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On Derivations of Basic Governing Equations for Solid Bodies in Bi-spherical Coordinate System

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Abstract
Traditionally solid mechanics problems are formulated in Cartesian, cylindrical and spherical coordinate system. Using such formulation and coordinate system, solutions of solid mechanics problems are obtained for specific geometries such as straight boundary, circular, cylindrical and spherical boundaries. However, such available coordinate system cannot describe many geometries in spherical or cylindrical coordinate system with inclusion of eccentricities, two spherical or cylindrical bodies in contact and parallel. In this article, author address this issue by giving complete original formulations and derivations of basic governing partial differential equations in Bi-spherical coordinate system. Bi-spherical coordinate system allow us to solve problems such as eccentric spheres, two parallel spheres (intersecting or non-intersecting) in solid mechanics which traditional spherical coordinate system cannot handle. Author develop the original equations in Bi-spherical coordinates in terms of useful quantities of stresses and strains components in three dimensions. The paper is limited to basic formulations and practical applications of such formulations can easily be expanded for getting solutions using any known analytical or numerical methods.

Keywords: - Bi-spherical coordinates, Elasticity, Deformable solids, Orthogonal Curvilinear Coordinates.

Introduction

Traditionally, equilibrium problems of solid bodies use only three coordinate systems, Cartesian, cylindrical and spherical to solve any problems. Though, conventional coordinate system, are capable of solving thousand number of examples accurately, they are insufficient for describing all boundaries and geometries of problems that occur in continuum mechanics and real-life practice of such problems.

Recently, by incorporating new coordinate system, a problem was formulated in bipolar cylindrical coordinate system, is an example tackling problems of eccentric cylinders, two parallel intersecting and non-intersection cylinders which cannot be handled by traditional cylindrical coordinate system [1]. The references show the basic information about these coordinate systems, however, basic governing equations representing the problems are not seen which is very tedious to derive and due to these difficulties, problems having complex geometries and boundaries remain unsolved. In this view, present paper focus to use one of such coordinate systems as Bi-spherical coordinate systems and to give derivations of basic governing equations for tackling static equilibrium problems occurring in Bi-spherical coordinate system. Its use is to tackle problems of eccentric sphere, two parallel sphere and many. Few applications of such formulation for transport phenomena is seen in ref. [2]. Summary list of all such new coordinate system is seen in reference [1,4].

Detailed derivations in Bi-spherical coordinate system

Bi-spherical coordinate system is a three-dimensional orthogonal coordinate system. It consists of two spheres, obtained by rotation of two-dimensional bipolar cylindrical coordinate about the axis which is joined by two focal points, results in family of spheres which are intersecting or non-intersecting with each other depending on the coordinates and geometrical dimensions given. The bi-spherical coordinates in three dimensional is seen in Figure 1. To develop equations in the system of Bi-spherical coordinate systems, symbols given in Appendix are used. are defined as Bi-spherical coordinates, (x1, x2, x3) are defined as Cartesian coordinates. It is required to express the curvilinear basis in terms of the Cartesian basis as follows and symbols used in the reference [3] (Figure 2).

