VIBRATION OF AN AXIALLY LOADED HETEROGENEOUS PINNED-PINNED BEAM WITH AN INTERMEDIATE ROLLER SUPPORT

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Dedicated to Professor Tibor Czibere on the occasion of his 90th birthday

Abstract. The present paper is devoted to the issue of what effect the axial load (compressive or tensile) has on the eigenfrequencies of a heterogeneous pinned-pinned beam with an intermediate roller support (called a PrsP beam). This problem is a three point boundary value problem (eigenvalue problem) associated with homogeneous boundary conditions. If the Green functions of the three point boundary value problem (BVP) are known the eigenvalue problem that provide the eigenfrequencies for the beam loaded axially can be transformed into an eigenvalue problem governed by a homogeneous Fredholm integral equation. The later eigenvalue problems can be reduced to an algebraic eigenvalue problem which then can be solved numerically by using an effective solution algorithm which is based on the boundary element method.

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1. INTRODUCTION

Beams are quite often used as various structural elements because of their favorable load-bearing capabilities. Various such problems have been solved [1–3]. Nowadays, nonhomogeneous (heterogeneous or inhomogeneous) curved beams are also becoming more and more common. The benefits of such structural members can be the reduced weight and the higher strength. A class of inhomogeneity, namely, cross-sectional inhomogeneity means that the material parameters, like the Young modulus, are functions of the cross-sectional coordinates.

As for vibrations of beams, there are some recent results that must be mentioned. Bizzi et al. give an analytical solution to the mode-shape equation of Timoshenko beams vibrating under gravity loading in [4]. Ondra and Titurus [5] study systems of beams with tendon loading. The effect of the number and location of the attachment points is investigated on the vibrations of these systems is investigated. Bozyigit et al. [6] aim to find the effectiveness of the Adomian decomposition method and the differential transformation method on the vibrations of axially loaded Timoshenko beams. Mirzabeigy and Madoliat [7] focus on the vibrations of axially loaded beams resting on variable elastic foundation while Wu and Chang [8] tackle the vibrations of

axially loaded multi-step Timoshenko beams carrying multiple concentrated elements. The continuous mass transfer matrix is used in the later article to find solutions.

At the same time, it seems that, meanwhile the Green function is commonly used for various straight beam problems. Lueschen et al [9] consider Timoshenko and Euler-Bernoulli beams - the Green function is given in closed-form for vibration problems. Foda and Abduljabbar [10] deal with the dynamic response of an Euler-Bernoulli beam finding the response of the beam to a moving mass. Kukla and Zamojska make some efforts in [11] to tackle the vibrations of an axially loaded tapered beam. The beam can consist of an arbitrary number of segments so the model can serve as approximation for continuously varying cross-sections. Failla and Santini [12] solve the bending problem of discontinuous beams - there are constraints along the beam axis and flexural stiffness jumps. An efficient algorithm is developed based on the Green function.

Based on the above literature research, in this article, it is our main objective to clarify what effect the axial force has on the eigenfrequencies of a pinned-pinned beam with an intermediate roller support. The material is linearly elastic, isotropic. However, the elastic coefficients can change over the cross-section – this material behavior is called cross-sectional inhomogeneity. The governing equations of the corresponding equilibrium problem constitute a three-point boundary value problem. Its solution can be given in a closed form provided that the Green function is known. By utilizing the definition of the Green function we calculate the elements which make possible to calculate it. With the Green function the three-point eigenvalue problem that provides eigenfrequencies as a function of the axial force is replaced by a homogeneous Fredholm integral equation in which the Green function is the kernel. Then the eigenvalue problem governed by the homogeneous Fredholm integral equation is reduced to an algebraic eigenvalue problem by using the boundary element technique.

The paper is organized in five sections. Section 2 presents the differential equation of the problem and the corresponding boundary and continuity conditions in a dimensionless formulation. Section 3 provides the definition of the Green function for three-point boundary value problems. It is worth mentioning that the definition is a constructive one since it makes the calculation of the Green functions possible for the cases of compressive and tensile forces possible. The calculations are detailed in Section 4. The numerical results are presented in Section 5. The last section contains the conclusions.

2. DIFFERENTIAL EQUATIONS

2.1. Governing equations. The pinned-pinned beam with an intermediate roller support (PrsP beam) is shown in Figure 1. The axial force N ($N > 0$) is compressive in this Figure. The beam has a uniform cross section throughout its length. The E -weighted centerline [13] (called centerline for short) of the beam coincides with the axis $\hat{x}$ of the coordinate system $\hat{x}, \hat{y}, \hat{z}$. Its origin is at the left end of the centerline. It is assumed that the coordinate plane $\hat{x}\hat{z}$ is a symmetry plane of the beam. It is also assumed that the modulus of elasticity E satisfies the relation $E(\hat{y}, \hat{z}) = E(-\hat{y}, \hat{z})$ over

Figure 1.

the cross section A , which means that (a) it is an even function of $\hat{y}$ and (b) it is independent of the coordinate $\hat{x}$. In this case we speak about cross sectional heterogeneity.

We remark that the E -weighted first moment $Q_{\hat{y}}$ of the cross section is zero in this coordinate system [13]:

$$Q_{\hat{y}} = \int_A \hat{y} E(\hat{y}, \hat{z}) dA = 0$$

The length of the beam is L , the position of the middle support is given by $\hat{b}$.

Equilibrium problems of Euler-Bernoulli beams with cross sectional heterogeneity are governed by the ordinary differential equation:

$$\frac{d^4 \hat{w}}{d\hat{x}^4} \pm \hat{\mathcal{N}} \frac{d^2 \hat{w}}{d\hat{x}^2} = \frac{\hat{f}_z}{I_{ey}}, \quad \hat{\mathcal{N}} = \frac{N}{I_{ey}} \quad (2.1)$$

where the sign of $\hat{\mathcal{N}}$ is (positive)[negative] if the axial force is (compressive)[tensile], $\hat{w}(x)$ is the vertical displacement of the material points on the centerline, $\hat{f}_z(\hat{x})$ is the intensity of the distributed load acting on the centerline and I_{ey} is defined by the equation

$$I_{ey} = \int_A E(\hat{y}, \hat{z}) z^2 dA, \quad (2.2)$$

where A is the area of the cross section. If E is constant the beam is homogeneous and

$$I_{ey} = IE, \quad I = \int_A z^2 dA \quad (2.3)$$

in which I is the moment of inertia. It is noted that the effect of the material composition on I_{ey} is demonstrated through examples in [14].

In what follows we shall use dimensionless variables defined by the following relations

$$\begin{aligned} x &= \hat{x}/L, & \xi &= \hat{\xi}/L, & w &= \hat{w}/L, \\ y &= \frac{d\hat{w}}{d\hat{x}} = \frac{dw}{dx}, & b &= \hat{b}/\ell, & \ell &= \frac{\hat{x}}{L} \Big|_{\hat{x}=L} = 1, \end{aligned} \quad (2.4)$$

where $\hat{\xi}$ is also a coordinate on the axis x introduced here for our later considerations. Applying dimensionless quantities to equation (2.1) we have

$$\frac{d^4 w}{dx^4} \pm \mathcal{N} \frac{d^2 w}{dx^2} = f_z, \quad \mathcal{N} = L^2 \hat{\mathcal{N}} = \frac{L^2 N}{I_{ey}} \quad f_z = \frac{L^3 \hat{f}_z}{I_{ey}} \quad (2.5)$$

Table 1.

Boundary conditions	
$w(0) = 0, \quad w^{(2)}(0) = 0$	$w(\ell) = 0, \quad w^{(2)}(\ell) = 0$
Continuity conditions	
$w(b-0) = 0, \quad w(b+0) = 0$ $w^{(1)}(b-0) = w^{(1)}(b+0)$ $w^{(2)}(b-0) = w^{(2)}(b+0)$	

Equation (2.5) is associated with the boundary and continuity conditions presented in Table 1.

