Last

4.1 Replacement First

Suppose the unit is replaced preventively at T or at cycle N , whichever occurs first. Then, the expected cost rate is (Nakagawa, 2014)

$$C_F(T, N) = \frac{c_T + (c_F - c_T) \int_0^T [1 - G^{(N)}(t)] dF(t) + (c_N - c_T) \int_0^T \bar{F}(t) dG^{(N)}(t)}{\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) dt}. \quad (7)$$

Clearly, $\lim_{N \rightarrow \infty} C_F(T, N) = C(T)$ in (2) and $\lim_{T \rightarrow \infty} C_F(T, N) = C(N)$ in (5).

When $c_T = c_N$, differentiating $C_F(T, N)$ with respect to T and setting it equal to zero,

$$\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) [h(T) - h(t)] dt = \frac{c_T}{c_F - c_T}. \quad (8)$$

The left-hand side increases with T to ∞ . Thus, there exists a unique T_F^* ($0 < T_F^* < \infty$) which satisfies (8), and

$$C_F(T_F^*, N) = (c_F - c_T) h(T_F^*). \quad (9)$$

The left-hand side of (8) increases strictly with N to that of (3), T_F^* decreases strictly to T^* in (3).

Forming the inequality $C_F(T, N + 1) - C_F(T, N) \geq 0$,

$$\int_0^T [1 - G^{(N)}(t)] \bar{F}(t) [Q(T, N) - h(t)] dt \geq \frac{c_T}{c_F - c_T}. \quad (10)$$

Substituting (8) for (10),

$$Q(T, N) \geq h(T),$$

which does not hold for any N , i.e., $N_F^* = \infty$. Thus, optimum policy to minimize $C_F(T, N)$ is $(T_F^* = T^*, N_F^* = \infty)$, where T^* is given in (3). This shows that when the costs of preventive replacement are the equal, age replacement is more economical than the random policy.

Furthermore, if $Q(T, N)$ increases with N to $h(T)$, then the left-hand side of (10) increases with N to that of (3). Thus, if $T > T^*$, then there exists a unique minimum N_F^* ($1 \leq N_F^* < \infty$) which satisfies (10), and decreases with T to N^* given in (6). Conversely, if $T \leq T^*$, then $N_F^* = \infty$.

4.2 Replacement Last

Suppose the unit is replaced at T or at cycle N , whichever occurs last. Then, the expected cost rate is (Nakagawa, 2014)

$$C_L(T, N) = \frac{c_T + (c_F - c_T)\{F(T) + \int_T^\infty [1 - G^{(N)}(t)] dF(t)\} + (c_N - c_T) \int_T^\infty \bar{F}(t) dG^{(N)}(t)}{\int_0^T \bar{F}(t) dt + \int_T^\infty [1 - G^{(N)}(t)] \bar{F}(t) dt}. \quad (11)$$

Clearly, $\lim_{T \rightarrow 0} C_L(T, N) = \lim_{T \rightarrow 0} C_F(T, N)$ and $\lim_{N \rightarrow 0} C_L(T, N) = \lim_{N \rightarrow 0} C_F(T, N)$.

When $c_T = c_N$, differentiating $C_L(T, N)$ with respect to T and setting it equal to zero,

$$\int_0^T \bar{F}(t)[h(T) - h(t)] dt - \int_T^\infty [1 - G^{(N)}(t)] \bar{F}(t)[h(t) - h(T)] dt = \frac{c_T}{c_F - c_T}. \quad (12)$$

The left-hand side of (12) increases with T to $h(\infty) = \infty$. Thus, there exists a unique T_L^* ($0 < T_L^* < \infty$) which satisfies (12), and

$$C_L(T_L^*, N) = (c_F - c_T)h(T_L^*). \quad (13)$$

Letting $L_F(T)$ and $L_L(T)$ be the left-hand sides of (8) and (12),

$$\begin{aligned} L_{LF}(T) &= L_L(T) - L_F(T) \\ &= \int_0^T G^{(N)}(t) \bar{F}(t) t [h(T) - h(t)] dt - \int_T^\infty [1 - G^{(N)}(t)] \bar{F}(t) [h(t) - h(T)] dt, \end{aligned}$$

which increases strictly with T from $L_{LF}(0) < 0$ to ∞ . Thus, there exists a unique T_A^* which satisfies $L_{LF}(T) = 0$, and T_A^* increases with N to ∞ . Thus, if $L_F(T_A^*) \geq c_T/(c_F - c_T)$, then $T_F^* \leq T_L^*$, and hence, from (9) and (13), replacement first is more economical than replacement last.