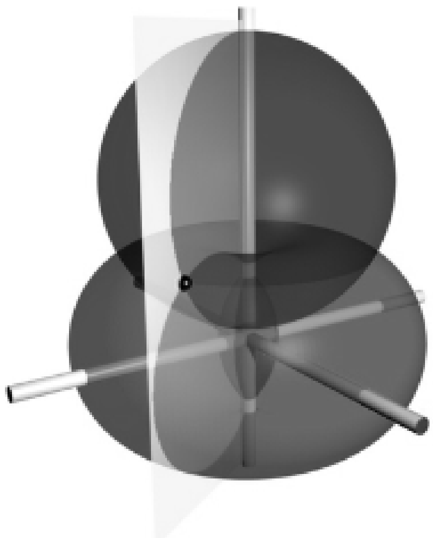

Figure 1 Bi-spherical Coordinate System σ, τ, ϕ

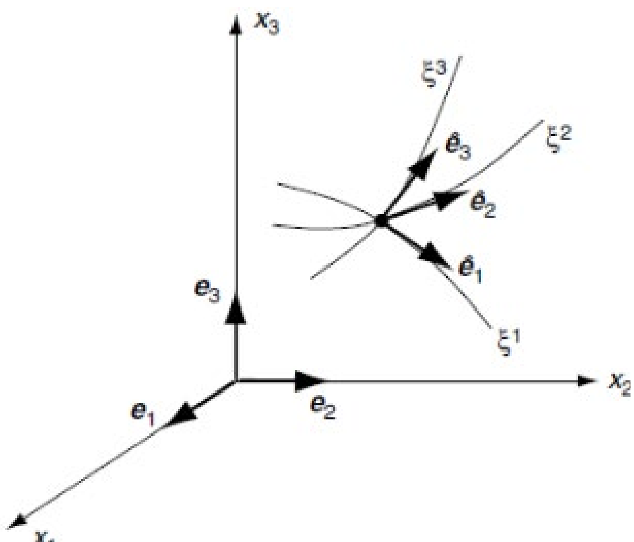

Figure 2 Curvilinear Coordinates [3].

$$\hat{e}_1 = \frac{1}{h_1}\frac{\partial x_k}{\partial \xi^1} e_k \quad \hat{e}_2 = \frac{1}{h_2}\frac{\partial x_k}{\partial \xi^2} e_k, \quad \hat{e}_3 = \frac{1}{h_3}\frac{\partial x_k}{\partial \xi^3} e_k \qquad (1)$$

In equation (1), k ranges the number of coordinate axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ (in a generalized symbols) or $\hat{e}_\sigma$, $\hat{e}_\tau$, $\hat{e}_\phi$ (defined in bi-spherical coordinates) are defined as the curvilinear basis in three dimensions. Also, in Bi-spherical coordinates, the scale factors are defined as follows [4],

$$h_1 = h_\sigma = \frac{a}{\cosh\tau - \cos\sigma}; h_2 = h_\tau = \frac{a}{\cosh\tau - \cos\sigma}; h_3 = h_\phi = \frac{a\sin\sigma}{\cosh\tau - \cos\sigma} \qquad (2)$$

Relation between cartesian and Bi-spherical systems are given by

$$x_1 = x = \frac{\sin\sigma}{\cosh\tau - \cos\sigma}\cos\phi, x_2 = y = \frac{\sin\sigma}{\cosh\tau - \cos\sigma}\sin\phi, x_3 = z = \frac{\sinh\tau}{\cosh\tau - \cos\sigma} \qquad (3)$$

To formulate the complete derivations, derivations of curvilinear basis is necessary. Also, Derivations of Cartesian coordinates with respect to the Bi-spherical coordinates can be easily expressed using the quotient rule of differentiation. Derivations of unit vectors, required as a part of formulation at a later stage is written as follows in three dimensions with symbols used in reference [3],

$$\begin{aligned}
&\frac{\partial c_1}{\partial \alpha_1} = -\frac{c_2 \partial h_1}{h_2 \partial \alpha_2} - \frac{c_3 \partial h_1}{h_3 \partial \alpha_3}, \frac{\partial c_1}{\partial \alpha_2} = \frac{c_2 \partial h_2}{h_1 \partial \alpha_1}, \frac{\partial c_1}{\partial \alpha_3} = \frac{c_3 \partial h_3}{h_1 \partial \alpha_1} \\
&\frac{\partial c_2}{\partial \alpha_2} = -\frac{c_3 \partial h_2}{h_3 \partial \alpha_3} - \frac{c_1 \partial h_2}{h_1 \partial \alpha_1}, \frac{\partial c_2}{\partial \alpha_3} = \frac{c_3 \partial h_3}{h_2 \partial \alpha_2}, \frac{\partial c_2}{\partial \alpha_1} = \frac{c_1 \partial h_1}{h_2 \partial \alpha_2} \\
&\frac{\partial c_3}{\partial \alpha_3} = -\frac{c_1 \partial h_3}{h_1 \partial \alpha_1} - \frac{c_2 \partial h_3}{h_2 \partial \alpha_2}, \frac{\partial c_3}{\partial \alpha_1} = \frac{c_1 \partial h_1}{h_3 \partial \alpha_3}, \frac{\partial c_3}{\partial \alpha_2} = \frac{c_2 \partial h_2}{h_3 \partial \alpha_3}
\end{aligned} \qquad (4)$$