We remark that the general solution of the homogeneous differential equation

$$w^{(4)} \pm \mathcal{N} w^{(2)} = 0, \quad w^{(n)} = \begin{cases} w & \text{if } n = 0 \\ \frac{d^n w}{dx^n} & \text{if } n = 1, 2, \dots \end{cases} \quad (2.6a)$$

is very simple, i.e.,

$$w = \sum_{\ell=1}^4 a_{\ell} w_{\ell}(x) = a_1 + a_2 x + a_3 \cos px + a_4 \sin px, \quad p = \sqrt{\mathcal{N}} \quad (2.6b)$$

if the sign of $\mathcal{N}$ is positive – $a_1, \dots, a_4$ are undetermined integration constants – and

$$w = \sum_{\ell=1}^4 a_{\ell} w_{\ell}(x) = a_1 + a_2 x + a_3 \cosh px + a_4 \sinh px, \quad p = \sqrt{\mathcal{N}}. \quad (2.6c)$$

if the sign of $\mathcal{N}$ is negative. Note that $a_1, \dots, a_4$ are different integration constants in equations (2.6b) and (2.6c).

If we know the Green function $G(x, \xi)$ for the boundary value problem determined by ODE (2.5) and the boundary and continuity conditions presented in Table 1 the solution for the dimensionless deflection $w(x)$ due to the dimensionless load $f_z(\xi)$ is given by the following integral:

$$w(x) = \int_0^{\ell} G(x, \xi) f_z(\xi) d\xi \quad (2.7)$$

The definition of the Green function we need is presented in Section 3.

3. GREEN FUNCTION FOR THREE-POINT BOUNDARY VALUE PROBLEMS

3.1. Definition. We shall consider the in-homogeneous ordinary differential equation

$$L[y(x)] = r(x), \quad (3.1)$$

where the differential operator of order $2k$ is defined by the following equation

$$L[y(x)] = \sum_{n=0}^{2k} p_n(x) y^{(n)}(x) \quad (3.2)$$

in which k is a natural number, the functions $p_n(x)$ and $r(x)$ are continuous and $p_{2k}(x) \neq 0$ if $x \in [0, \ell]$ ($\ell > 0$). Moreover let b be an inner point in the interval $[0, \ell]$: $b = \ell_1$, $\ell - b = \ell_2$ and $\ell_1 + \ell_2 = \ell$.

We assume that the inhomogeneous differential equation (3.1) is associated with homogeneous boundary and continuity conditions given by the following equations:

$$\begin{aligned} U_{0r}[y] &= \sum_{n=0}^{2k} \alpha_{nrI} y_I^{(n-1)}(0) = 0, & r = 1, 2, \dots, k \\ U_{br}[y] &= \sum_{n=0}^{2k} \beta_{nrI} y_I^{(n-1)}(b) - \sum_{n=0}^{2k} \beta_{nrII} y_{II}^{(n-1)}(b) = 0, & r = 1, 2, \dots, 2k \\ U_{\ell r}[y] &= \sum_{n=0}^{2k} \gamma_{nrII} y_{II}^{(n-1)}(\ell) = 0, & r = 1, 2, \dots, k \end{aligned} \quad (3.3)$$

The Latin subscripts I and II refer successively to the intervals $[0, b]$ and $[b, \ell]$: y_I and y_{II} are the solutions to the differential equation in the interval I and II while α_{nrI} , β_{nrI} , β_{nrII} and γ_{nrII} are arbitrary constants.

In accordance with equation (2.7) solution of the three-point boundary value problem (3.1), (3.2) and (3.3) is sought in the following form:

$$y(x) = \int_0^\ell G(x, \xi) r(\xi) d\xi, \quad (3.4)$$

where the function $G(x, \xi)$ is the Green function defined by the following formulas and properties [15]:

Formulas:

$$G(x, \xi) = \begin{cases} G_{1I}(x, \xi) & \text{if } x, \xi \in [0, b], \\ G_{2I}(x, \xi) & \text{if } x \in [b, \ell] \text{ and } \xi \in [0, b], \\ G_{1II}(x, \xi) & \text{if } x \in [0, b] \text{ and } \xi \in [b, \ell], \\ G_{2II}(x, \xi) & \text{if } x, \xi \in [b, \ell]. \end{cases} \quad (3.5)$$

Properties:

1. The function $G_{1I}(x, \xi)$ is a continuous function of x and ξ if $0 \leq x \leq \xi \leq b$ and $0 \leq \xi \leq x \leq b$. In addition it is $2k$ times differentiable with respect to x and the derivatives

$$\frac{\partial^n G_{1I}(x, \xi)}{\partial x^n} = G_{1I}(x, \xi)^{(n)}(x, \xi), \quad n = 1, 2, \dots, 2k \quad (3.6)$$

are also continuous functions of x and ξ in the triangles $0 \leq x \leq \xi \leq b$ and $0 \leq \xi \leq x \leq b$.

2. Let ξ be fixed in $[0, b]$. The function $G_{1I}(x, \xi)$ and its derivatives

$$G_{1I}^{(n)}(x, \xi) = \frac{\partial^n G_{1I}(x, \xi)}{\partial x^n}, \quad n = 1, 2, \dots, 2k - 2 \quad (3.7)$$

should be continuous for $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{1I}^{(n)}(\xi + \varepsilon, \xi) - G_{1I}^{(n)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{1I}^{(n)}(\xi + 0, \xi) - G_{1I}^{(n)}(\xi - 0, \xi) \right] = 0 \quad n = 0, 1, 2, \dots, 2k - 2 \end{aligned} \quad (3.8a)$$

The derivative $G_{1I}^{(2k-1)}(x, \xi)$ should, however, have a jump if $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{1I}^{(2k-1)}(\xi + \varepsilon, \xi) - G_{1I}^{(2k-1)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{1I}^{(2k-1)}(\xi + 0, \xi) - G_{1I}^{(2k-1)}(\xi - 0, \xi) \right] = \frac{1}{p_{2k}(\xi)} \quad . \end{aligned} \quad (3.8b)$$

In contrast to this, $G_{2I}(x, \xi)$ and its derivatives

$$G_{2I}^{(n)}(x, \xi) = \frac{\partial^n G_{2I}(x, \xi)}{\partial x^n}, \quad n = 1, 2, \dots, 2k \quad (3.9)$$

are all continuous functions for any x in $[b, \ell]$.

3. Let ξ be fixed in $[b, \ell]$. The function $G_{1II}(x, \xi)$ and its derivatives

$$G_{1II}^{(n)}(x, \xi) = \frac{\partial^n G_{1II}(x, \xi)}{\partial x^n}, \quad n = 1, 2, \dots, 2k \quad (3.10)$$

are all continuous functions for any x in $[0, b]$.

4. Though the function $G_{2II}(x, \xi)$ and its derivatives

$$G_{2II}^{(n)}(x, \xi) = \frac{\partial^n G_{2II}(x, \xi)}{\partial x^n}, \quad n = 1, 2, \dots, 2k - 2 \quad (3.11)$$

are also continuous for $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{2II}^{(n)}(\xi + \varepsilon, \xi) - G_{2II}^{(n)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{2II}^{(n)}(\xi + 0, \xi) - G_{2II}^{(n)}(\xi - 0, \xi) \right] = 0 \quad n = 0, 1, 2, \dots, 2k - 2 \end{aligned} \quad (3.12a)$$

the derivative $G_{2II}^{(2k-1)}(x, \xi)$ should, however, have a jump if $x = \xi$:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left[G_{2II}^{(2k-1)}(\xi + \varepsilon, \xi) - G_{2II}^{(2k-1)}(\xi - \varepsilon, \xi) \right] &= \\ &= \left[G_{2II}^{(2k-1)}(\xi + 0, \xi) - G_{2II}^{(2k-1)}(\xi - 0, \xi) \right] = \frac{1}{p_{2k}(\xi)} \quad . \end{aligned} \quad (3.12b)$$

5. Let α be an arbitrary but finite non-zero constant. For a fixed $\xi \in [0, \ell]$ the product $G(x, \xi)\alpha$ as a function of x ($x \neq \xi$) should satisfy the homogeneous differential equation

$$M[G(x, \xi)\alpha] = 0.$$

6. The product $G(x, \xi)\alpha$ as a function of x should satisfy both the boundary conditions and the continuity conditions

$$\begin{aligned} U_{0r}[G] &= \sum_{n=1}^{2k} \alpha_{nrI} G^{(n-1)}(0) = 0, \quad r = 1, \dots, k \\ U_{br}[G] &= \sum_{n=1}^{2k} \left(\beta_{nrI} G^{(n-1)}(b-0) - \beta_{nrII} G^{(n-1)}(b+0) \right) = 0, \quad r = 1, \dots, 2k \quad (3.13) \\ U_{\ell r}[G] &= \sum_{n=1}^{2k} \gamma_{nrII} G^{(n-1)}(\ell) = 0, \quad r = 1, \dots, k \end{aligned}$$

The above continuity conditions should be satisfied by the function pairs $G_{1I}(x, \xi)$, $G_{2I}(x, \xi)$ and $G_{1II}(x, \xi)$, $G_{2II}(x, \xi)$ as well.