If $L_F(T_A^*) < c_T/(c_F - c_T)$, then $T_L^* < T_F^*$, and replacement last is more economical. This means that if the ratio of $c_T/(c_F - c_T)$ is large, we adopt replacement last, and if N is large, replacement first is adopted.

5. Replacement Overtime

We suppose the unit is replaced preventively at the first completion of working cycles Y_j ($j = 1, 2, \dots$) over T . We call this policy as *replacement overtime*, and the expected cost rate is (Nakagawa and Zhao, (2015).

$$C_O(T) = \frac{c_O + (c_F - c_O) \sum_{j=0}^\infty \int_0^T [\int_{T-t}^\infty F(t+u) dG(u)] dG^{(j)}(t)}{\sum_{j=0}^\infty \int_0^T [\int_0^\infty \bar{G}(u) \bar{F}(t+u) du] dG^{(j)}(t)}, \quad (14)$$

where c_O = replacement cost over T with $c_O < c_F$. Optimum T_O^* satisfies

$$\begin{aligned} \tilde{Q}(T) \sum_{j=0}^{\infty} \int_0^T [\int_0^{\infty} \bar{G}(u) \bar{F}(t+u) du] dG^{(j)}(t) \\ - \sum_{j=0}^{\infty} \int_0^T \left[\int_{T-t}^{\infty} F(t+u) dG(u) \right] dG^{(j)}(t) = \frac{c_O}{c_F - c_O}, \end{aligned} \quad (15)$$

where

$$\tilde{Q}(T, N) \equiv \frac{\int_T^{\infty} [G^{(N)}(t) - G^{(N+1)}(t)] dF(t)}{\int_T^{\infty} [G^{(N)}(t) - G^{(N+1)}(t)] \bar{F}(t) dt},$$

and $\tilde{Q}(T) \equiv \tilde{Q}(T, 0)$.

When $G(t) = 1 - e^{-\theta t}$, (15) is

$$\tilde{Q}_1(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_O}{c_F - c_O}, \quad (16)$$

where

$$\tilde{Q}_1(T) \equiv \frac{\int_T^{\infty} e^{-\theta t} dF(t)}{\int_T^{\infty} e^{-\theta t} \bar{F}(t) dt}$$

increases with T to $h(\infty) = \infty$. Thus, there exists a unique T_O^* ($0 < T_O^* < \infty$) which satisfies (16), and the resulting cost rate is

$$c_O(T_O^*) = (c_F - c_O) \tilde{Q}_1(T_O^*) = \frac{c_O + (c_F - c_O) F(T_O^*)}{\int_0^{T_O^*} \bar{F}(t) dt}. \quad (17)$$

Noting that $\tilde{Q}_1(T) > h(T)$ for $0 \leq T < \infty$, $T_O^* < T^*$, and hence, from (2) and (17), $c_O(T_O^*) > C(T^*)$ when $c_O = c_T$, i.e., age replacement is more economical than the overtime policy.

5.1 Replacement Overtime First

Suppose the unit is replaced over T ($0 \leq T \leq \infty$) or at cycle N ($N = 1, 2, \dots$), whichever occurs first. Then, the expected cost rate is (Nakagawa and Zhao, X. 2015).

$$C_{OF}(T, N) = \frac{c_O + (c_F - c_O) \sum_{j=0}^{N-1} \int_0^T \{ \int_0^{\infty} [F(t+u) - F(t)] dG(u) \} dG^{(j)}(t) + (c_N - c_O) \int_0^T \bar{F}(t) dG^{(N)}(t)}{\sum_{j=0}^{N-1} \int_0^T [\int_0^{\infty} \bar{G}(u) \bar{F}(t+u) du] dG^{(j)}(t)}. \quad (18)$$

Clearly, $\lim_{T \rightarrow \infty} C_{OF}(T, N) = C(N)$ in (5) and $\lim_{N \rightarrow \infty} C_{OF}(T, N) = C_O(T)$ in (14).