Here, substituting the symbols according to Bi-spherical coordinate system symbols in $c_1 = e_\sigma, c_2 = e_\tau, c_3 = e_\phi, \alpha_1 = \sigma$ and $\alpha_2 = \tau, \alpha_3 = \phi$ Eq. (4), and h_1, h_2, h_3, which are the scale factors as given in equation (2),

$$\begin{aligned}
&\frac{\partial c_1}{\partial \alpha_1} = \frac{\partial e_\sigma}{\partial \sigma} = \frac{-e_\tau \sinh\tau}{\cosh\tau - \cos\sigma} \\
&\frac{\partial c_2}{\partial \alpha_2} = \frac{\partial e_\tau}{\partial \tau} = \frac{e_\sigma \sin\sigma}{\cosh\tau - \cos\sigma} \\
&\frac{\partial c_3}{\partial \alpha_3} = \frac{\partial e_\varphi}{\partial \varphi} = e_\sigma \left\{\frac{\cosh\tau\cos\sigma - 1}{\cos\sigma - \cosh\tau}\right\} + e_\tau \left\{\frac{\sin\sigma\sinh\tau}{\cosh\tau - \cos\sigma}\right\} \\
&\frac{\partial c_1}{\partial \alpha_2} = \frac{\partial e_\sigma}{\partial \tau} = \frac{-e_\tau \sin\sigma}{\cosh\tau - \cos\sigma} \\
&\frac{\partial c_2}{\partial \alpha_3} = \frac{\partial e_\tau}{\partial \varphi} = \frac{-e_\varphi \sin\sigma\sinh\tau}{\cosh\tau - \cos\sigma} \\
&\frac{\partial c_3}{\partial \alpha_1} = \frac{\partial e_\varphi}{\partial \sigma} = 0 \\
&\frac{\partial c_1}{\partial \alpha_3} = \frac{\partial e_\sigma}{\partial \varphi} = -\frac{e_\varphi(\cosh\tau\cos\sigma - 1)}{(\cos\sigma - \cosh\tau)} \\
&\frac{\partial c_2}{\partial \alpha_1} = \frac{\partial e_\tau}{\partial \sigma} = -\frac{e_\sigma \sinh\tau}{(\cosh\tau - \cos\sigma)} \\
&\frac{\partial c_3}{\partial \alpha_2} = \frac{\partial e_\varphi}{\partial \tau} = 0
\end{aligned} \qquad (5)$$

Equation. (5) are formulations of unit vectors/curvilinear basis, is significant in the formulations of the general equations of the equilibrium.

Derivations of Equilibrium Equations (Divergence)

In this section, author develop an important governing equation, partial differential equations in bi-spherical coordinate system, includes the components of stress in three directions of Bi-spherical system. For derivations of equations of equilibrium, for convenience Divergence is written to represent the stress tensor as follows with symbols in reference [5].

F1, F2 and F3 are the components of the forces. The values of scale factors h1, h2 and h3 are substituted and $(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3)=(\sigma,\tau,\varphi)$ Eq. (6) is obtained as follows. Here, Bi-spherical coordinates are three dimensional coordinates, and hence all three coordinates (σ,τ,φ) will be considered.

$$\nabla \boldsymbol{F}=\frac{1}{\boldsymbol{h}_1\boldsymbol{h}_2\boldsymbol{h}_3}\left[\frac{\partial(\boldsymbol{h}_1\boldsymbol{h}_3\boldsymbol{F}_1)}{\partial \boldsymbol{u}_1}+\frac{\partial(\boldsymbol{h}_3\boldsymbol{h}_1\boldsymbol{F}_2)}{\partial \boldsymbol{u}_2}+\frac{\partial(\boldsymbol{h}_1\boldsymbol{h}_3\boldsymbol{F}_3)}{\partial \boldsymbol{u}_3}\right]$$