REMARK 1. It can be proved by utilizing the previous definition that integral (3.4) satisfies differential equation and boundary conditions (3.3) [15].

REMARK 2. If the boundary value problem defined by (3.1) and (3.3) is self adjoint then the Green function is symmetric [15]:

$$G(x, \xi) = G(\xi, x) \quad (3.14)$$

It can be checked with ease utilizing the corresponding definition presented in [15] that the three-point boundary value problem defined by differential equation (2.6a) and the boundary and continuity condition given in Table 1 is self-adjoint.

3.2. Calculation of the Green functions.

3.2.1. *Introductory remarks.* Let us denote the linearly independent particular solutions of the homogeneous ordinary differential equation

$$L[y(x)] = 0 \quad (3.15)$$

by

$$y_1(x), y_2(x), y_3(x), \dots, y_{2k}(x). \quad (3.16)$$

Since the general solution is a linear combination of the particular solutions it can be given in the following form:

$$y(x) = \sum_{n=1}^{2k} \mathcal{A}_n y_n(x) \quad (3.17)$$

where the coefficients $\mathcal{A}_n$ are arbitrary integration constants. The Green function should satisfy the homogeneous differential equation (3.15). Therefore it follows that it can be given as a linear combination of the particular solutions $y_n(x)$, i.e., by equation (3.17). It is a further problem how to find the coefficients $\mathcal{A}_n$. Subsections 3.3 and 3.4 deal with this issue.

3.3. Calculation of the Green function if $\xi \in [0, b]$. It is obvious that the integration constants $\mathcal{A}_n$ should be different in the two triangular domains $0 \leq x \leq \xi \leq b$

Figure 2.

and $0 \leq \xi \leq x \leq b$. For this reason we shall apply the following assumptions:

(a)

$$\begin{aligned} G_{1I}(x, \xi) &= \sum_{\ell=1}^{2k} (a_{\ell I}(\xi) + b_{\ell I}(\xi)) y_{\ell}(x), & x \leq \xi; \\ G_{1I}(x, \xi) &= \sum_{\ell=1}^{2k} (a_{\ell I}(\xi) - b_{\ell I}(\xi)) y_{\ell}(x), & x \geq \xi \end{aligned} \quad x \in [0, b] \quad (3.18)$$

and

(b)

$$G_{2I}(x, \xi) = \sum_{\ell=1}^{2k} c_{\ell I}(\xi) y_{\ell}(x), \quad x \in [b, \ell] \quad (3.19)$$

where the coefficients $a_{\ell I}(\xi)$, $b_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$ are, in fact, unknown functions.

Note that the above representation for $G_{1I}(x, \xi)$ and $G_{2I}(x, \xi)$ ensures the fulfillment of Property 5 of the definition.

Continuity conditions (3.8a) yield the following equations

$$\sum_{\ell=1}^{2k} b_{\ell I}(\xi) y_{\ell}^{(n)}(\xi) = 0, \quad n = 0, 1, 2, \dots, 2k-2 \quad (3.20a)$$

As regards the discontinuity condition (3.8b) we get

$$\frac{1}{p_{2k}(\xi)} = \left[G^{(2k-1)}(\xi + 0, \xi) - G^{(2k-1)}(\xi - 0, \xi) \right] = -2 \sum_{\ell=1}^k b_{\ell I}(\xi) y_{\ell}^{(2k-1)}(\xi).$$

Hence

$$\sum_{\ell=1}^{2k} b_{\ell I}(\xi) y_{\ell}^{(2k-1)}(\xi) = -\frac{1}{2p_{2k}(\xi)}. \quad (3.20b)$$

Fulfillment of equations (3.20a) and (3.20b) ensures the fulfillment of Properties 1, 2 and 4 of the definition.

Continuity and discontinuity conditions (3.8) result in the inhomogeneous linear system of equations (3.20) for the unknowns $b_{\ell}(\xi)$ ($\ell = 1, 2, \dots, 2k$). Its determinant assumes the following form:

$$\begin{vmatrix} y_1(\xi) & y_2(\xi) & \dots & y_k(\xi) \\ y_1^{(1)}(\xi) & y_2^{(1)}(\xi) & \dots & y_k^{(1)}(\xi) \\ \dots & \dots & \dots & \dots \\ y_1^{(2k-1)}(\xi) & y_2^{(2k-1)}(\xi) & \dots & y_{2k}^{(2k-1)}(\xi) \end{vmatrix}. \quad (3.21)$$

This determinant is the Wronskian¹ [16] of differential equation (3.15) which is not zero since the particular solutions $y_{\ell}(x)$ are linearly independent. This means that we can always find a unique solution for the coefficients $b_{\ell I}(\xi)$ in representation (3.18) of the Green function.

Utilizing the boundary and continuity conditions (3.13) the following equations can be obtained for the unknown coefficients $a_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$:

$$U_{0r}[G] = \sum_{n=1}^{2k} \alpha_{nrI} \sum_{\ell=1}^{2k} (a_{\ell I}(\xi) + b_{\ell I}(\xi)) z_{\ell}(0)^{(n-1)} = 0 \quad r = 1, 2, \dots, k \quad (3.22a)$$

$$U_{br}[G] = \sum_{n=1}^{2k} \left(\beta_{nrI} \sum_{\ell=1}^{2k} (a_{\ell I}(\xi) - b_{\ell I}(\xi)) z_{\ell}(b)^{(n-1)} - \beta_{nrII} \sum_{\ell=1}^{2k} c_{\ell I}(\xi) z_{\ell}(b)^{(n-1)} \right) = 0, \quad r = 1, 2, \dots, 2k \quad (3.22b)$$

$$U_{cr}[G] = \sum_{n=1}^{2k} \gamma_{nrII} \sum_{\ell=1}^{2k} c_{\ell I}(\xi) z_{\ell}(c)^{(n-1)} = 0, \quad r = 1, 2, \dots, k \quad (3.22c)$$

or

$$\begin{aligned} \sum_{\ell=1}^{2k} a_{\ell I} U_{ar}[z_{\ell}] &= - \sum_{\ell=1}^{2k} b_{\ell I} U_{ar}[z_{\ell}] & r = 1, 2, \dots, k \\ \sum_{\ell=1}^{2k} a_{\ell I} U_{brI}[z_{\ell}] - \sum_{\ell=1}^{2k} c_{\ell I} U_{brII}[z_{\ell}] &= \sum_{\ell=1}^{2k} b_{\ell I} U_{brI}[z_{\ell}] & r = 1, 2, \dots, 2k \\ \sum_{\ell=1}^{2k} c_{\ell I} U_{cr}[z_{\ell}] &= 0 & r = 1, 2, \dots, k \end{aligned} \quad (3.23)$$

¹Józef Hoene-Wroński (1776-1853)

Solutions for $a_{\ell I}(\xi)$ and $c_{\ell I}(\xi)$ exist if and only if the determinant of the linear equation system (3.23) is different from zero. The linear independence of the boundary and continuity conditions assures in general that the equation system (3.23) is solvable.

Fulfillment of equations (3.23) ensures the fulfillment of Property 6 of the definition.

3.4. Calculation of the Green function if $\xi \in [b, \ell = 1]$. Now we shall assume that:

(a)

$$\begin{aligned} G_{1I}(x, \xi) &= \sum_{\ell=1}^{2k} (a_{\ell II}(\xi) + b_{\ell II}(\xi)) y_{\ell}(x), & x \leq \xi; \\ G_{1I}(x, \xi) &= \sum_{\ell=1}^{2k} (a_{\ell II}(\xi) - b_{\ell II}(\xi)) y_{\ell}(x), & x \geq \xi \end{aligned} \quad x \in [b, \ell = 1] \quad (3.24)$$

and

(b)

$$G_{2I}(x, \xi) = \sum_{\ell=1}^{2k} c_{\ell I}(\xi) y_{\ell}(x), \quad x \in [0, b] \quad (3.25)$$

where the coefficients $a_{\ell II}(\xi)$, $b_{\ell II}(\xi)$ and $c_{\ell II}(\xi)$ are again unknown functions.

