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Research Article

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Four-Dimensional Differential Operator

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Abstract

The hyper-exponential function is a great family of special functions. When we configure a complex plane with a vector of an arbitrary direction in three-dimensional space as the real axis and time as the imaginary axis, by using the variables on this complex plane the hyper-exponential function of second-order proves to be a harmonic function. Based on this, I use quaternion to define my own four-dimensional differential operator and show its characteristics.

1. Wave Equation

The hyper-exponential function of second-order is used to show a solution that satisfies the wave equation.

We set the following:

$l, m, n \in R$

c is constant.

$$v := \{\, lx + my + nz \pm ct \mid (x,y,z) \in R^3 \,, t \in R \,, l^2 + m^2 + n^2 = 1 \,\}$$

$$F(v) = Exph_j^2(v; f(v)) \qquad (\, j = 0, 1 \,)$$

$$\frac{\partial^2}{\partial x^2} = l^2 \frac{d^2}{dv^2} \qquad \frac{\partial^2}{\partial y^2} = m^2 \frac{d^2}{dv^2} \qquad \frac{\partial^2}{\partial z^2} = n^2 \frac{d^2}{dv^2} \qquad \frac{\partial^2}{\partial t^2} = c^2 \frac{d^2}{dv^2}$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2 \partial t^2}\right) F(v) = (l^2 + m^2 + n^2 - 1) \frac{d^2 F(v)}{dv^2}$$

$\therefore$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2 \partial t^2}\right) F(v) = 0 \qquad ---①$$

2. Harmonic Function

We set the following:

$$r := \{\, lx + my + nz \mid (x,y,z) \in R^3 \,, l^2 + m^2 + n^2 = 1 \,\}$$

$t, \tau, c_0 \in R$

$$v = r \pm ct \,,\quad c = i\, c_0 \,,\quad \tau = c_0 t \,, v = r \pm i\tau$$

$$\frac{\partial^2}{\partial x^2} = l^2 \frac{\partial^2}{\partial r^2} \qquad \frac{\partial^2}{\partial y^2} = m^2 \frac{\partial^2}{\partial r^2} \qquad \frac{\partial^2}{\partial z^2} = n^2 \frac{\partial^2}{\partial r^2}$$

From the above ①,

$$\left\{(l^2 + m^2 + n^2) \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right\} F(v) = \left\{\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right\} F(v) = 0$$

$p(r,\tau) \in R$, $q(r,\tau) \in R$

$$F(v) = p(r,\tau) + iq(r,\tau)$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right)F(v) = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right)\{\, p(r,\tau) + iq(r,\tau)\,\} = 0$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right)p(r,\tau) = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \tau^2}\right)q(r,\tau) = 0$$

Therefore, the hyper-exponential function of second-order is a harmonic function

$\therefore$

$$i\frac{\partial F}{\partial r} = \frac{\sigma F}{\partial \tau} \qquad ---②$$

3. Four-Dimensional Differential Operator

We set the following:

$$I \times I = J \times J = K \times K = -1$$

$$I \times J = K,\ \ J \times K = I,\ \ K \times I = J$$

$$J \times I = -K,\ \ K \times J = -I,\ \ I \times K = -J$$

$$c = ic_0\ ,\tau = c_0 t\ ,ct = ic_0 t = i\,\tau\ ,i^2 = -1$$

$$\left(\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}J + \frac{\partial}{\partial z}K + \frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}J + \frac{\partial}{\partial z}K - \frac{\partial}{\partial \tau}\right) \qquad ---③$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2}\right)$$

$$+\left\{\left(\frac{\partial^2}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)I + \left(\frac{\partial^2}{\partial z\partial x} - \frac{\partial^2}{\partial x\partial z}\right)J + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)K\right\}$$

$$+\left\{\left(\frac{\partial^2}{\partial \tau\partial x} - \frac{\partial^2}{\partial x\partial \tau}\right)I + \left(\frac{\partial^2}{\partial \tau\partial y} - \frac{\partial^2}{\partial y\partial \tau}\right)J + \left(\frac{\partial^2}{\partial \tau\partial z} - \frac{\partial^2}{\partial z\partial \tau}\right)K\right\}$$

$$\left(\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}J + \frac{\partial}{\partial z}K - \frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial x}I + \frac{\partial}{\partial y}J + \frac{\partial}{\partial z}K + \frac{\partial}{\partial \tau}\right) \qquad ---④$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2}\right)$$

$$+\left\{\left(\frac{\partial^2}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)I + \left(\frac{\partial^2}{\partial z\partial x} - \frac{\partial^2}{\partial x\partial z}\right)J + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)K\right\}$$

$$-\left\{\left(\frac{\partial^2}{\partial \tau\partial x} - \frac{\partial^2}{\partial x\partial \tau}\right)I + \left(\frac{\partial^2}{\partial \tau\partial y} - \frac{\partial^2}{\partial y\partial \tau}\right)J + \left(\frac{\partial^2}{\partial \tau\partial z} - \frac{\partial^2}{\partial z\partial \tau}\right)K\right\}$$

