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Research Article

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A Real Approximation to Riemann Hypothesis

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Abstract
This work present the requisites to define ζ(s) like an analytic function on real numbers; as well as a numerical evaluation made computationally to obtain criteria of convergence. By the other hand its discusses some algebraic identities to establish where the function zeta Riemann gives place to ζ(s) = 0.

1. Introduction

Riemann Hypothesis Establish Next Assessment
The Riemann hypothesis asserts that all interesting solutions of the equation ζ(s) = 0 lie on a certain vertical straight line. Cite: Clay Mathematics Institute; December of 2023.
Riemann function is defined as next:

$$\zeta(s) = \Sigma_{n=1}^{\infty} \frac{1}{n^s}$$

Where $s = \sigma + it \in C$ such that $\Re(s) > 1$
Remember that the meaning of a number powered by a complex number have any definition√. For that reason lets use the meaning established on *Appendix 1* ; then $i = \pm\sqrt{-1} = \pm 1$ and therefore for the development of this work

$$s = \sigma \pm t$$

Like the problem were established with the idea to define such equation with complex powers then I will develop next argumentation using a famous corollary about analytic functions.

2. Methodology

Definition. Formally, a function f is real analytic on an open set D in the real line if for any $s_0 \in D$ one can write

$$f(s) = \Sigma_{n=0}^{\infty} a_n (s-s_0)^n = a_0 + a_1(s-s_0) + a_2(s-s_0)^2 + a_3(s-s_0)^3 ...$$

in which the coefficients a_0, a_1, ... are real numbers and the series is convergent to $f(s)$ for s in a neighbourhood of s_0.

Proof
Lets take the series of powers

$$S = \Sigma_{n=0}^{\infty} a_n (s-s_0)^n = a_0 + a_1(s-s_0) + a_2(s-s_0)^2 + a_3(s-s_0)^3 ...$$

when $r = s - s_0$ and $n \to \infty$; S must fulfilled the condition $|r| < 1$ to converge. Rewriting S expression with r we have:

Rewriting S expression with r we have:

$$S = \Sigma_{n=0}^{\infty} a_n r^n = a_0 + a_1 r + a_2 r^2 + a_3 r^3 ...$$

Now consider next equation:

$$a_n = \frac{1}{(n+1)^s (s-s_0)^n} = \frac{1}{(n+1)^s r^n}$$

Substituting on S equation we have:

$$S = \Sigma_{n=0}^{\infty}(\frac{1}{(n+1)^s r^n})r^n = \frac{1}{1^s r^0} + \frac{1}{(1+1)^s r^1}r + \frac{1}{(2+1)^s r^2}r^2 + \frac{1}{(3+1)^s r^3}r^3...$$

Simplifying we have:

$$S = \Sigma_{n=0}^{\infty}\frac{1}{(n+1)^s}r^n = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...$$

with $\eta = n + 1$ and substituting on S we have:

$$S = \Sigma_{\eta-1=0}^{\infty}\frac{1}{(\eta-1+1)^s} = \Sigma_{\eta=1}^{\infty}\frac{1}{\eta^s} = \zeta(s)$$

This equation is also know as "Riemann zeta function"; being the particular case of series of powers when all terms $(s-s_0)^n$ are equal to 1 (hint: another particular case of series of powers is the "geometric series" on this case all coefficients a_n are equal).

Lets prove that Zeta Riemann function is a monotone decreasing sequence,

$$\{\frac{1}{\eta^s}\}$$

evaluating $\eta \in \{1, 2, 3, ..., i, ...\}$ we have:

$\frac{1}{1^s}, \frac{1}{2^s}, \frac{1}{3^s}, ..., \frac{1}{i^s}, ...$

like

$\frac{1}{1^s} > \frac{1}{2^s} > \frac{1}{3^s} > ... > \frac{1}{i^s} > ...$

Hence the sequence is decreasing

Now taking the limit of η when tends to infinite we have:

$lim_{\eta\to\infty}\{\frac{1}{\eta^s}\} = 0$

By definition of quotient between ∞.