Note that the above representation for $G_{1II}(x, \xi)$ and $G_{2II}(x, \xi)$ ensures the fulfillment of Property 5 of the definition.

Continuity conditions (3.12a) lead to the following equations:

$$\sum_{\ell=1}^{2k} b_{\ell II}(\xi) y_{\ell}^{(n)}(\xi) = 0, \quad n = 0, 1, 2, \dots, 2k - 2 \quad (3.26a)$$

As regards discontinuity condition (3.12b) we obtain

$$\frac{1}{p_{2k}(\xi)} = \left[G^{(2k-1)}(\xi + 0, \xi) - G^{(2k-1)}(\xi - 0, \xi) \right] = -2 \sum_{\ell=1}^k b_{\ell II}(\xi) y_{\ell}^{(2k-1)}(\xi).$$

Hence

$$\sum_{\ell=1}^{2k} b_{\ell II}(\xi) y_{\ell}^{(2k-1)}(\xi) = -\frac{1}{2p_{2k}(\xi)}. \quad (3.26b)$$

Continuity and discontinuity conditions (3.12) yield the inhomogeneous linear system of equations (3.26) for the unknowns $b_{\ell II}(\xi)$ ($\ell = 1, 2, \dots, 2k$). Since these equations are formally the same as equations (3.20) it follows that their determinant is again the Wronskian (3.21) and this fact guarantees their solvability. It also follows that $b_{\ell II} = b_{\ell I}$.

Note that fulfillment of equations (3.26a) and (3.26b) ensures the fulfillment of Properties 1, 3 and 4 of the definition.

Utilizing now the boundary and continuity conditions (3.13) leads to the following linear equations for the unknown coefficients $a_{\ell II}(\xi)$ and $c_{\ell II}(\xi)$:

$$U_{0r}[G] = \sum_{n=1}^{2k} \alpha_{nrI} \sum_{\ell=1}^{2k} c_{\ell II}(\xi) z_{\ell}(a)^{(n-1)} = 0 \quad r = 1, 2, \dots, k \quad (3.27a)$$

$$U_{br}[G] = \sum_{n=1}^{2k} \left(\beta_{nrI} \sum_{\ell=1}^{2k} c_{\ell II}(\xi) z_{\ell}(b)^{(n-1)} - \beta_{nrII} \sum_{\ell=1}^{2k} (a_{\ell II}(\xi) + b_{\ell II}(\xi)) z_{\ell}(b)^{(n-1)} \right) = 0, \quad r = 1, 2, \dots, 2k \quad (3.27b)$$

$$U_{\ell r}[G] = \sum_{n=1}^{2k} \gamma_{nrII} \sum_{\ell=1}^{2k} (a_{\ell II}(\xi) - b_{\ell II}(\xi)) z_{\ell}(c)^{(n-1)} = 0, \quad r = 1, 2, \dots, k \quad (3.27c)$$

or

$$\sum_{\ell=1}^{2k} a_{\ell II} U_{0r}[z_{\ell}] = 0 \quad r = 1, 2, \dots, k \quad (3.28a)$$

$$\sum_{\ell=1}^{2k} a_{\ell II} U_{brI}[z_{\ell}] - \sum_{\ell=1}^{2k} b_{\ell II} U_{brII}[z_{\ell}] = \sum_{\ell=1}^{2k} c_{\ell II} U_{brII}[z_{\ell}] \quad r = 1, 2, \dots, 2k \quad (3.28b)$$

$$\sum_{\ell=1}^{2k} b_{\ell II} U_{\ell r}[z_{\ell}] = \sum_{\ell=1}^{2k} c_{\ell II} U_{\ell r}[z_{\ell}] \quad r = 1, 2, \dots, k \quad (3.28c)$$

The linear equation system (3.27) is solvable for $a_{\ell II}(\xi)$ and $b_{\ell II}(\xi)$ if its determinant is different from zero. Since the boundary and continuity conditions are linearly independent equation system (3.27) is, in general, solvable.

If the determinants of the equation systems (3.23) and (3.28) are different from zero then there Green function exists which satisfies the properties of the definition given in Section 3.1.

Fulfillment of equations (3.28) ensures the fulfillment of Property 6 of the definition.

4. CALCULATION OF THE GREEN FUNCTION FOR PRSP BEAMS

4.1. PrsP beams subjected to a compressive force. If the axial force is compression it follows from equation (2.5) (or (2.6a)) that equilibrium problems of PrsP beams are governed by the differential equation

$$L[w(x)] = w^{(4)} + \mathcal{N}w^{(2)} = f_z \quad (4.1)$$

associated with the boundary and continuity conditions in Table 1.

4.1.1. *Green function if $\xi \in [0, b]$.* In the sequel we shall follow the line of thought presented in Subsection 3.3. Recalling that the general solution of the homogeneous differential equation $L[w(x)] = 0$ is given by (2.6b) the continuity and discontinuity conditions (3.20) result in the following equation system for $b_{\ell I}$:

$$\begin{bmatrix} 1 & \xi & \cos p\xi & \sin p\xi \\ 0 & 1 & -p \sin p\xi & p \cos p\xi \\ 0 & 0 & -p^2 \cos p\xi & -p^2 \sin p\xi \\ 0 & 0 & p^3 \sin p\xi & -p^3 \cos p\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \quad (4.2)$$

REMARK 3. It has been taken into account here that $w(x)$ corresponds to $y(x)$, $k = 2$ and $p_{2k} = 1$.

Solution of this equation system assumes the following form:

$$\begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} p\xi \\ -p \\ -\sin p\xi \\ \cos p\xi \end{bmatrix}. \quad (4.3)$$

We proceed with the boundary and continuity conditions (3.23). Recalling the boundary and continuity conditions in Table 1 and Remark 3. we obtain the following equation system:

(a) Boundary conditions at $x = 0$:

$$\begin{aligned} a_{1I}w_1(0) + a_{2I}w_2(0) + a_{3I}w_3(0) + a_{4I}w_4(0) &= \\ &= -b_{1I}w_1(0) - b_{2I}w_2(0) - b_{3I}w_3(0) - b_{4I}w_4(0), \end{aligned} \quad (4.4a)$$

$$\begin{aligned} a_{1I}w_1^{(2)}(0) + a_{2I}w_2^{(2)}(0) + a_{3I}w_3^{(2)}(0) + a_{4I}w_4^{(2)}(0) &= \\ &= -b_{1I}w_1^{(2)}(0) - b_{2I}w_2^{(2)}(0) - b_{3I}w_3^{(2)}(0) - b_{4I}w_4^{(2)}(0). \end{aligned} \quad (4.4b)$$

(b) Continuity conditions at $x = b$:

$$\begin{aligned} a_{1I}w_1(b) + a_{2I}w_2(b) + a_{3I}w_3(b) + a_{4I}w_4(b) &= \\ &= b_{1I}w_1(b) + b_{2I}w_2(b) + b_{3I}w_3(b) + b_{4I}w_4(b), \end{aligned} \quad (4.5a)$$

$$c_{1I}w_1(b) + c_{2I}w_2(b) + c_{3I}w_3(b) + c_{4I}w_4(b) = 0, \quad (4.5b)$$

$$\begin{aligned} a_{1I}w_1^{(1)}(b) + a_{2I}w_2^{(1)}(b) + a_{3I}w_3^{(1)}(b) + a_{4I}w_4^{(1)}(b) &- \\ - c_{1I}w_1^{(1)}(b) - c_{2I}w_2^{(1)}(b) - c_{3I}w_3^{(1)}(b) - c_{4I}w_4^{(1)}(b) &= \\ &= b_{1I}w_1^{(1)}(b) + b_{2I}w_2^{(1)}(b) + b_{3I}w_3^{(1)}(b) + b_{4I}w_4^{(1)}(b), \end{aligned} \quad (4.5c)$$

$$\begin{aligned} a_{1I}w_1^{(2)}(b) + a_{2I}w_2^{(2)}(b) + a_{3I}w_3^{(2)}(b) + a_{4I}w_4^{(2)}(b) &- \\ - c_{1I}w_1^{(2)}(b) - c_{2I}w_2^{(2)}(b) - c_{3I}w_3^{(2)}(b) - c_{4I}w_4^{(2)}(b) &= \\ &= b_{1I}w_1^{(2)}(b) + b_{2I}w_2^{(2)}(b) + b_{3I}w_3^{(2)}(b) + b_{4I}w_4^{(2)}(b). \end{aligned} \quad (4.5d)$$