When $G(t) = 1 - e^{-\theta t}$ and $c_O = c_N$, optimum T_{OF}^* satisfies

$$\tilde{Q}_1(T) \sum_{j=0}^{N-1} \int_0^T \frac{(\theta t)^j}{j!} e^{-\theta t} \bar{F}(t) dt - \sum_{j=0}^{N-1} \int_0^T \frac{(\theta t)^j}{j!} e^{-\theta t} dF(t) = \frac{c_O}{c_F - c_O}. \quad (19)$$

The left-hand side of (19) increases with T to $h(\infty)$. Thus, there exists a unique T_{OF}^* ($0 < T_{OF}^* < \infty$) which satisfies (19), and the resulting cost rate is

$$C_{OF}(T_{OF}^*, N) = (c_F - c_O) \tilde{Q}_1(T_{OF}^*). \quad (20)$$

Noting that the left-hand side of (19) increases strictly with N to that of (16), T_{OF}^* decreases strictly with N to T_O^* given in (16).

Forming the inequality $C_{OF}(T, N+1) - C_{OF}(T, N) \geq 0$,

$$\begin{aligned} & Q_2(T, N-1) \sum_{j=0}^{N-1} \left[\frac{(\theta T)^j}{j!} \int_T^\infty e^{-\theta t} \bar{F}(t) dt + \int_0^T \frac{(\theta t)^j}{j!} \bar{F}(t) dt \right] \\ & - \sum_{j=0}^{N-1} \left[\frac{(\theta T)^j}{j!} \int_T^\infty e^{-\theta t} dF(t) + \int_0^T \frac{(\theta t)^j}{j!} e^{-\theta t} dF(t) \right] \geq \frac{c_O}{c_F - c_O}, \end{aligned} \quad (21)$$

where

$$Q_2(T, N) \equiv \frac{\int_0^T (\theta t)^N \left[\int_t^\infty e^{-\theta u} dF(u) \right] dt}{\int_0^T (\theta t)^N \left[\int_t^\infty e^{-\theta u} \bar{F}(u) du \right] dt} < \tilde{Q}_1(T)$$

increases strictly with N to $\tilde{Q}_1(T)$ and increases with T (Nakagawa, 2014). Substituting (19) for (21),

$$Q_2(T, N) \geq \tilde{Q}_1(T),$$

which does not hold for any N , i.e., $N_{OF}^* = \infty$. Thus, optimum policy to minimize $C_{OF}(T, N)$ is $(T_{OF}^* = T_O^*, N_{OF}^* = \infty)$, where T_O^* is given in (16). This shows that when the costs of preventive replacement are equal, replacement overtime is more economical than random policy.

Furthermore, noting that $Q_2(T, N)$ increase with N to $\tilde{Q}_1(T)$ (Nakagawa and Zhao, 2015), the left-hand side of (21) increase strictly with N to that of (16). Thus, if $T > T_O^*$, then there exists a unique minimum N_{OF}^* ($1 \leq N_{OF}^* < \infty$) which satisfies (21), and decrease with T to N^* given in (6). Conversely, if $T \leq T_O^*$, then $N_{OF}^* = \infty$.

5.2 Replacement Overtime Last

Suppose the unit is replaced over time T or at cycle N , whichever occurs last. Then, the expected cost rate is (Nakagawa and Zhao, 2015).

$$C_{OL}(T, N) = \frac{c_O + (c_N - c_O) \int_T^\infty \bar{F}(t) dG^{(N)}(t) + (c_F - c_O) \left(\int_0^\infty F(t) dG^{(N)}(t) + \sum_{j=N}^\infty \int_0^T \left\{ \int_0^\infty [F(t+u) - F(t)] dG(u) \right\} dG^{(j)}(t) \right)}{\int_0^\infty [1 - G^{(N)}(t)] \bar{F}(t) dt + \sum_{j=N}^\infty \int_0^T \left[\int_0^\infty \bar{G}(u) \bar{F}(t+u) du \right] dG^{(j)}(t)}. \quad (22)$$

Clearly, $\lim_{T \rightarrow 0} C_{OL}(T, N) = C(N)$ in (5) and $\lim_{N \rightarrow 0} C_{OL}(T, N) = C_O(T)$ in (14).