Considering that index i increase by one unit at time, then we can conclude that the sequence is monotone.

Finally with $\eta > 0$ therefore ηs is positive $\forall s \in \Re$; hence $\frac{1}{\eta^s} > 0$

Due to early properties about $\zeta(s)$ series; Cauchy condensation test can be used to evaluate convergence of this series.

Lets consider the series

$\Sigma_{\eta=0}^{\infty}2^{\eta}\frac{1}{\eta^{2\eta}} = 2^0\frac{1}{0^{2(0)}} + 2^1\frac{1}{1^{2(1)}} + 2^2\frac{1}{2^{2(2)}} + 2^3\frac{1}{3^{2(3)}} + 2^4\frac{1}{4^{2(4)}} + ...$

$= 1\frac{1}{1} + 2\frac{1}{1} + 4\frac{1}{2^4} + 8\frac{1}{3^6} + 16\frac{1}{4^8} + 32\frac{1}{5^{10}} + ...$

$= 3.2512193300475...$

And lets name this number like ζ_{test}.

The next condition establish if $\zeta(s)$ series converge:

$\Sigma_{\eta=1}^{\infty}\frac{1}{\eta^s} \leq \Sigma_{\eta=0}^{\infty}2^{\eta}\frac{1}{\eta^{2\eta}}$

Like $\Sigma_{\eta=0}^{\infty} 2^{\eta} \frac{1}{\eta^{2\eta}} = \zeta_{test}$ (i.e. converge to the number ζ_{test})

Therefore $\zeta(s)$ is convergent.

Next table shows the result to evaluate ζ_{test} from $\eta = 0$ until $\eta = 9$.

n	2^n	n^{2n}	$\frac{1}{n^{2n}}$	$2^n \frac{1}{n^{2n}}$
0	1	1	1	1
1	2	1	1	2
2	4	16	0.0625	0.25
3	8	6561	0.000152415790276	0.001219326322207
4	16	4294967296	2.3283064365387E-10	3.72529029846191E-09
5	32	2.3283064365387E+022	4.294967296E-23	1.37438953472E-21
6	64	6.33402866629733E+049	1.57877403574267E-50	1.01041538287531E-48
7	128	1.48781564719761E+108	6.72126282502512E-109	8.60321641603216E-107
8	256	1.55251809230071E+231	6.44114876959713E-232	1.64893408501687E-229
9	512	NUM!	NUM!	NUM!

Table 1. Evaluation of ζ_{test} for different values of η.

Can be established that $\zeta(s)$ series converges to ζ_{test} because the coefficients of each term are for main value $2^{\eta}\ 1/\eta^{2\eta}$ to small when $\eta = 8$; to know a power of around 10^{-229}; on LibreOffice Calc is not possible to obtain more values for $\eta > 8$ (note: the value $0^0 = 1$ is not defined on this software and is necessary to introduce number 1 at hand).

Early situation can be understood like; the series increase when next term is added; for $\eta > 8$ ζ_{test} series increase an amount infinitely small, reaching practically a constant value. Hence $\zeta(s)$ converge because ζ_{test} is constant.

Some useful number to compare early result are:

$$\zeta_{test} - \pi = 0.109626676457704...$$

$$\frac{\zeta_{test}}{\pi} = 1.03489525490596...$$

$$e^{\zeta_{test}} = 25.8218060334812...$$

$$\pi^{\zeta_{test}} = 41.3374113001018...$$

Lets note that converge criteria to an open set $D(s, s_0) = \{s|\ d(s, s_0) < \epsilon\}$; must be established taking into account the value of ζ_{test}.