(c) Boundary conditions at $x = \ell$:

$$c_{1I}w_1(\ell) + c_{2I}w_2(\ell) + c_{3I}w_3(\ell) + c_{4I}w_4(\ell) = 0, \quad (4.6a)$$

$$c_{1I}w_1^{(2)}(\ell) + c_{2I}w_2^{(2)}(\ell) + c_{3I}w_3^{(2)}(\ell) + c_{4I}w_4^{(2)}(\ell) = 0. \quad (4.6b)$$

Using (2.6b) boundary conditions (4.4) at $x = 0$ yield

$$a_{1I} + a_{3I} = -b_{1I} - b_{3I}, \quad a_{3I} = -b_{3I}.$$

Hence

$$a_{1I} = -b_{1I} = -\frac{1}{2p^2}\xi, \quad (4.7a)$$

$$a_{3I} = \frac{\sin p\xi}{2p^3}. \quad (4.7b)$$

As regards continuity conditions (4.5) at $x = b$ and boundary conditions (4.6) at $x = \ell$ substituting (2.6b) yields

$$\begin{bmatrix} b & \sin pb & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & \cos pb & \sin pb \\ 1 & p \cos pb & 0 & -1 & p \sin pb & -p \cos pb \\ 0 & -\sin pb & 0 & 0 & \cos pb & \sin pb \\ 0 & 0 & 1 & \ell & \cos p\ell & \sin p\ell \\ 0 & 0 & 0 & 0 & \cos p\ell & \sin p\ell \end{bmatrix} \begin{bmatrix} a_{2I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} =$$

$$= \frac{1}{2p^3} \begin{bmatrix} 2p\xi - pb - \sin p\xi \cos pb - \sin p(\xi - b) \\ 0 \\ -p + 2p \sin p\xi \sin pb + p \cos p\xi \cos pb \\ 2 \sin p\xi \cos pb - \cos p\xi \sin pb \\ 0 \\ 0 \end{bmatrix}$$

from where

$$\begin{aligned} a_{2I} &= \\ &= \frac{1}{2p^3 \mathcal{D}_c} [(p(\ell - b) \sin p\ell - \sin bp \sin p(\ell - b)) (2p\xi - bp - \sin p(\xi - b) - \sin p\xi \cos bp) + \\ &\quad - (\ell - b) (\sin bp \sin p(\ell - b)) (-p + 2p \sin p\xi \sin bp + p \cos p\xi \cos bp) + \\ &\quad + (p(\ell - b) \sin bp \cos (p\ell - bp) - \sin bp \sin (p\ell - bp)) (2 \sin p\xi \cos bp - \cos p\xi \sin bp)], \end{aligned} \quad (4.7c)$$

$$\begin{aligned} a_{4I} &= \\ &= \frac{1}{2p^3 \mathcal{D}_c} [-(\cos p\xi \sin pb - 2 \sin p\xi \cos pb) (b \sin p(\ell - b) - pb(\ell - b) \cos p(\ell - b)) - \\ &\quad + (\ell - b) (\sin p(\ell - b)) (\sin p(\xi - b) - 2p\xi + pb + \sin p\xi \cos pb) - \\ &\quad - b(\ell - b) \sin p(\ell - b) (p - 2p \sin p\xi \sin pb - p \cos p\xi \cos pb)], \end{aligned} \quad (4.7d)$$

$$c_{1I} = \frac{1}{2p^3\mathcal{D}_c} [-\ell (\sin p(\ell - b) \sin pb) (\sin p(\xi - b) - 2p\xi + pb + \sin p\xi \cos pb) - \\ + \ell (\sin p(\ell - b)) (pb \cos pb - \sin pb) (\cos p\xi \sin pb - 2 \sin p\xi \cos pb) + \\ + b\ell (\sin p(\ell - b) \sin pb) (p - 2p \sin p\xi \sin pb - p \cos p\xi \cos pb)], \quad (4.7e)$$

$$c_{2I} = \frac{1}{2p^3\mathcal{D}_c} [(\sin p(\ell - b) \sin pb) (\sin p(\xi - b) - 2p\xi + pb + \sin p\xi \cos pb) + \\ + (\sin p(\ell - b)) (\sin pb - pb \cos pb) (\cos p\xi \sin pb - 2 \sin p\xi \cos pb) + \\ + b(\sin pb \sin p(\ell - b)) (p \cos p\xi \cos pb - p + 2p \sin p\xi \sin pb)], \quad (4.7f)$$

$$c_{3I} = \frac{1}{2p^3\mathcal{D}_c} [(\ell - b) (\sin p\ell \sin pb) (\sin p(\xi - b) - 2p\xi + pb + \sin p\xi \cos pb) - \\ (\ell - b) (\sin p\ell) (\sin pb - pb \cos pb) (\cos p\xi \sin pb - 2 \sin p\xi \cos pb) + \\ + b(\ell - b) (\sin pb \sin p\ell) (p \cos p\xi \cos pb - p + 2p \sin p\xi \sin pb)], \quad (4.7g)$$

$$c_{4I} = \frac{1}{2p^3\mathcal{D}_c} [-(\ell - b) \cos p\ell \sin pb (\sin p(\xi - b) - 2p\xi + pb + \sin p\xi \cos pb) + \\ + (\ell - b) \cos p\ell (\sin pb - pb \cos pb) (\cos p\xi \sin pb - 2 \sin p\xi \cos pb) - \\ - b(\ell - b) (\cos p\ell \sin pb) (p \cos p\xi \cos pb - p + 2p \sin p\xi \sin pb)], \quad (4.7h)$$

in which

$$\mathcal{D}_c = pb(\ell - b) \sin p\ell - \ell (\sin pb) \sin p(\ell - b). \quad (4.8)$$

With $a_{\ell I}$, $b_{\ell I}$ and $c_{\ell I}$

$$G(x, \xi) = \begin{cases} G_{1I}(x, \xi) = \sum_{\ell=1}^4 (a_{\ell I}(\xi) \pm b_{\ell I}(\xi)) w_{\ell}(x) & \text{if } x \in [0, \ell], \\ G_{2I}(x, \xi) = \sum_{\ell=1}^4 c_{\ell I}(\xi) w_{\ell}(x) & \text{if } x \in [b, \ell]. \end{cases} \quad (4.9)$$

is the Green function if $\xi \in [0, b]$. The sign is positive if $x < \xi$ and negative if $x > \xi$.

4.1.2. *Green function if $\xi \in [b, \ell]$.* The equation system for $b_{\ell II}$ coincides with equation system (4.2). Hence $b_{\ell II} = b_{\ell I}$ and the solutions are given by equation (4.3).