$$\nabla F=\frac{1}{h_1h_2h_3}\left[\begin{array}{l}\frac{\partial}{\partial\sigma}\left\{F_1\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a\sin\sigma}{\cosh\tau-\cos\sigma}\right]\right\}+\frac{\partial}{\partial\tau}\left\{F_2\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a\sin\sigma}{\cosh\tau-\cos\sigma}\right]\right\}\\+\frac{\partial}{\partial\varphi}\left\{F_3\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\right\}\end{array}\right]$$

$$\textbf{\textit{first term:}}\frac{\partial}{\partial\sigma}\left\{F_1\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a\sin\sigma}{\cosh\tau-\cos\sigma}\right]\right\}=\frac{\partial F_1}{\partial\sigma}\left[\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}\right]+F_1\left\{\frac{\partial}{\partial\sigma}\left(\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}\right)\right\}$$

$$=\frac{\partial F_1}{\partial\sigma}\left[\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}\right]+F_1\left\{\frac{\sin^2\sigma-\cosh\tau\cos\sigma+1}{(\cos\sigma-\cosh\tau)^3}\right\}$$

$$\sec ond\ \ term:\frac{\partial}{\partial\tau}\left\{F_2\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a\sin\sigma}{\cosh\tau-\cos\sigma}\right]\right\}=\frac{\partial F_2}{\partial\tau}\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}+F_2\frac{\partial}{\partial\tau}\left\{\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}\right\}$$

$$=\frac{\partial F_2}{\partial\tau}\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2}+F_2\left\{\frac{\cos\tau(\cosh\tau-\cos\sigma)-2\sin\tau\sinh\tau}{(\cosh\tau-\cos\sigma)^3}\right\}$$

$$Third\ \ term:\frac{\partial}{\partial\varphi}\left\{F_3\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\left[\frac{a}{\cosh\tau-\cos\sigma}\right]\right\}=\frac{\partial F_3}{\partial\varphi}\frac{a^2}{(\cosh\tau-\cos\sigma)^2}+F_3\frac{\partial}{\partial\varphi}\left\{\frac{a^2}{(\cosh\tau-\cos\sigma)^2}\right\}$$

$$=\frac{\partial F_3}{\partial\varphi}\frac{a^2}{(\cosh\tau-\cos\sigma)^2} \tag{6}$$

Symbols for the stresses are defined and can be written as the traction components as follows $\boldsymbol{\sigma}=\boldsymbol{e}_\sigma\boldsymbol{T}_\sigma+\boldsymbol{e}_\tau\boldsymbol{T}_\tau+\boldsymbol{e}_\varphi\boldsymbol{T}_\varphi$ and

$$\begin{aligned}F_1&=T_\sigma=\sigma_\sigma\hat{e}_\sigma+\tau_{\sigma\tau}\hat{e}_\tau+\tau_{\sigma\varphi}\hat{e}_\varphi\\F_2&=T_\tau=\sigma_\tau\hat{e}_\tau+\tau_{\sigma\tau}\hat{e}_\sigma+\tau_{\tau\varphi}\hat{e}_\varphi\\F_3&=T_\varphi=\sigma_\varphi\hat{e}_\varphi+\tau_{\sigma\varphi}\hat{e}_\sigma+\tau_{\tau\varphi}\hat{e}_\tau\end{aligned} \tag{7}$$

Also Writing,

$$\frac{\partial F_1}{\partial\sigma}=\frac{\partial T_\sigma}{\partial\sigma}=\frac{\partial\sigma_\sigma}{\partial\sigma}\hat{e}_\sigma+\frac{\partial\hat{e}_\sigma}{\partial\sigma}\sigma_\sigma+\frac{\partial\tau_{\sigma\tau}}{\partial\sigma}\hat{e}_\tau+\frac{\partial\hat{e}_\tau}{\partial\sigma}\tau_{\sigma\tau}+\frac{\partial\tau_{\sigma\varphi}}{\partial\sigma}\hat{e}_\varphi+\frac{\partial\hat{e}_\varphi}{\partial\sigma}\tau_{\sigma\varphi}$$
$$\frac{\partial F_2}{\partial\tau}=\frac{\partial T_\tau}{\partial\tau}=\frac{\partial\sigma_\tau}{\partial\tau}\hat{e}_\tau+\frac{\partial\hat{e}_\tau}{\partial\tau}\sigma_\tau+\frac{\partial\tau_{\sigma\tau}}{\partial\tau}\hat{e}_\sigma+\frac{\partial\hat{e}_\sigma}{\partial\tau}\tau_{\sigma\tau}+\frac{\partial\tau_{\tau\varphi}}{\partial\tau}\hat{e}_\varphi+\frac{\partial\hat{e}_\varphi}{\partial\tau}\tau_{\tau\varphi}$$
$$\frac{\partial F_3}{\partial\varphi}=\frac{\partial T_\varphi}{\partial\tau}=\frac{\partial\sigma_\varphi}{\partial\tau}\hat{e}_\varphi+\frac{\partial\hat{e}_\varphi}{\partial\tau}\sigma_\varphi+\frac{\partial\tau_{\sigma\varphi}}{\partial\tau}\hat{e}_\sigma+\frac{\partial\hat{e}_\sigma}{\partial\tau}\tau_{\sigma\varphi}+\frac{\partial\tau_{\tau\varphi}}{\partial\tau}\hat{e}_\tau+\frac{\partial\hat{e}_\tau}{\partial\tau}\tau_{\tau\varphi} \tag{8}$$