When $G(t) = 1 - e^{-\theta t}$ and $c_O = c_N$, optimum T_{OL}^* satisfies

$$\begin{aligned} & \tilde{Q}_1(T) \sum_{j=0}^{N-1} \int_T^{\infty} \frac{(\theta t)^j}{j!} e^{-\theta t} \bar{F}(t) dt - \sum_{j=0}^{N-1} \int_T^{\infty} \frac{(\theta t)^j}{j!} e^{-\theta t} dF(t) \\ & + \tilde{Q}_1(T) \int_0^T \bar{F}(t) dt - F(T) = \frac{c_O}{c_F - c_O}. \end{aligned} \quad (23)$$

The left-hand side increase with T to $h(\infty) = \infty$. Thus, there exists a unique T_{OL}^* ($0 < T_{OL}^* < \infty$) which satisfies (23), and

$$C_{OL}(T_{OL}^*, N) = (c_F - c_O) \tilde{Q}_1(T_{OL}^*). \quad (24)$$

Noting that the left-hand side of (23) increase with N from that of (16), T_{OL}^* decreases with N from T_O^* given in (16).

Forming the inequality $C_{OL}(T, N+1) - C_{OL}(T, N) \geq 0$,

$$\begin{aligned} & \tilde{Q}_2(T, N-1) \left[\int_0^T \bar{F}(t) dt + \sum_{j=0}^{N-1} \int_T^{\infty} \frac{(\theta t)^j}{j!} e^{-\theta t} \bar{F}(t) dt + \sum_{j=N}^{\infty} \frac{(\theta t)^j}{j!} \int_T^{\infty} e^{-\theta t} \bar{F}(t) dt \right] - \\ & \sum_{j=N}^{\infty} \left[\frac{(\theta t)^j}{j!} \int_T^{\infty} e^{-\theta t} dF(t) - \int_T^{\infty} \frac{(\theta t)^j}{j!} e^{-\theta t} dF(t) \right] - 1 \geq \frac{c_O}{c_F - c_O}, \end{aligned} \quad (25)$$

where

$$\tilde{Q}_2(T, N) \equiv \frac{\int_T^{\infty} (\theta t)^N \left[\int_t^{\infty} e^{-\theta u} dF(u) \right] dt}{\int_T^{\infty} (\theta t)^N \left[\int_t^{\infty} e^{-\theta u} \bar{F}(t) du \right] dt} < \tilde{Q}_1(T),$$

increases with N from $\tilde{Q}_1(T)$ and increases with T to $h(\infty)$. Substituting (23) for (25),

$$\tilde{Q}_2(T, N-1) \geq \tilde{Q}_1(T),$$

which always holds for any T , i.e., $N_{OL}^* = 0$. Thus, optimum policy to minimize $C_{OL}(T, N)$ is $(T_{OL}^* = T_O^*, N_{OL}^* = \infty)$, where T_O^* is given in (16). This shows that when the costs of preventive replacement are equal, replacement overtime is more economical than random policy.

Furthermore, noting that $\tilde{Q}_2(T, N)$ increases with N from $\tilde{Q}_1(T)$, the left-hand side of (25) increase with N from

$$\begin{aligned} & \tilde{Q}_2(T, 0) \left[\int_0^T \bar{F}(t) dt + \int_T^{\infty} e^{-\theta(t-T)} \bar{F}(t) dt \right] + \int_T^{\infty} [1 - e^{-\theta(t-T)}] dF(t) - 1 \\ & \geq \tilde{Q}_1(T) \int_0^T \bar{F}(t) dt - F(T), \end{aligned}$$

which agrees with that of (16). Thus, if $T < T_O^*$, then there exists a unique and minimum N_{OL}^* ($1 \leq N_{OL}^* < \infty$) which satisfies (25), and decreases with T from N^* given in (6). Conversely, if $T \geq T_O^*$, then $N_{OL}^* = 0$.

6. Replacement Middle

Suppose the unit is replaced at T , Y_1 or Y_2 , whichever occurs first, where each Y_i ($i = 1, 2$) has distribution $G_i(t) \equiv \Pr\{Y_i \leq t\}$. Then, the expected cost rate is

$$C_{2F}(T) = \frac{c_T + (c_1 - c_T) \int_0^T \bar{F}(t) \bar{G}_2(t) dG_1(t) + (c_2 - c_T) \int_0^T \bar{F}(t) \bar{G}_1(t) dG_2(t) + (c_F - c_T) \int_0^T \bar{G}_1(t) \bar{G}_2(t) dF(t)}{\int_0^T \bar{F}(t) \bar{G}_1(t) \bar{G}_2(t) dt}, \quad (26)$$

where c_i = replacement cost at Y_i with $c_i < c_F$ ($i = 1, 2$) and c_T and c_F are given in (2).