The boundary conditions in Table 1 for $x = 0$ yield:

$$c_{1II} = c_{3II} = 0.$$

The continuity conditions at $x = b$ and the boundary conditions at $x = \ell$ lead to the following equation system – details of the calculation are all omitted:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b & \sin pb \\ 1 & b & \cos pb & \sin pb & 0 & 0 \\ 0 & 1 & -p \sin pb & p \cos pb & -1 & -p \cos pb \\ 0 & 0 & -\cos pb & -\sin pb & 0 & \sin pb \\ 1 & \ell & \cos p\ell & \sin p\ell & 0 & 0 \\ 0 & 0 & -\cos p\ell & -\sin p\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4I} \\ c_{2II} \\ c_{4II} \end{bmatrix} =$$

$$= \frac{1}{2p^3} \begin{bmatrix} 0 \\ -p\xi + pb + \sin p(\xi - b) \\ p - p \cos p(\xi - b) \\ -\sin p(\xi - b) \\ p\xi - p\ell - \sin p(\xi - \ell) \\ \sin p(\xi - \ell) \end{bmatrix}.$$

The solutions needed for calculating the Green function are gathered here:

$$\begin{aligned} a_{1II} = \frac{1}{2p^3\mathcal{D}_c} & [(bp\ell \sin p\ell - \ell \sin p(\ell - b) \sin pb) (\sin p(\xi - b) - p\xi + pb) + \\ & + (\sin p(\ell - \xi)) (pb^2 \sin p\ell - pbl \sin bp) - pb^2 (\sin p\ell) (\sin p(\ell - \xi) - p\ell + p\xi) + \\ & + \ell (\sin p(\xi - b) \sin p(\ell - b)) (\sin pb - pb \cos pb) + \\ & + bl (\sin p(\ell - b) \sin pb) (p - p \cos p(\xi - b))], \quad (4.10a) \end{aligned}$$

$$\begin{aligned} a_{2II} = \frac{1}{2p^3\mathcal{D}_c} & [\sin p(\ell - \xi) (\sin p(\ell - b) \sin pb - pb (\sin p\ell - \sin pb)) + \\ & + (\sin p(\ell - b) \sin pb - pb \sin p\ell) (\sin p(\xi - b) - p\xi + pb) + \\ & + (\sin p(\ell - b) \sin pb - pb \sin p\ell) (\sin p(\xi - \ell) - p\xi + p\ell) - \\ & - (\sin p(\xi - b) \sin p(\ell - b)) (\sin pb - pb \cos pb) - \\ & - b (\sin p(\ell - b) \sin pb) (p - p \cos p(\xi - b))], \quad (4.10b) \end{aligned}$$

$$\begin{aligned} a_{3II} = \frac{1}{2p^3\mathcal{D}_c} & [-b (\sin p\ell \sin pb) (\sin p(\xi - b) - p\xi + pb) - \\ & - (\sin p(\ell - \xi) \sin pb) (b \sin p\ell - \ell \sin pb) + \\ & + b (\sin p\ell \sin pb) (\sin p(\ell - \xi) + p\xi - p\ell) - \\ & - (\sin p(\xi - b) \sin p\ell) (\sin pb - pb \cos pb) (\ell - b) - \\ & - b (\ell - b) (\sin p\ell \sin pb) (p - p \cos p(\xi - b))], \quad (4.10c) \end{aligned}$$

$$\begin{aligned} a_{4II} = \frac{1}{2p^3\mathcal{D}_c} & [\sin p(\ell - \xi) ((\sin pb) (b \cos p\ell - \ell \cos pb) + pb(\ell - b)) + \\ & + b (\cos p\ell \sin pb) (\sin p(\xi - b) - p\xi + pb) + \\ & + b (\cos p\ell \sin pb) (\sin p(\xi - \ell) - p\xi + p\ell) + \\ & + (\sin p(\xi - b) \cos p\ell) (\sin pb - pb \cos pb) (\ell - b) + \\ & + b(\ell - b) (\cos p\ell \sin pb) (p - p \cos p(\xi - b))], \quad (4.10d) \end{aligned}$$

$$c_{1II} = 0, \quad (4.10e)$$

$$\begin{aligned} c_{2II} = \frac{1}{2p^3\mathcal{D}_c} & [-(\sin p(\xi - \ell)) (\sin p(\ell - b) \sin pb - p(\ell - b) \sin pb) + \\ & - \sin p(\xi - b) (\sin p(\ell - b) \sin pb - p(\ell - b) \cos p(\ell - b) \sin bp) - \\ & + (\sin p(\ell - b) \sin pb) (\sin p(\xi - \ell) - p\xi + p\ell) + \end{aligned}$$

$$+ (\sin p(\ell - b) \sin pb) (\sin p(\xi - b) - p\xi + pb) + \\ + (\ell - b) (\sin p(\ell - b) \sin pb) (p - p \cos p(\xi - b))], \quad (4.10f)$$

$$c_{3II} = 0, \quad (4.10g)$$

$$c_{4II} = \frac{1}{2p^3 \mathcal{D}_c} [-b (\sin p(\ell - b)) (\sin p(\xi - b) - p\xi + pb) + \\ + \sin p(\xi - b) (b \sin p(\ell - b) - pb(\ell - b) \cos p(\ell - b)) + \\ + (\sin p(\xi - \ell)) (b \sin p(\ell - b) - pb(\ell - b)) - \\ - b (\sin p(\ell - b)) (\sin p(\xi - \ell) - p\xi + p\ell) - \\ - b(\ell - b) (\sin p(\ell - b)) (p - p \cos p(\xi - b))]. \quad (4.10h)$$

With $a_{\ell II}$, $b_{\ell II}$ and $c_{\ell II}$

$$G(x, \xi) = \begin{cases} G_{1II}(x, \xi) = \sum_{\ell=1}^4 c_{\ell II}(\xi) w_{\ell}(x) & \text{if } x \in [0, b], \\ G_{2II}(x, \xi) = \sum_{\ell=1}^4 (a_{\ell II}(\xi) \pm b_{\ell II}(\xi)) w_{\ell}(x) & \text{if } x \in [b, \ell]. \end{cases} \quad (4.11)$$

is the Green function if $\xi \in [b, \ell]$. The sign is positive if $x < \xi$ and negative if $x > \xi$.

The Green function given by equations (4.9) and (4.11) satisfies the symmetry condition $G(x, \xi) = G(\xi, x)$, $x, \xi \in [0, \ell]$. Fulfillment of the symmetry condition was checked numerically for a number of point pairs x, ξ .

4.2. PrsP beams subjected to a tensile force. If the axial force is tensile equation according to equation (2.5) (or (2.6a)) equilibrium problems of PrsP beams are governed by the differential equation

$$L[w(x)] = w^{(4)} - \mathcal{N}w^{(2)} = f_z \quad (4.12)$$

for which the boundary and continuity conditions are presented in Table 1.

4.2.1. Green function if $\xi \in [0, b]$. In the present subsection we shall apply the solution steps detailed in Subsection 4.1.1. Recalling that the general solution of the homogeneous differential equation $L[w(x)] = 0$ is given by (2.6c) the continuity and discontinuity conditions (3.20) lead to the following equation system for $b_{\ell I}$:

$$\begin{bmatrix} 1 & \xi & \cosh p\xi & \sinh p\xi \\ 0 & 1 & p \sinh p\xi & p \cosh p\xi \\ 0 & 0 & p^2 \cosh p\xi & p^2 \sinh p\xi \\ 0 & 0 & p^3 \sinh p\xi & p^3 \cosh p\xi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \quad (4.13)$$

from where

$$\begin{bmatrix} b_{1I} \\ b_{2I} \\ b_{3I} \\ b_{4I} \end{bmatrix} = \frac{1}{2p^3} \begin{bmatrix} -p\xi \\ p \\ \sinh p\xi \\ -\cosh p\xi \end{bmatrix}. \quad (4.14)$$

REMARK 4. It has been taken again into account in the calculations that $w(x)$ corresponds to $y(x)$, $k = 2$ and $p_{2k} = 1$.

We proceed with boundary and continuity conditions (3.23). Recalling the boundary and continuity conditions in Table 1 and Remark 4. the equation system we obtain is formally the same as the equation system given by (4.4), (4.5) and (4.6) – the solution for w is, however, different – see (2.6c). After substituting it we have

$$a_{1I} = -b_{1I} = \frac{1}{2p^3} p\xi,$$

$$a_{3I} = -b_{3I} = -\frac{1}{2p^3} \sinh p\xi$$

and

$$\begin{bmatrix} b & \sinh pb & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & b & \cosh pb & \sinh pb \\ 1 & p \cosh pb & 0 & -1 & -p \sinh pb & -p \cosh pb \\ 0 & \sinh pb & 0 & 0 & -\cosh pb & -\sinh pb \\ 0 & 0 & 1 & \ell & \cosh p\ell & \sinh p\ell \\ 0 & 0 & 0 & 0 & \cosh p\ell & \sinh p\ell \end{bmatrix} \begin{bmatrix} a_{2I} \\ a_{4I} \\ c_{1I} \\ c_{2I} \\ c_{3I} \\ c_{4I} \end{bmatrix} =$$

$$= \frac{1}{2p^3} \begin{bmatrix} -2p\xi + pb + \sinh(p\xi - bp) + \sinh p\xi \cosh pb \\ 0 \\ p + 2p \sinh p\xi \sinh pb - p \cosh p\xi \cosh pb \\ 2 \sinh p\xi \cosh pb - \cosh p\xi \sinh pb \\ 0 \\ 0 \end{bmatrix}$$