Substituting the relevant quantities for the derivatives of the unit vector basis functions, compiling and substituting the derivations into the divergence equations. Combining above three terms, simplifying and Re writing equation (6), the terms with the unit vectors, e_σ e_τ and e_φ are collected separately, simplification of above formulae leads us to the expressions in Bi-spherical coordinate systems,

$$\hat{e}_\sigma\frac{1}{h_1h_2h_3}\left[\alpha\frac{\partial\sigma_\sigma}{\partial\sigma}-\tau_{\sigma\tau}\frac{\alpha\sinh\tau}{\cosh\tau-\cos\sigma}+\sigma_\sigma\beta+\alpha\frac{\partial\tau_{\sigma\tau}}{\partial\tau}+\gamma\tau_{\sigma\tau}+\xi\frac{\partial\tau_{\sigma\varphi}}{\partial\varphi}+\alpha\frac{\sigma_\tau\sin\sigma}{\cosh\tau\cos\sigma}+\sigma_\varphi\xi\left\{\frac{\cosh\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}\right]$$
$$+\hat{e}_\tau\frac{1}{h_1h_2h_3}\left[\frac{-\sigma_\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\alpha+\alpha\frac{\partial\tau_{\sigma\tau}}{\partial\sigma}+\tau_{\sigma\tau}\beta+\alpha\frac{\partial\sigma_\tau}{\partial\tau}-\frac{\alpha\tau_{\sigma\tau}\sin\sigma}{\cosh\tau-\cos\sigma}+\gamma\sigma_\tau+\xi\frac{\partial\tau_{\tau\varphi}}{\partial\varphi}+\sigma_\varphi\xi\frac{\sin\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\right]$$
$$+\hat{e}_\varphi\frac{1}{h_1h_2h_3}\left[\frac{\partial\tau_{\sigma\varphi}}{\partial\sigma}\alpha+\tau_{\sigma\varphi}\beta+\alpha\frac{\partial\tau_{\tau\varphi}}{\partial\tau}+\xi\frac{\partial\sigma_\varphi}{\partial\varphi}-\tau_{\sigma\varphi}\xi\left\{\frac{\cosh\tau\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}-\frac{\tau_{\tau\varphi}\xi\sin\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\right]$$

Where

$$\alpha=\frac{a^2\sin\sigma}{(\cosh\tau-\cos\sigma)^2};\beta=\frac{\sin^2\sigma-\cosh\tau\cos\sigma+1}{(\cos\sigma-\cosh\tau)^3};\xi=\frac{a^2}{(\cosh\tau-\cos\sigma)^2}$$
$$\gamma=\frac{\cos\tau(\cosh\tau-\cos\sigma)-2\sin\tau\sinh\tau}{(\cosh\tau-\cos\sigma)^3} \tag{9}$$