The value of s has been approximated manually calculating $\zeta(s)$ on Libre-Office Calc using a laptop Dell Vostro 14 - 3000 series with a memory of 7.5 GiB; processor Intel Core i5-7200 CPU 2.5 GHz x 4; graphics Mesa Intel HD Graphics 620 (KBL GT2); disk capacity of 1.0 TB and Ubuntu 20.04.6 LTS for different values of η and comparing its result vs the value of ζ_{test}. The value given was s = 1.4. Higher values for η were not considered due to a freeze on the graphic interface.

	$\eta = 5000$	$\eta = 10000$	$\eta = 20000$
$\zeta(s = 1.3)$	3.67302377623045	3.72163325170118	3.76111828801966
$\zeta(s = 1.4)$	3.02268924184343	3.04275137310375	3.05795640541761
$\zeta(s = 1.5)$	2.58409249158088	2.59237584867299	2.59823338983623

Table 2. Values of $\zeta(s)$ for different values of s and η

For $s < 1.4$, $\zeta(s)$ converge sooner, i.e. for $\eta < 5000$; while for $s > 1.4$; $\zeta(s)$ converge because for $\eta > 20000$; this result suggest that Cauchy condensation test can be satisfied fully; i.e. when η tends to infinite.

The criterion used on this work was next:

S convergence to $\zeta(s)$ for a minimum value of $s = 1.4$

Lets call this value like constant of convergence or $s_{convergence}$.

Like was established before; for any value of $s \leq s_{convergence}$ $\zeta(s)$ not converge. If convergence criteria is modified to consider all points that fulfilled Cauchy condensation test; when η is finite ($\eta < 5000$); $\zeta(s)$ convergence dominion could be extended

$\zeta(s = 0.1); \eta = 3$	3.1878966365702
$\zeta(s = -0.9); \eta = 11$	3.22114303824938

Table 3. Values of $\zeta(s)$ for different values of s and η small.

Now lets develop topological criteria about open sets to satisfy convergence of $\zeta(s)$.

Lets take two values s_1 and s_2 on the zone of convergence such that:

$$s_{convergence} \leq s_1 \leq s_1$$

$$s_{convergence} \leq s_2 \leq s_2$$

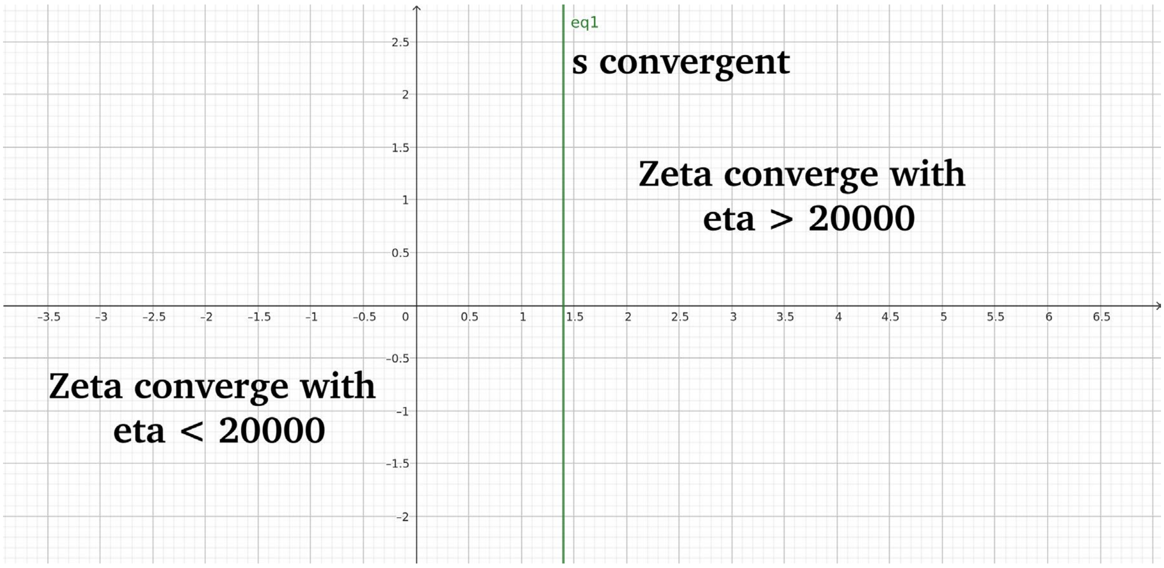
On convergence zone s_1 and s_2 exist to be part of $\Re$ and satisfy convergence regarding $s_{convergence}$ criteria.