When $c_i = c_T$, optimum T_{2F}^* to minimize $C_{2F}(T)$ satisfies

$$\int_0^T \bar{F}(t) \bar{G}_1(t) dG_2(t) [h(T) - h(t)] dt = \frac{c_T}{c_F - c_T}. \quad (27)$$

The left-hand side of (27) increases with T from 0 to $h(\infty) = \infty$. Thus, there exists a unique T_{2F}^* ($0 < T_{2F}^* < \infty$) which satisfies (27), and the resulting cost rate is

$$C_{2F}(T_{2F}^*) = (c_F - c_T) h(T_{2F}^*). \quad (28)$$

Next, suppose the unit is replaced at times T , Y_1 or Y_2 , whichever occurs last. Then, the expected cost rate is

$$C_{2L}(T) = \frac{c_T + (c_1 - c_T) \int_T^\infty \bar{F}(t) G_2(t) dG_1(t) + (c_2 - c_T) \int_T^\infty \bar{F}(t) G_1(t) dG_2(t) + (c_F - c_T) [1 - \int_T^\infty G_1(t) G_2(t) dF(t)]}{\int_0^T \bar{F}(t) dt + \int_T^\infty [1 - G_1(t) G_2(t)] \bar{F}(t) dt}. \quad (29)$$

When $c_i = c_T$, optimum T_{2L}^* to minimize $C_{2L}(T)$ satisfies

$$\int_0^T \bar{F}(t) [h(T) - h(t)] dt - \int_T^\infty [1 - G_1(t) G_2(t)] \bar{F}(t) [h(t) - h(T)] dt = \frac{c_T}{c_F - c_T}. \quad (30)$$

The left-hand side of (30) increases strictly with T to $h(\infty) = \infty$. Thus, there exists a finite and unique T_{2L}^* ($0 < T_{2L}^* < \infty$) which satisfies (30), and the resulting cost rate is

$$C_{2L}(T_{2L}^*) = (c_F - c_T) h(T_{2L}^*). \quad (31)$$

Finally, suppose the unit is replaced at times T , Y_1 or Y_2 , whichever occurs middle. We call this policy as *replacement middle* (Zhao et al., 2015), and the expected cost rate is

$$C_{2M}(T) = \frac{c_T + (c_1 - c_T) \left[\int_0^T \bar{F}(t) G_2(t) dG_1(t) + \int_T^\infty \bar{F}(t) \bar{G}_2(t) dG_1(t) \right] + (c_2 - c_T) \left[\int_0^T \bar{F}(t) G_1(t) dG_2(t) + \int_T^\infty \bar{F}(t) \bar{G}_1(t) dG_2(t) \right] + (c_F - c_T) \left\{ \int_0^T [1 - G_1(t) G_2(t)] dF(t) + \int_T^\infty \bar{G}_1(t) \bar{G}_2(t) dF(t) \right\}}{\int_0^T [1 - G_1(t) G_2(t)] \bar{F}(t) dt + \int_T^\infty \bar{F}(t) \bar{G}_1(t) \bar{G}_2(t) dt}. \quad (32)$$

when $c_i = c_T$, optimum T_{2M}^* to minimize $C_{2M}(T)$ satisfies

$$\int_0^T \bar{F}(t)[1 - G_1(t)G_2(t)][h(T) - h(t)]dt - \int_T^\infty \bar{F}(t)\bar{G}_1(t)\bar{G}_2(t)[h(t) - h(T)]dt = \frac{c_T}{c_F - c_T}. \quad (33)$$

The left-hand side of (33) increases with T to ∞ . Thus, there exists a unique T_{2M}^* ($0 < T_{2M}^* < \infty$) which satisfies (33), and

$$C_{2M}(T_{2M}^*) = (c_F - c_T)h(T_{2M}^*). \quad (34)$$

6.1 Comparison of Three Policies

Let $L_F(T)$, $L_L(T)$ and $L_M(T)$ be the left-hand sides of (27), (30) and (33), respectively. From (27) and (30),

$$L_{LF}(T) \equiv L_L(T) - L_F(T) = \int_0^T [1 - \bar{G}_1(t)\bar{G}_2(t)][h(T) - h(t)]dt - \int_T^\infty \bar{F}(t)[1 - G_1(t)G_2(t)][h(t) - h(T)]dt,$$

which increases with T from $L_{LF}(0) < 0$ to ∞ . Thus, there exists a unique $\tilde{T}_{LF}$ ($0 < \tilde{T}_{LF} < \infty$) which satisfies $L_{LF}(T) = 0$.