To obtain the equilibrium equations, making the individual terms in three directions, σ,τ,φ as zero gives us the general equilibrium equations in three dimensional Bi-spherical coordinate system. These governing differential equations can readily use for solving problems with eccentric spherical geometry and problems with family of spheres interacting with each other

$$\frac{1}{h_1h_2h_3}\left[\alpha\frac{\partial\sigma_\sigma}{\partial\sigma}-\tau_{\sigma\tau}\frac{\alpha\sinh\tau}{\cosh\tau-\cos\sigma}+\sigma_\sigma\beta+\alpha\frac{\partial\tau_{\sigma\tau}}{\partial\tau}+\gamma\tau_{\sigma\tau}+\xi\frac{\partial\tau_{\sigma\varphi}}{\partial\varphi}+\alpha\frac{\sigma_\tau\sin\sigma}{\cosh\tau\cos\sigma}+\sigma_\varphi\xi\left\{\frac{\cosh\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}\right]=0$$
$$\frac{1}{h_1h_2h_3}\left[\frac{-\sigma_\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\alpha+\alpha\frac{\partial\tau_{\sigma\tau}}{\partial\sigma}+\tau_{\sigma\tau}\beta+\alpha\frac{\partial\sigma_\tau}{\partial\tau}-\frac{\alpha\tau_{\sigma\tau}\sin\sigma}{\cosh\tau-\cos\sigma}+\gamma\sigma_\tau+\xi\frac{\partial\tau_{\tau\varphi}}{\partial\varphi}+\sigma_\varphi\xi\frac{\sin\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\right]=0$$
$$\frac{1}{h_1h_2h_3}\left[\frac{\partial\tau_{\sigma\varphi}}{\partial\sigma}\alpha+\tau_{\sigma\varphi}\beta+\alpha\frac{\partial\tau_{\tau\varphi}}{\partial\tau}+\xi\frac{\partial\sigma_\varphi}{\partial\varphi}-\tau_{\sigma\varphi}\xi\left\{\frac{\cosh\tau\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}-\frac{\tau_{\tau\varphi}\xi\sin\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\right]=0 \tag{10}$$

Strain Displacement Equations

Gradient $$\nabla\phi=\nabla E=\frac{1}{h_1}\frac{\partial E}{\partial\alpha_1}\hat{e}_\alpha+\frac{1}{h_2}\frac{\partial E}{\partial\alpha_2}\hat{e}_\tau+\frac{1}{h_3}\frac{\partial E}{\partial\alpha_3}\hat{e}_\varphi \tag{11}$$

Substituting the values as $\alpha_1 = \sigma, \alpha_2 = \tau, \alpha_3 = \varphi$ and scale parameters from equation (2), Displacement vector and defining strain tensor as follows,

$$E = u_\sigma \hat{e}_\sigma + u_\tau \hat{e}_\tau + u_\varphi \hat{e}_\varphi \quad e = \begin{bmatrix} u_{\sigma\sigma} & u_{\sigma\tau} & u_{\sigma\varphi} \\ u_{\tau\sigma} & u_{\tau\tau} & u_{\tau\varphi} \\ u_{\varphi\sigma} & u_{\varphi\tau} & u_{\varphi\varphi} \end{bmatrix}$$

Upon substituting the above equations of displacement vector and performing the derivative operations, we get following equations,

$$\begin{aligned}
\frac{\partial E}{\partial \alpha_1} &= \frac{\partial E}{\partial \sigma} = \frac{\partial u_\sigma}{\partial \sigma}\hat{e}_\sigma + \frac{\partial \hat{e}_\sigma}{\partial \sigma}u_\sigma + \frac{\partial u_\tau}{\partial \sigma}\hat{e}_\tau + \frac{\partial \hat{e}_\tau}{\partial \sigma}u_\tau + \frac{\partial u_\varphi}{\partial \sigma}\hat{e}_\varphi + \frac{\partial \hat{e}_\varphi}{\partial \sigma}u_\varphi \\
\frac{\partial E}{\partial \alpha_2} &= \frac{\partial E}{\partial \tau} = \frac{\partial u_\sigma}{\partial \tau}\hat{e}_\sigma + \frac{\partial \hat{e}_\sigma}{\partial \tau}u_\sigma + \frac{\partial u_\tau}{\partial \tau}\hat{e}_\tau + \frac{\partial \hat{e}_\tau}{\partial \tau}u_\tau + \frac{\partial u_\varphi}{\partial \tau}\hat{e}_\varphi + \frac{\partial \hat{e}_\varphi}{\partial \tau}u_\varphi \\
\frac{\partial E}{\partial \alpha_3} &= \frac{\partial E}{\partial \varphi} = \frac{\partial u_\sigma}{\partial \varphi}\hat{e}_\sigma + \frac{\partial \hat{e}_\sigma}{\partial \varphi}u_\sigma + \frac{\partial u_\tau}{\partial \varphi}\hat{e}_\tau + \frac{\partial \hat{e}_\tau}{\partial \varphi}u_\tau + \frac{\partial u_\varphi}{\partial \varphi}\hat{e}_\varphi + \frac{\partial \hat{e}_\varphi}{\partial \varphi}u_\varphi
\end{aligned} \tag{12}$$