Subtracting second term of first we have:

$$0 \leq d(s_1, s_2) = s_1 - s_2 \leq s_1 - s_2 = \epsilon$$

Hence S converge to $\zeta(s)$; $\forall s \in D(s, s_0) = \{s | \ d(s, s_0) < \epsilon \ ; \zeta(s) \leq \zeta_{test}$ and $s \geq s_{convergence}\}$

Therefore $\zeta(s)$ is analytical on the conditions described early.



Graph 1. Cartesian plane and zones of convergence finite (left) and infinite(right) for $\zeta(s)$ regarding Cauchy condensation test.

Principle of Analytic Continuation, or Identity Theorem. Let f and g be analytic in a region A. Suppose that there is a sequence $s_1, s_2, \ldots$ of distinct points of A converging to $s_0 \in A$, such that $f(s_n) = g(s_n)$ for all $n = 1, 2, 3, \ldots$ Then $f = g$ on all of A. The conclusion is valid, in particular if $f = g$ on some neighbourhood of some point in A.

Consider first the equation of the k-sphere:

$$r^2 = \Sigma_{i=1}^{k} (x_i - c_i)^2$$

Clearing r from k-sphere equation and substituting on $\zeta(s)$ we have:

$$\zeta(s) = \Sigma_{r=1}^{\infty} \frac{1}{r^s} = \Sigma_{r=1}^{\infty} \frac{1}{(\sqrt{\Sigma_{i=1}^{k} (x_i - c_i)^2})_r^s} \ \ldots(1)$$

where every sub-index r indicate the integer value of the radius for each term; such that $x_i \in [c_i, \text{r}]$ with $i, r \in \{1, 2, \ldots\}$ for each term. Lets call this function $\zeta_{k-sphere}(s)$.

Now lets prove this function is analytic by rewriting such equation as the general term of series of power:

$S_{k-sphere} =$

$\Sigma_{r=1}^{\infty} \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_r^s} = \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_1^s} + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_2^s} + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_3^s}$
$+ \ldots$

$= \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_1^s} \cdot 1 + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_2^s} \cdot 1 + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_3^s} \cdot 1 + \ldots$

$= \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_1^s} \cdot 1 + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_2^s} \frac{(s-s_0)}{(s-s_0)} + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_3^s} \frac{(s-s_0)^2}{(s-s_0)^2} + \ldots$

$= a_0 + a_1(s - s_0) + a_2(s - s_0)^2 + \ldots$

$= \Sigma_{i=0}^{\infty} a_i (s - s_0)^i$

as r tends to zero to satisfy the requisite to converge; $|s - s_0| < 1$ condition must be fulfilled.

By direct comparison test we have:

$$0 \leq \zeta_{k-sphere}(s) \leq \zeta(s)$$

In particular

$$\zeta_{k-sphere}(s) = \zeta(s)$$

Like $\zeta(s)$ converges by Cauchy condensation test hence $\zeta_{k-sphere}(s)$ converges.

Now taking

$|s - s_0| < \epsilon$

and establish early condition for two different points s_1 and $s_2 \in \Re$ we have:

$-\epsilon_1 < s_1 - s_0 < \epsilon_1$

$-\epsilon_2 < s_2 - s_0 < \epsilon_2$

Clearing s_1 and s_2

$-\epsilon_1 + s_0 < s_1 < \epsilon_1 + s_0$

$-\epsilon_2 + s_0 < s_2 < \epsilon_2 + s_0$

Subtracting both terms we have

$-\epsilon_1 + \epsilon_2 < s_1 - s_2 < \epsilon_1 - \epsilon_2$

Also

$-(\epsilon_1 - \epsilon_2) < s_1 - s_2 < \epsilon_1 - \epsilon_2$

Later

$d(s_1, s_2) = |s_1 - s_2| < \epsilon_1 - \epsilon_2 = \epsilon^*$

Therefore

$S_{k-sphere}$ converge to $\zeta_{k-sphere}(s)$ to an open set $\in \Re$ defined by direct comparison test like:

$\forall s \in D(s, s_0) = \{s |\ d(s, s_0) < \epsilon \text{ and } |s - s_0| < 1\} \in \Re$

Hence $\zeta_{k-sphere}(s)$ is analytic.

Lets take now

$$\Sigma_{r=1}^{\infty}\frac{1}{r^s} = \Sigma_{r=1}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})_r^s}$$

Note that like we´ve been substituted k-sphere equation into η term as well as defined conditions to establish convergence for both $\zeta(s)$ and $\zeta_{k-sphere}(s)$; convergence to an open set $D \in \Re$ must be established with the conditions given for both equations.

Lets define the arbitrary sequence $s_1, s_2, \ldots$ of m distinct points of $\Re$ converging to $s_0 = s_{convergene} \in \Re$

Let be

$\{s_m\} = s_{convergence} \cdot (m^{\frac{1}{m}})$

the term of such series.

Like

$m^{\frac{1}{m}} = 1$ when $m \to \infty$

hence

s_m converge to $s_{convergence}$

therefore $\zeta(s_m) = \zeta_{k-sphere}(s_m)$ for all $m = 1, 2, 3, \ldots$

Like $d(s, s_0) = |s - s_0| < 1$

and

$d(s, s_0) < \epsilon$

must be concluded that

$\epsilon = 1$

therefore

$D(s, s_0) = \{s|\ d(s, s_0) < \epsilon = 1\ ;\ \zeta(s) \leq \zeta_{test}$ and $s \geq s_{convergence}\}$

i.e.

$\zeta(s) = \zeta_{k-sphere}(s)$

on a neighbourhood of $s \in \Re$

The meaning of $\zeta(s) = 0$

Lets consider that $(c_1, c_2, c_3, \ldots, c_k)$ are the point of the center of some k-sphere $\in \Re$ and $(x_1, x_2, x_3, \ldots, x_k) \in [c, r]$ is some point inside such geometrical object such that

$$r = (\sqrt{\Sigma_{i=1}^{k}(x_i - c_i)^2})^s \in N$$

Note that all points $(x_1, x_2, x_3, \ldots, x_k) \in N$; that satisfied early condition defines the boundary (frontier, perimeter, surface, etc...) of such k-sphere.

Valuating $\zeta_{k-sphere}(s)$ with this conditions we have:

$\Sigma_{r=1}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_r^s} = \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_1^s} + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_2^s} + \frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_3^s} + \ldots$

$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots$

$= \Sigma_{r=1}^{\infty}\frac{1}{r^s} = \zeta(s)$

when $s=0$ and taking $a^0=1; \forall a \in N$ we have:

$\zeta_{k-sphere}(s=0)=1+1+1+...=\zeta(s=0)$

By the other hand if we take the limit of s when tend to infinite we have:

$lim_{s\to\infty}\zeta_{k-sphere}(s)=$

$lim_{s\to\infty}(\Sigma_{r=1}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_r^s})=lim_{s\to\infty}(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_1^s})+lim_{s\to\infty}(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_2^s})+$
$lim_{s\to\infty}(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_3^s})+...$

$=lim_{s\to\infty}(\frac{1}{1^s})+lim_{s\to\infty}(\frac{1}{2^s})+lim_{s\to\infty}(\frac{1}{3^s})+...$