From (27) and (33),

$$L_{MF}(T) \equiv L_M(T) - L_F(T) = \int_0^T [G_1(t)\bar{G}_2(t) + \bar{G}_1(t)G_2(t)][h(T) - h(t)]dt - \int_T^\infty \bar{F}(t)\bar{G}_1(t)\bar{G}_2(t)[h(t) - h(T)]dt,$$

which increases strictly with T from $L_{MF}(0) < 0$ to ∞ . Thus, there exists unique $\tilde{T}_{MF}$ ($0 < \tilde{T}_{MF} < \infty$) which satisfies $L_{MF}(T) = 0$.

From (30) and (33)

$$L_{LM}(T) \equiv L_L(T) - L_M(T) = \int_0^T \bar{F}(t)G_1(t)G_2(t)[h(T) - h(t)]dt - \int_T^\infty \bar{F}(t)[G_1(t)\bar{G}_2(t) + \bar{G}_1(t)G_2(t)][h(t) - h(T)]dt,$$

which increases strictly with T from $L_{LM}(0) < 0$ to ∞ . Thus, there exists a finite and unique $\tilde{T}_{LM}$ ($0 < \tilde{T}_{LM} < \infty$) which satisfies $L_{LM}(T) = 0$.

The above discussions show the following comparative results:

- (i) If $L_M(\tilde{T}_{MF}) \geq c_T/(c_F - c_T)$ and $L_M(\tilde{T}_{LM}) \geq c_T/(c_F - c_T)$, then the ranking is replacement first, middle, last.
- (ii) If $L_M(\tilde{T}_{MF}) < c_T/(c_F - c_T) < L_M(\tilde{T}_{LM})$ and $L_F(\tilde{T}_{LF}) \geq c_T/(c_F - c_T)$, then the ranking is replacement middle, first, last.
- (iii) If $L_M(\tilde{T}_{LM}) < c_T/(c_F - c_T) < L_M(\tilde{T}_{MF})$ and $L_F(\tilde{T}_{LF}) < c_T/(c_F - c_T)$, then the ranking is replacement middle, last, first.
- (iv) If $L_M(\tilde{T}_{MF}) < c_T/(c_F - c_T)$ and $L_M(\tilde{T}_{LM}) \leq c_T/(c_F - c_T)$, then the ranking is replacement last, middle, first.

7. General Age Replacement

Suppose the unit is replaced at random times Y_i , where Y_i ($i = 1, 2, \dots, n$) are independent with each other and have general distributions $G_i(t)$ with finite means $1/\theta_i$, respectively. Denoting $Y_m \equiv \min\{Y_1, Y_2, \dots, Y_n\}$, it has a distribution

$$\Pr\{Y_m \leq t\} = 1 - \prod_{i=1}^n \bar{G}_i(t), \quad (35)$$

and denoting $Y_M \equiv \max\{Y_1, Y_2, \dots, Y_n\}$, it has a distribution

$$\Pr\{Y_M \leq t\} = \prod_{i=1}^n G_i(t). \quad (36)$$

Note that (35) and (36) correspond to the respective failure distributions of a series system and a parallel system with n units when $G_i(t)$ is the failure distribution of unit i (Nakagawa, 2008), so that we call the above policies as *redundant replacement*.

Next, we consider the following four replacement policies:

- (a) Replacement first: The unit is replaced at time T or at time Y_m , whichever occurs first. Then, the unit is replaced at time $Y_F \equiv \min\{T, Y_m\}$ with a general distribution

$$G_F(t) \equiv \Pr\{Y_F \leq t\} = \begin{cases} 1 - \prod_{i=1}^n \bar{G}_i(t) & t < T, \\ 1 & t \geq T. \end{cases} \quad (37)$$

- (b) Modified replacement first: The unit is replaced at time T or at time Y_M , whichever occurs first. Then, the unit is replaced at time $\tilde{Y}_F \equiv \min\{T, Y_M\}$ with a general distribution

$$\tilde{G}_F(t) \equiv \Pr\{\tilde{Y}_F \leq t\} = \begin{cases} \prod_{i=1}^n G_i(t) & t < T, \\ 1 & t \geq T. \end{cases} \quad (38)$$

- (c) Replacement last: The unit is replaced at time T or at time Y_M , whichever occurs last. Then, the unit is replaced at time $Y_L \equiv \max\{T, Y_M\}$ with a general distribution

$$G_L(t) \equiv \Pr\{\tilde{Y}_F \leq t\} = \begin{cases} 0 & t < T, \\ \prod_{i=1}^n G_i(t) & t \geq T. \end{cases} \quad (39)$$

(d) Modified replacement last: The unit is replaced at time T or at time Y_m , whichever occurs last.