Upon substitution of derivatives of the unit vectors appears in above three equations and simplifying the equations further, Collecting all the terms together and Separating out the terms with the unit vectors e_σ, e_τ and e_φ gives,

$$\begin{aligned}
&\hat{e}_\sigma\hat{e}_\sigma\left\{\frac{\cosh\tau - \cos\sigma}{a}\frac{\partial u_\sigma}{\partial \sigma} - u_\tau\frac{\sinh\tau}{a}\right\} + \\
&\hat{e}_\sigma\hat{e}_\tau\left\{-\frac{u_\sigma \sinh\tau}{\cosh\tau - \cos\sigma}\frac{1}{h_1} + \frac{\partial u_\tau}{\partial \sigma}\frac{1}{h_1} + \frac{1}{h_2}\frac{\partial u_\sigma}{\partial \tau} + \frac{1}{h_2}\frac{u_\tau \sin\sigma}{\cosh\tau - \cos\sigma}\right\} + \\
&\hat{e}_\sigma\hat{e}_\varphi\left\{\frac{\partial u_\varphi}{\partial \sigma}\frac{1}{h_1} + \frac{1}{h_3}\frac{\partial u_\sigma}{\partial \varphi} + \frac{1}{h_3}u_\varphi\left\{\frac{\cosh\tau\cos\sigma - 1}{\cos\sigma - \cosh\tau}\right\}\right\} + \\
&\hat{e}_\tau\hat{e}_\tau\left\{\frac{-1}{h_2}\frac{\sin\sigma}{\cosh\tau - \cos\sigma}u_\sigma + \frac{\partial u_\tau}{\partial \tau}\right\} + \\
&\hat{e}_\varphi\hat{e}_\varphi\left\{\frac{-1}{h_3}u_\sigma\left\{\frac{\cosh\tau\cos\sigma - 1}{\cos\sigma - \cosh\tau}\right\} - \frac{1}{h_3}\frac{\sin\sigma\sinh\tau}{\cosh\tau - \cos\sigma}u_\tau + \frac{\partial u_\varphi}{\partial \varphi}\frac{1}{h_3}\right\} + \\
&\hat{e}_\tau\hat{e}_\varphi\left\{\frac{u_\varphi}{h_3}\left\{\frac{\sin_\sigma \sinh\tau}{\cosh\tau - \cos\sigma}\right\} + \frac{1}{h_3}\frac{\partial u_\tau}{\partial \varphi} + \frac{\partial u_\varphi}{\partial \tau}\frac{1}{h_2}\right\}
\end{aligned} \tag{13}$$

Placing the above results into the strain displacement form $\boldsymbol{e} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]$, where e is the strain matrix and $\nabla \boldsymbol{u}$ is the displacement gradient matrix and $(\nabla \boldsymbol{u})^T$ is its transpose, gives the desired relations in three dimensional Bi-spherical coordinate equations. The individual scalar equations are given by

$$u_{\sigma\sigma}=\left\{\frac{\cosh\tau-\cos\sigma}{a}\frac{\partial u_\sigma}{\partial\sigma}-u_\tau\frac{\sinh\tau}{a}\right\}$$

$$u_{\sigma\tau}=\left\{-\frac{u_\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\frac{1}{h_1}+\frac{\partial u_\tau}{\partial\sigma}\frac{1}{h_1}+\frac{1}{h_2}\frac{\partial u_\sigma}{\partial\tau}+\frac{1}{h_2}\frac{u_\tau\sin\sigma}{\cosh\tau-\cos\sigma}\right\}$$