$=lim_{s\to\infty}(\Sigma_{r=1}^{\infty}\frac{1}{r^s})=lim_{s\to\infty}\zeta(s)$

In specific

$lim_{s\to\infty}\zeta_{k-sphere}(s)=$

$lim_{s\to\infty}(\Sigma_{r=1}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_r^s})=(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_1^{lim_{s\to\infty}}})+(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_2^{lim_{s\to\infty}}})+$
$(\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_3^{lim_{s\to\infty}}})+...$

$=(\frac{1}{1^{lim_{s\to\infty}}})+(\frac{1}{2^{lim_{s\to\infty}}})+(\frac{1}{3^{lim_{s\to\infty}}})+...$

$=lim_{s\to\infty}(\Sigma_{r=1}^{\infty}\frac{1}{r^s})=lim_{s\to\infty}\zeta(s)$

Simplifying we have:

$lim_{s\to\infty}\zeta_{k-sphere}(s)=1+0+0+...=lim_{s\to\infty}\zeta(s)$

Hence

$lim_{s\to\infty}\zeta_{k-sphere}(s)=lim_{s\to\infty}\zeta(s)=1$

with

$\zeta(s)=\zeta_1(s)+\Sigma_{i=2}^{\infty}\frac{1}{r^s}$

calling second term like $\zeta(s)$ therefore:

$lim_{s\to\infty}\zeta(s)=0$

besides with

$\zeta_{k-sphere}(s)=\zeta_{k-sphere_1}(s)+\Sigma_{r=2}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_r^s}$

calling second term like $\zeta_{k-sphere}(s)$

$lim_{s\to\infty}\zeta_{k-sphere}(s)=0$

therefore

$lim_{s\to\infty}\zeta(s)=lim_{s\to\infty}\zeta_{k-sphere}(s)=0$

to fulfilled condition to be analytic and converge to an open set $D(s, s_0) \in \Re$; can be used the former requirements defined to establish the theorem of analytic continuation (identity theorem) taking into account that:

S converge to $\zeta(s); \forall s \in$

$D(s,s_0)=\{s|\ d(s,s_0)<\epsilon=1\ ;\ \zeta(s)\leq\zeta_{test}\ ;\ lim_{s\to\infty}\zeta(s)=lim_{s\to\infty}\zeta_{k-sphere}(s)=0$ and $s\geq s_{convergence}\}$

i.e. for a k-sphere located everywhere inside the zone of convergence of s; the condition $\zeta(s) = 0$ is satisfied; to know the condition of Riemann Hypothesis.

In particular when

$s=\sigma+it=\sigma\pm t$

with $\sigma=\frac{1}{2}$

$t \in \Re$ defines the zone of convergence for the open set $D(s, s_0)$ when theorem of function analytic for $\zeta(s)$ is satisfied.

Note that this condition maintains without consideration about the dimensionality of such k-sphere.

In particular s cannot be equal to 1/2 due to convergence criteria about sconvergence.

In other words every $s_1, s_2, \ldots \in \Re^k$ that tends to infinite and defines the power of the r term; i.e. the radius of a k -sphere in $\Re^k$ is a zero of $\zeta(s)$.

About the obtainment of $\zeta(s)$ and its relation with $\zeta_{k-sphere}(s)$.

Lets take k-sphere equation

$r^2=\Sigma_{i=1}^{k}(x_i-c_i)^2$

Now taking the inverse multiplicative from both sides we have

$\frac{1}{r^2}=\frac{1}{\Sigma_{i=1}^{k}(x_i-c_i)^2}$

Applying square root from both sides we have

$\frac{1}{r}=\frac{1}{\sqrt{\Sigma_{r=1}^{k}(x_i-c_i)^2}}$

And powering to s we have

$\frac{1}{r^s}=\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})^s}$

Giving values to $r\in\{1,2,3,...\}$ we have

$\frac{1}{1^s}=\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})^s}$

$\frac{1}{2^s}=\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})^s}$

$\frac{1}{3^s}=\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})^s}$

. . .

and adding all terms we will have:

$\zeta(s)=\Sigma_{r=1}^{\infty}\frac{1}{r^s}=\Sigma_{r=1}^{\infty}\frac{1}{(\sqrt{\Sigma_{i=1}^{k}(x_i-c_i)^2})_r^s}=\zeta_{k-sphere}(s)$

From early procedure we can see that $\zeta(s)$ have an identity linked with ksphere.