from where

$$a_{2I} = \frac{1}{2p^3 \mathcal{D}_t} \times$$

$$[(p(\ell - b) \sinh p\ell - \sinh pb \sinh p(\ell - b))(pb - 2p\xi + \sinh p(\xi - b) + \sinh p\xi \cosh pb) -$$

$$- (\ell - b) \sinh pb \sinh p(\ell - b)(p + 2p \sinh p\xi \sinh pb - p \cosh p\xi \cosh pb) -$$

$$- (p(\ell - b) \sinh pb \cosh p(\ell - b) - \sinh pb \sinh p(\ell - b)) \times$$

$$(2 \sinh p\xi \cosh pb - \cosh p\xi \sinh pb)], \quad (4.15a)$$

$$a_{4I} =$$

$$= \frac{1}{2p^3 \mathcal{D}_t} [-(\cosh p\xi \sinh pb - 2 \sinh p\xi \cosh pb)(b \sinh p(\ell - b) + pb(\ell - b) \cosh p(\ell - b)) +$$

$$- (\ell - b)(\sinh p(\ell - b))(\sinh p(\xi - b) - 2p\xi + pb + \sinh p\xi \cosh pb) +$$

$$+ b(\ell - b)(\sinh p(\ell - b))(p + 2p \sinh p\xi \sinh pb - p \cosh p\xi \cosh pb)], \quad (4.15b)$$

$$c_{1I} = \frac{1}{2p^3 \mathcal{D}_t} [\ell(\sinh p(\ell - b) \sinh pb)(\sinh p(\xi - b) - 2p\xi + pb + \sinh p\xi \cosh pb) -$$

$$- \ell(\sinh p(\ell - b))(pb \cosh pb - \sinh pb)(\cosh p\xi \sinh pb - 2 \sinh p\xi \cosh pb) +$$

$$- b\ell(\sinh p(\ell - b) \sinh pb)(p + 2p \sinh p\xi \sinh pb - p \cosh p\xi \cosh pb)], \quad (4.15c)$$

$$c_{2I} = \frac{1}{2p^3 \mathcal{D}_t} [- (\sinh p (\ell - b) \sinh pb) (\sinh p (\xi - b) - 2p\xi + pb + \sinh p\xi \cosh pb) + \\ - \sinh p (\ell - b) (\sinh pb - pb \cosh pb) (\cosh p\xi \sinh pb - 2 \sinh p\xi \cos pb) - \\ - b (\sinh pb \sinh p (\ell - b)) (p \cosh p\xi \cosh pb - p - 2p \sinh p\xi \sinh pb)], \quad (4.15d)$$

$$c_{3I} = \frac{1}{2p^3 \mathcal{D}_t} [- (\ell - b) (\sinh p\ell \sinh pb) (\sinh p (\xi - b) - 2p\xi + pb + \sinh p\xi \cosh pb) + \\ - (\ell - b) (\sinh p\ell) (\sinh pb - pb \cosh pb) (\cosh p\xi \sinh pb - 2 \sinh p\xi \cos pb) \\ - b (\ell - b) (\sinh pb \sinh p\ell) (p \cosh p\xi \cosh pb - p - 2p \sinh p\xi \sinh pb)], \quad (4.15e)$$

$$c_{4I} = \frac{1}{2p^3 \mathcal{D}} [(\ell - b) (\cosh p\ell \sinh pb) (\sinh p (\xi - b) - 2p\xi + pb + \sinh p\xi \cosh pb) + \\ + (\ell - b) \cosh p\ell (\sinh pb - pb \cosh pb) (\cosh p\xi \sinh pb - 2 \sinh p\xi \cos pb) + \\ + b (\ell - b) (\cosh p\ell \sinh pb) (p \cosh p\xi \cosh pb - p - 2p \sinh p\xi \sinh pb)] \quad (4.15f)$$

in which

$$\mathcal{D}_t = bp (\ell - b) \sinh p\ell - \ell \sinh bp \sinh p (\ell - b). \quad (4.16)$$

With $a_{\ell I}$, $b_{\ell I}$ and $c_{\ell I}$ derived above for the case of the tensile axial force equation (4.9) can be used to find the Green function if $\xi \in [0, b]$.

4.2.2. *Green function if $\xi \in [b, \ell]$.* The equation system for $b_{\ell II}$ coincides formally with equation system (4.13). Consequently, $b_{\ell II} = b_{\ell I}$ and the solutions are given by equation (4.14).

The boundary conditions in Table 1 for $x = 0$ yield:

$$c_{1II} = c_{3II} = 0$$

The continuity conditions at $x = b$ and the boundary conditions at $x = \ell$ lead to the following equation system – details of the calculation are all omitted:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & b & \sinh pb \\ 1 & b & \cosh pb & \sinh pb & 0 & 0 \\ 0 & 1 & p \sinh pb & p \cosh pb & -1 & -p \cosh pb \\ 0 & 0 & \cosh pb & \sinh pb & 0 & -\sinh pb \\ 1 & \ell & \cosh p\ell & \sinh p\ell & 0 & 0 \\ 0 & 0 & \cosh p\ell & \sinh p\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{1II} \\ a_{2II} \\ a_{3II} \\ a_{4I} \\ c_{2II} \\ c_{4II} \end{bmatrix} = \\ = \frac{1}{2p^3} \begin{bmatrix} 0 \\ p\xi - pb - \sinh p (\xi - b) \\ -p + p \cosh p (\xi - b) \\ -\sinh p (\xi - b) \\ -p\xi + p\ell + \sinh p (\xi - \ell) \\ \sinh p (\xi - \ell) \end{bmatrix}$$

The solutions obtained for $a_{1II}, \dots, a_{4II}$ and $c_{1II}, \dots, c_{4II}$ and needed for calculating the Green function are presented here:

$$a_{1II} = \frac{1}{2p^3\mathcal{D}_t} [(pb\ell \sinh p\ell - \ell \sinh pb \sinh p(\ell - b))(-\sinh p(\xi - b) + p\xi - pb) + \\ + (\sinh p(\xi - \ell))(pb^2 \sinh p\ell - pb\ell \sinh pb) - pb^2(\sinh p\ell)(\sinh p(\xi - \ell) + p\ell - p\xi) - \\ - \ell(\sinh p(\xi - b) \sinh p(\ell - b))(\sinh pb - pb \cosh pb) - \\ - b\ell(\sinh p(\ell - b) \sinh pb)(p - p \cosh p(\xi - b))] , \quad (4.17a)$$

$$a_{2II} = \frac{1}{2p^3\mathcal{D}} [-\sinh p(\ell - \xi)(\sinh p(\ell - b) \sinh pb + pb(-\sinh p\ell + \sinh pb)) + \\ + (\sinh p(\ell - b) \sinh pb - pb \sinh p\ell)(-\sinh p(\xi - b) + p\xi - pb) - \\ - (\sinh p(\ell - b) \sinh pb - pb \sinh p\ell)(\sinh p(\xi - \ell) + p\ell - p\xi) + \\ + (\sinh p(\xi - b) \sinh p(\ell - b))(\sinh pb - pb \cosh pb) + \\ + b(\sinh pb \sinh p(\ell - b))(p - p \cosh p(\xi - b))] , \quad (4.17b)$$

$$a_{3II} = \frac{1}{2p^3\mathcal{D}_t} [b(\sinh p\ell \sinh pb)(\sinh p(\xi - b) - p\xi + pb) - \\ - (\sinh p(\ell - \xi) \sinh pb)(b \sinh p\ell - \ell \sinh pb) - \\ - b(\sinh p\ell \sinh pb)(\sinh p(\ell - \xi) + p\xi - p\ell) - \\ - (\sinh p(\xi - b) \sinh p\ell)(\sinh pb - pb \cosh pb)(\ell - b) - \\ - b(\ell - b)(\sinh p\ell \sinh pb)(p - p \cosh p(\xi - b))] , \quad (4.17c)$$