Then, the unit is replaced at time $\tilde{Y}_L \equiv \max\{T, Y_m\}$ with a general distribution

$$\tilde{G}_L(t) \equiv \Pr\{Y_L \leq t\} = \begin{cases} 0 & t < T, \\ 1 - \prod_{i=1}^n \bar{G}_i(t) & t \geq T. \end{cases} \quad (40)$$

From the definitions of Y_i and $\tilde{Y}_i$ ($i = F, L$), $Y_F \leq \tilde{Y}_F \leq \tilde{Y}_L \leq Y_L$, and hence, $G_F(t) \geq \tilde{G}_F(t) \geq \tilde{G}_L(t) \geq G_L(t)$.

7.1 Replacement First

Suppose the unit is replaced at time Y_F in case (a). Then, replacing $G(t)$ with $G_F(t)$ in (37), the expected cost rate is (Chen et al., 2016)

$$C_{GF}(T) = \frac{c_R + (c_F - c_R) \int_0^T \prod_{i=1}^n \bar{G}_i(t) dF(t)}{\int_0^T \bar{F}(t) \prod_{i=1}^n \bar{G}_i(t) dt}, \quad (41)$$

where c_R = replacement cost at time Y_i with $c_R < c_F$.

Optimum T_F^* to minimize $C_{GF}(T)$ satisfies

$$\int_0^T \bar{F}(t) [h(T) - h(t)] \prod_{i=1}^n \bar{G}_i(t) dt = \frac{c_R}{c_F - c_R}. \quad (42)$$

The left-hand side of (42) increases with T to $h(\infty) = \infty$. Thus, there exists a unique T_F^* ($0 < T_F^* < \infty$) which satisfies (42), and the resulting cost rate is

$$C_{GF}(T_F^*) = (c_F - c_R) h(T_F^*) \quad (43)$$

Next, suppose the unit is replaced at time $\tilde{Y}_F$ in case (b). Then, replacing $G(t)$ with $G_F(t)$ in (38), the expected cost rate is (Chen et al., 2016)

$$\tilde{C}_{GF}(T) = \frac{c_R + (c_F - c_R) \int_0^T [1 - \prod_{i=1}^n G_i(t)] dF(t)}{\int_0^T \bar{F}(t) [1 - \prod_{i=1}^n G_i(t)] dt}. \quad (44)$$

Optimum $\tilde{T}_F^*$ to minimize $\tilde{C}_{GF}(T)$ satisfies

$$\int_0^T \bar{F}(t) [h(T) - h(t)] [1 - \prod_{i=1}^n G_i(t)] dt = \frac{c_R}{c_F - c_R}. \quad (45)$$

The left-hand side of (45) increases with T to $h(\infty) = \infty$. Thus, there exists a unique $\tilde{T}_F^*$ ($0 < \tilde{T}_F^* < \infty$) which satisfies (42), and the resulting cost rate is

$$\tilde{C}_{GF}(\tilde{T}_F^*) = (c_F - c_R)h(\tilde{T}_F^*). \quad (46)$$

Comparing (42) and (45)

$$\int_0^T \bar{F}(t)[h(T) - h(t)] \left[1 - \prod_{i=1}^n G_i(t) \right] dt - \int_0^T \bar{F}(t)[h(T) - h(t)] \prod_{i=1}^n \bar{G}_i(t) dt \geq 0,$$

which follows that $\tilde{T}_F^* \leq T_F^*$. Thus, from (43) and (46), $\tilde{C}_{GF}(\tilde{T}_F^*) \leq C_{GF}(T_F^*)$, i.e., modified replacement first is economical than replacement first.