$$u_{\sigma\varphi}=\left\{\frac{\partial u_\varphi}{\partial\sigma}\frac{1}{h_1}+\frac{1}{h_3}\frac{\partial u_\sigma}{\partial\varphi}+\frac{1}{h_3}u_\varphi\left\{\frac{\cosh\tau\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}\right\}$$

$$u_{\tau\tau}=\left\{\frac{-1}{h_2}\frac{\sin\sigma}{\cosh\tau-\cos\sigma}u_\sigma+\frac{\partial u_\tau}{\partial\tau}\right\}$$

$$u_{\varphi\varphi}=\left\{\frac{-1}{h_3}u_\sigma\left\{\frac{\cosh\tau\cos\sigma-1}{\cos\sigma-\cosh\tau}\right\}-\frac{1}{h_3}\frac{\sin\sigma\sinh\tau}{\cosh\tau-\cos\sigma}u_\tau+\frac{\partial u_\varphi}{\partial\varphi}\frac{1}{h_3}\right\}$$

$$u_{\tau\varphi}=\left\{\frac{u_\varphi}{h_3}\left\{\frac{\sin_\sigma\sinh\tau}{\cosh\tau-\cos\sigma}\right\}+\frac{1}{h_3}\frac{\partial u_\tau}{\partial\varphi}+\frac{\partial u_\varphi}{\partial\tau}\frac{1}{h_2}\right\}$$

The above equations are the strain displacement relations in Bi-spherical coordinate system in three dimensions.

Conclusions

In this paper, author presented an original and a detailed systematic derivation of governing partial differential equations in bi-spherical coordinate system comprising of a stress and strain tensor. A governing differential equations and physical quantities of strain are derived based on the following factors represents the exact theory of elasticity in Bi-spherical coordinate system (1) the orthogonal curvilinear coordinate systems, (2) scale factors, (3) Cartesian coordinates in terms of bi-spherical coordinates and (4) curvilinear basis in terms of Cartesian basis, and (5) formulations of curvilinear basis in terms of Cartesian basis and its derivatives, are formulated in Bi-spherical coordinate systems in three dimensional system. Equations are very useful in tackling hundreds of static equilibrium problems of solid spherical bodies which traditional spherical coordinate system cannot tackle. The equations are generalized, and can conveniently be used for many real life practical solid mechanics problems such as eccentric hole in three-dimensional solid sphere, elastic half space with spherical geometry, family of interacting spherical body subjected to various loadings and boundary conditions and parallel spheres and many more with convenience for future numerical and analytical solutions.

Appendix – Mathematical symbols used in the formulation of Bi-Spherical Coordinate System

σ,τ,ϕ	Coordinate in bi-spherical system
x1, x2, x3	Direction in Cartesian coordinates
$h_\sigma,h_\tau,h_\varphi$	Scale factors in Bi-spherical system
h_1,h_2,h_3	Scale factors in generalized system
$\hat{e}_1\ \hat{e}_2\ \hat{e}_3$	Curvilinear basis in generalized symbol
$\hat{e}_\sigma\ \hat{e}_\tau\ \hat{e}_\phi$	Curvilinear basis in generalized symbol
F1, F2 and F3	components of the Traction in generalized symbol
$\boldsymbol{T}_\sigma,\boldsymbol{T}_\tau,\boldsymbol{T}$	Components of traction
ξ^1,ξ^2,ξ^3	orthogonal curvilinear coordinates

$\sigma_\sigma, \tau_{\sigma\tau}, \tau_{\sigma\varphi}$ $\sigma_\tau, \tau_{\sigma\tau}, \tau_{\tau\varphi}$ $\sigma_\varphi, \tau_{\sigma\varphi}, \tau_{\tau\varphi}$	Stress tensor in bi-spherical coordinate system
$u_{\sigma\sigma}, u_{\sigma\tau}, u_{\sigma\varphi}, u_{\tau\tau}, u_{\varphi\varphi}, u_{\tau\varphi}$	Normal and shearing Strain in in bi-spherical coordinate system

Conflict of Interest
There is not conflict of interest applicable to this article

Data Availability Statement
My manuscript has no associate data

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