From the above ③ and ④, the following operator ☒ is defined

$$☒ := -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c^2\partial t^2}\right)$$

$$+\left\{\left(\frac{\partial^2}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)I + \left(\frac{\partial^2}{\partial z\partial x} - \frac{\partial^2}{\partial x\partial z}\right)J + \left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)K\right\}$$

$$\pm\left\{\left(\frac{\partial^2}{\partial \tau\partial x} - \frac{\partial^2}{\partial x\partial \tau}\right)I + \left(\frac{\partial^2}{\partial \tau\partial y} - \frac{\partial^2}{\partial y\partial \tau}\right)J + \left(\frac{\partial^2}{\partial \tau\partial z} - \frac{\partial^2}{\partial z\partial \tau}\right)K\right\}$$

From the above ①,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{c^2\partial t^2}\right)F(v)=0$$

From the above ②,

$$\left(\frac{\partial^2}{\partial\tau\partial x}-\frac{\partial^2}{\partial x\partial\tau}\right)F(v)=-l\,i\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\tau^2}\right)F(v)=0$$

$$\left(\frac{\partial^2}{\partial\tau\partial y}-\frac{\partial^2}{\partial y\partial\tau}\right)F(v)=-m\,i\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\tau^2}\right)F(v)=0$$

$$\left(\frac{\partial^2}{\partial\tau\partial z}-\frac{\partial^2}{\partial z\partial\tau}\right)F(v)=-n\,i\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial\tau^2}\right)F(v)=0$$

$$\frac{\partial^2F(v)}{\partial\tau\partial x}-\frac{\partial^2F(v)}{\partial x\partial\tau}$$
$$=l\frac{\partial^2F(v)}{\partial\tau\partial r}-i\frac{\partial^2F(v)}{\partial x\partial r}$$
$$=-li\frac{\partial^2F(v)}{\partial\tau^2}-li\frac{\partial^2F(v)}{\partial r^2}$$
$$=-li\left(\frac{\partial^2F(v)}{\partial r^2}+\frac{\partial^2F(v)}{\partial\tau^2}\right)$$
$$=0$$

$\therefore$

If

$$\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)F(v)=\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)F(v)=\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)F(v)=0\,,$$

then

$\boxtimes F(v)=0$.

Otherwise,

$$\boxtimes F(v)=\left\{\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)I+\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)J+\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)K\right\}F(v)\,.$$

4. Application of Three-Dimension

If $v\coloneqq\{\,lx+my+nz\mid(x,y,z)\in R^3\,,l^2+m^2+n^2=1\,\}$, *We set the following*:

$$\frac{\partial F(v)}{\partial x}=A_x\,,\frac{\partial F(v)}{\partial y}=A_y\,,\frac{\partial F(v)}{\partial z}=A_z\,.$$

Therefore,

$$\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)F(v)=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\,,$$

$$\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)F(v)=\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\,,$$

$$\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)F(v)=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\,.$$

Therefore,

$$\left\{\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)I+\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)J+\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)K\right\}F(v)$$

$$=\begin{vmatrix} I & J & K\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

Therefore,
the result of the operation is as follows:

If

$$\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)F(v)=\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)F(v)=\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)F(v)=\mathbf{0}\,,$$

then

$$\boxtimes F(v)=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F(v)=-(l^2+m^2+n^2)\frac{d^2F(v)}{dv^2}=-f(v)F(v)$$

$\therefore$

$$\boxtimes F(v)=-\nabla^2F(v)=-f(v)F(v)\,.$$

Otherwise,

$$\boxtimes F(v)=-f(v)F(v)+$$
$$\left\{\left(\frac{\partial^2}{\partial y\partial z}-\frac{\partial^2}{\partial z\partial y}\right)I+\left(\frac{\partial^2}{\partial z\partial x}-\frac{\partial^2}{\partial x\partial z}\right)J+\left(\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2}{\partial y\partial x}\right)K\right\}F(v)$$

$\therefore$

$$\boxtimes F(v)=-\nabla^2F(v)+\begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}=-f(v)F\,(v)+\begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.$$

5. Conclusion

1. *In case of four-dimension, the variables are three real numbers and one imaginary number. The result of the operation consists of three imaginary numbers and one real number.*

$$R^3\times(C\setminus R)\xrightarrow{v}C\xrightarrow{F}C\xrightarrow{\boxtimes}H$$

2. *In case of three-dimension, the variables are three real numbers. The result of the operation consists of three imaginary numbers and one real number.*

$$R^3\xrightarrow{v}R\xrightarrow{F}R\xrightarrow{\boxtimes}H$$