7.2 Replacement Last

Suppose the unit is replaced at time Y_L in case (c). Then, replacing $G(t)$ with $G_L(t)$ in (39), the expected cost rate is (Chen et al., 2016)

$$C_{GL}(T) = \frac{c_R + (c_F - c_R) \left[1 - \int_0^\infty \prod_{i=1}^n G_i(t) dF(t) \right]}{\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{F}(t) [1 - \prod_{i=1}^n G_i(t)] dt}. \quad (47)$$

Optimum T_L^* to minimize $C_{GL}(T)$ satisfies

$$\int_0^T \bar{F}(t)[h(T) - h(t)] dt - \int_T^\infty \bar{F}(t)[h(T) - h(t)] [1 - \prod_{i=1}^n G_i(t)] dt = \frac{c_R}{c_F - c_R}. \quad (48)$$

The left-hand side of (48) increases strictly with T to ∞ . Thus, there exists a unique T_L^* ($0 < T_L^* < \infty$) which satisfies (48), and the resulting cost rate is

$$C_{GL}(T_L^*) = (c_F - c_R)h(T_L^*). \quad (49)$$

Next, suppose the unit is replaced at time $\tilde{Y}_L$ in case (d). Then, replacing $G(t)$ with $\tilde{G}_L(t)$ in (39), the expected cost rate is (Chen et al., 2016)

$$\tilde{C}_{GL}(T) = \frac{c_R + (c_F - c_R) \left\{ 1 - \int_0^\infty [1 - \prod_{i=1}^n \tilde{G}_i(t)] dF(t) \right\}}{\int_0^T \bar{F}(t) dt + \int_T^\infty \bar{F}(t) \prod_{i=1}^n \tilde{G}_i(t) dt}. \quad (50)$$

Optimum $\tilde{T}_L^*$ to minimize $\tilde{C}_{GL}(T)$ satisfies

$$\int_0^T \bar{F}(t)[h(T) - h(t)] dt - \int_T^\infty \bar{F}(t)[h(t) - h(T)] \prod_{i=1}^n \tilde{G}_i(t) dt = \frac{c_R}{c_F - c_R}. \quad (51)$$

The left-hand side of (51) increases with T to $h(\infty) = \infty$. Thus, there exists a unique $\tilde{T}_L^*$ ($0 < \tilde{T}_L^* < \infty$) which satisfies (51), and the resulting cost rate is

$$\tilde{C}_{GL}(\tilde{T}_L^*) = (c_F - c_R)h(\tilde{T}_L^*). \quad (52)$$

Comparing (48) and (52),

$$\int_T^\infty \bar{F}(t)[h(t) - h(T)] \left[1 - \prod_{i=1}^n G_i(t) \right] dt - \int_T^\infty \bar{F}(t)[h(t) - h(T)] \prod_{i=1}^n \bar{G}_i(t) dt \geq 0,$$

which follows that $\tilde{T}_L^* \leq T_L^*$. Thus, from (49) and (52), $\tilde{C}_{GL}(\tilde{T}_L^*) \leq C_{GL}(T_L^*)$, i.e., modified replacement last is economical than replacement last.

Finally, it is assumed that $c_{Ri} (i = 1, 2, \dots, n)$ are respective replacement costs at times Y_i . Then, the expected cost rate in (41) becomes

$$C_{GF}(T) = \frac{c_F \int_0^T \prod_{i=1}^n \bar{G}_i(t) dF(t) + c_T \bar{F}(T) \prod_{i=1}^n \bar{G}_i(t) + \sum_{i=1}^n c_{Ri} \int_0^T \bar{F}(t) \prod_{i=1, j \neq i}^n \bar{G}_i(t) dG_i(t)}{\int_0^T \bar{F}(t) \prod_{i=1}^n \bar{G}_i(t) dt}. \quad (53)$$

Similarly, $\tilde{C}_{GF}(T)$ in (44), $C_{GL}(T)$ in (47) and $\tilde{C}_{GL}(T)$ in (50) can be obtained.

8. Conclusions

Age replacement model has been already challenged that the constant planned T can be a random variable Y to meet the random perforation need. In this paper, the so called replacement that is planned at time T and random replacement planned at random time Y have been introduced. Another random replacement that is planned at working cycles N are also obtained. If we need to consider both age and random replacement policies with working cycles, we next have discussed replacement first, replacement last, replacement overtime and replacement middle. The motivations of these methods can be found in our literatures, so that we give the expected cost rates directly and discuss their optimum policies to minimize them analytically.

Another contribution of the paper is to give comparisons for these replacement policies. For examples, it has been shown that when the costs of preventive replacement are equal, age replacement is more economical than the random policy, age replacement is more economical than replacement overtime, replacement overtime is more economical than the random policy. Also, we have compared replacement first and replacement last analytically and showed that both have advantages in cost rates.

In order to formulate general replacement models of replacement first and last, we have obtained general distributions of replacement times, and the models of general replacement become our final target of the paper.

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