Endnote: If well analytic function requisite would be proved by Cauchy criterion must be understood that if the power by a complex number is not a mathematical operation with a clear definition, then this operation cannot be developed.

Same situation happens for the ”theoretical problem” to define a negative length for some geometrical object. To end this work its present a brief discussion about square, Cartesian plane and the meaning of $i=\sqrt{-1}$.

Appendix 1. The geometrical meaning of $i = \sqrt{-1}$.

Next operation is a famous result of ”The System of Complex Numbers”

$$i^2 = i \cdot i = (0, 1) \cdot (0, 1)$$

$$= (0 \cdot 0 - 1 \cdot 1, 1 \cdot 0 + 0 \cdot 1)$$

$$= (-1, 0) = -1$$

Early result is established using the rule of complex multiplication defined as:
$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$

In particular note that the result of such multiplication without to consider any other ”complex algebraic operation” to establish the number *i* is

$$i^2 = (-1, 0)$$

By the other hand; lets note that to define the result of

$$(-1, 0) = -1 \; ...(\alpha)$$

is necessary to use a ”projection function”; i.e. early result is expressed more properly by

$$f_{projection}(a, b) = (a)$$

Hence for α

$$f_{projection}(-1, 0) = (-1)$$

This detail has not been disregarded in the elaboration of this work.

Due to this problem an alternative explanation has been developed to argument the value of number i and its use to define ”power operation”.

Equation to obtain the area of an square can be used sometimes to represent geometrically the square root operation.

If we considerate that

$$A_{Square} = Side \cdot Side = Side^2$$

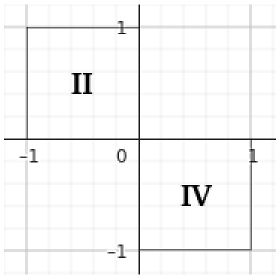
Then clearing ”Side” variable using square root we have:

$$\sqrt{Area} = Side$$

This means that the result to request the ”square root” to a number that represent the area of an square gives like result the length of the side that forms such geometrical object.

This operation can be used without to take into account a system of reference to be draw; i.e. can be used to measure squares drawn with help of compass, straightedge and other geometrical tools of design.

In specific when we use the Cartesian plane to draw an square we could find the problem show on next diagram:



Graph 2. Theoretical Squares of Negatives Sides on Cartesian System. Geogebra 2023.

We can see that draw an square into quadrants II and IV shows the problem to deal with at least one side length of negative value.

On the elaboration of this work has been sought to not pass by any "mathematical operation" that could be important to define some "mathematical procedure" not established formally; for the author of this work this situation is particularly important; taking into consideration only mathematical operations that could have some geometrical and algebraic representation; for example the equation to describe conic sections; in the sense that is usually considered on algebraic geometry.

For this reason; to considerate without a proof the use of the operation "ab- solute value" to define the length of such sides into the Cartesian plane has been avoided.

Instead has been considered that the length of such sides when is established negative; is a useful result (and more elemental) product to use the "scale" of the Cartesian plane.

The geometrical representation of the square, endowed with the algebraic equation to define the length of the side of an square described early, gives like result next relations:

$A_{QII} = \text{Area of square of quadrant II} = Side_x \cdot Side_y = (-1) \cdot (1) = -1$

also considering euclidean distance we have:

$(-1, 0)$ and $(0, 0)$

$\sqrt{(-1-0)^2 + (0-0)^2} = -1$

$\sqrt{(-1-0)^2} = -1$

$\sqrt{(-1)^2} = -1$

$A_{QIV} = \text{Area of square of quadrant IV} = Side_y \cdot Side_x = (-1) \cdot (1) = -1$

also considering euclidean distance we have:

$(0, -1)$ and $(0, 0)$

$\sqrt{(0-0)^2 + (-1-0)^2} = -1$

$\sqrt{(-1-0)^2} = -1$

$\sqrt{(-1)^2} = -1$

While when we obtain the square root for the first two equations we have:

$\sqrt{Area_{QII}} = \sqrt{Side_x \cdot Side_y} = \sqrt{(-1) \cdot (1)} = \sqrt{-1}$

$\sqrt{Area_{QIV}} = \sqrt{Side_y \cdot Side_x} = \sqrt{(-1) \cdot (1)} = \sqrt{-1}$

Remember that

$i = \sqrt{-1}$

By early argumentation was established that

$i = \pm(-1) = \pm 1$

For the elaboration of this work Argand´s discussion about complex number i was considered "incomplete" and "product of a theoretical misinterpretation about the use of the Cartesian plane".

In specific can be argue that the meaning of the term $\sqrt{(-1,0)}$ is not defined when is requested the square root of a 2-tuple.

This criterion establish that Complex Analysis program must be reappraise to avoid future mistakes; for example: the requisites to define an analytic function establish somewhere that a "pretended" Cauchy criterion about partial derivatives must be satisfied; when this differentiation principle around complex functions is not very well defined and therefore is not workable.

Same situation occurs about "zero of analytic functions" and other equation "commonly used" in complex analysis.

Appendix 2. The identity $n \cdot Ln(x) = Ln \cdot (x^n)$

I will show why is not possible to use Ln to clear coefficients of power f
some equation.

$Ln(x^n) = n\ Ln(x)$ ...(1)

Remember that

$Ln(x) = \int_1^x \frac{dt}{t}$

(1) could be rewrite like

$\int_1^{x^n} \frac{dt}{t} = n \int_1^x \frac{dt}{t}$

Using definition of Riemann´s Integral on early equality we have:

Definition Riemann´s Integral is such that

$\int_a^b f(x)dx = lim_{m\to\infty} \Sigma_{i=1}^m f(x_i)\Delta x$

where

$x_i \in x_0 = a, x_1, x_2, ..., x_m = b$; $\Delta x = \frac{b-a}{m}$
Applying this definition to Ln property we have

$\Delta x_{left} = \frac{x^n-1}{m}$ and $\Delta x_{right} = \frac{x-1}{m}$

while

$x_{i_{left}} = \frac{(x^n-1)i}{m}$ and $x_{i_{right}} = \frac{(x-1)i}{m}$

substituting on Riemann sum we have

$lim_{m\to\infty} \Sigma_{i=1}^m \frac{1}{\frac{(x^n-1)i}{m}} (\frac{x^n-1}{m}) = n\ lim_{m\to\infty} \Sigma_{i=1}^m \frac{1}{\frac{(x-1)i}{m}} (\frac{x-1}{m})$

expanding we have

$lim_{m\to\infty} (\frac{1}{\frac{(x^n-1)1}{m}} (\frac{x^n-1}{m}) + \frac{1}{\frac{(x^n-1)2}{m}} (\frac{x^n-1}{m}) + \frac{1}{\frac{(x^n-1)3}{m}} (\frac{x^n-1}{m}) + ...)$

$=$

$n(lim_{m\to\infty} (\frac{1}{\frac{(x-1)1}{m}} (\frac{x-1}{m}) + \frac{1}{\frac{(x-1)2}{m}} (\frac{x-1}{m}) + \frac{1}{\frac{(x-1)3}{m}} (\frac{x-1}{m}) + ...))$

simplifying

$lim_{m\to\infty}(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...)$

$=$

$n(lim_{m\to\infty}(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...))$

therefore

$(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...)=n(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+...)$

in particular

$1=n\ ;\ \frac{1}{2}=\frac{n}{2};\ ...$

Being false $\forall n\in\Re$ when $n\neq 1$

This definition is commonly used on complex analysis and in particular in analysis of Riemann functions to clear coefficients of powers.

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