$$a_{4II} = \frac{1}{2p^3\mathcal{D}_t} [-\sinh p(\ell - \xi)((\sinh pb)(b \cosh p\ell - \ell \cosh pb) + pb(\ell - b)) - \\ - b(\cosh p\ell \sinh pb)(\sinh p(\xi - b) - p\xi + pb) - \\ - b(\cosh p\ell \sinh pb)(\sinh p(\xi - \ell) - p\xi + p\ell) - \\ - (\sinh p(\xi - b) \cosh p\ell)(\sinh pb - pb \cosh pb)(\ell - b) - \\ - b(\ell - b)(\cosh p\ell \sinh pb)(p - p \cosh p(\xi - b))] , \quad (4.17d)$$

$$c_{1II} = 0 , \quad (4.17e)$$

$$c_{2II} = \frac{1}{2p^3\mathcal{D}_t} [(\sinh p(\xi - \ell))(\sinh pb \sinh p(\ell - b) - p(\ell - b) \sinh pb) + \\ + \sinh p(\xi - b)(\sinh p(\ell - b) \sinh pb - p(\ell - b) \sinh pb \cosh p(\ell - b)) - \\ - (\sinh p(\ell - b) \sinh pb)(\sinh p(\xi - \ell) - p\xi + p\ell) - \\ - (\sinh p(\ell - b) \sinh pb)(\sinh p(\xi - b) - p\xi + pb) + \\ - (\ell - b)(\sinh p(\ell - b) \sinh pb)(p - p \cosh p(\xi - b))] , \quad (4.17f)$$

$$c_{3II} = 0 , \quad (4.17g)$$

$$c_{4II} = \frac{1}{2p^3\mathcal{D}} [b \sinh p(\ell - b)(\sinh p(\xi - b) - p\xi + pb) - \\ - \sinh p(\xi - b)(b \sinh p(\ell - b) - pb(\ell - b) \cosh p(\ell - b)) -$$

$$\begin{aligned}
& -(\sinh p(\xi - \ell))(b \sinh p(\ell - b) - pb(\ell - b)) + \\
& + b(\sinh p(\ell - b))(\sinh p(\xi - \ell) - p\xi + p\ell) + \\
& + b(\ell - b)(\sinh p(\ell - b))(p - p \cosh p(\xi - b))]. \quad (4.17h)
\end{aligned}$$

With $a_{\ell II}$, $b_{\ell II} = b_{\ell I}$ and $c_{\ell I}$ presented above for the tensile axial force equation (4.11) can be used to find the Green function if $\xi \in [0, b]$. According to a numerical check the Green function satisfies the symmetry condition $G(x, \xi) = G(\xi, x)$.

4.2.3. Graphical representation of the Green function. The dimensionless critical force $\mathcal{N}_{\text{crit}}(p_{\text{crit}})$ as function of the location of the intermediate roller support is given by the polynomial [17]:

$$\begin{aligned}
\sqrt{\mathcal{N}_{\text{crit}}}/\pi &= p_{\text{crit}}/\pi = \\
&= -25.6511182911b^5 + 18.3289788300b^4 - 4.70118403874b^3 + \\
&+ 1.39533914036b^2 + 0.928843353123b + 1.43050335236, \quad b \in [0, 0.5]. \quad (4.18)
\end{aligned}$$

Note that for symmetry reasons $b \in [0, 0.5]$.

Assume that $b = 0.5$ and $\xi = 0.75$. Assume further that the axial force is compressive. Figure 3 shows the Green function provided that $p = 0.4p_{\text{crit}}$ and $p = 0.8p_{\text{crit}}$. Since the Green function is the dimensionless vertical displacement due to a dimensionless unit force applied to the beam at $\xi = 0.75$ it follows that the bending moment caused by the compressive force $\mathcal{N}$ has the same sign as the bending moment caused by the dimensionless unit force. Its magnitude increases with $\mathcal{N}$. The same is valid for the magnitude of the Green function. Figure 3 clearly shows this change.

Figure 3.

Figure 4.

Assume again that $b = 0.5$ and $\xi = 0.75$ but the axial force is tensile. Figure 4 shows the Green function provided that $p = 0.4p_{\text{crit}}$ and $p = 0.8p_{\text{crit}}$. This time there is a sign difference between the bending moments due to the dimensionless unit force and the tensile force. Hence the magnitude of the Green function decreases as the axial force $\mathcal{N}$ increases. Figure 4 clearly represents this phenomenon.

According to equation (3.4) the dimensionless vertical displacement w due to the dimensionless vertical force system f_z can be calculated by performing the integral in

$$w(x) = \int_{\xi=0}^{\xi=1} G(x, \xi) f_z(\xi) d\xi. \quad (4.19)$$

5. COMPUTATIONAL RESULTS

5.1. Integral equation of the problem. We shall assume that the inertia forces resulting from the longitudinal motion in the axial direction are neglected. It is also assumed that the moments of the inertia forces obtained from the rotation of cross section are negligible. Under these assumptions the dimensionless amplitude of the vibrations is denoted by w . It can be seen on the basis of Subsection 8.16 in [18], (4.1) and (4.12) that the amplitude w should fulfill the differential equation

$$L[w(x)] = w^{(4)} \pm \mathcal{N}w^{(2)} = \lambda w = f_z, \quad (5.1)$$

$$p^2 = \mathcal{N} = \frac{L^2 N}{I_{ey}}, \quad \lambda = \frac{\rho_a A \omega^2 L^4}{I_{ey}},$$

where ρ_a is the average density over the cross section of the beam, ω is the natural circular frequency while λ is the eigenvalue sought. Differential equation (5.1) is

associated with the homogeneous boundary and continuity conditions presented in Table 1. These constitute an eigenvalue problem with λ as the eigenvalue. If we substitute $\lambda w(\xi)$ for $f_z(\xi)$ in (4.19) we obtain the homogeneous Fredholm integral equation

$$w(x) = \lambda \int_{\xi=0}^{\ell=1} G(x, \xi) w(\xi) d\xi \quad (5.2)$$

for which the kernel $G(x, \xi)$ is presented in Subsection 4.1 if the axial force is compressive and in Subsection 4.2 if the axial force is tensile. The above integral equation determines an eigenvalue problem equivalent to the eigenvalue problem determined by differential equation (5.1) and the homogeneous boundary and continuity conditions presented in Table 1. Making use of the boundary element algorithm presented in [15] – see Subsection 7.2 – the eigenvalue problem determined by (5.2) can be reduced to an algebraic eigenvalue problem which can be solved numerically. A Fortran 90 code has been developed and applied to find numerical solutions for the eigenvalue λ . The interval $[0, \ell = 1]$ was divided into 12 elements and a quadratic isoparametric approximation was used over the elements in the code we developed.

If the axial force is zero the lowest dimensionless eigenvalue and circular frequency will be denoted by $\check{\lambda}_1$ and $\check{\omega}_1$.

5.2. Numerical results if b tends to zero. Table 2 represents the computational results if $b \rightarrow 0$. The quotient $\sqrt{\mathcal{N}_{\text{crit}}}/\pi$ is computed using equation (4.18). The value of $\check{\lambda}_1$ is given by equation (105) in [15]:

$$\begin{aligned} \sqrt{\check{\lambda}_1}/\pi^2 = & 402.66423 \times (1.0 - b)^5 - 1594.73367 \times (1.0 - b)^4 + \\ & + 2494.995 \times (1.0 - b)^3 - 1918.4007 \times (1.0 - b)^2 + \\ & + 716.4361 \times (1.0 - b) - 99.3904, \quad b \in [0, 0.5], \quad (5.3) \end{aligned}$$

or can be taken from Table 2 in [15]. This is also valid for Tables 3,...,7 which have the same structure as Table 2. The quotient $\omega_1^2/\check{\omega}_1^2 = \lambda_1/\check{\lambda}_1$ is computed for $\mathcal{N}/\mathcal{N}_{\text{crit}} = 0.00, 0.10, \dots, 0.90$ – see columns 2, 3 and 5 in Tables 2,...,7.

The computed values of $\omega_1^2/\check{\omega}_1^2 = \lambda_1/\check{\lambda}_1$ are denoted by diamonds in Figures 5,...,9. The difference between two subsequent values of $\omega_1^2/\check{\omega}_1^2$ is also included in these tables – see columns 4 and 6. If the function $\omega_1^2/\check{\omega}_1^2(\mathcal{N}/\mathcal{N}_{\text{crit}})$ is [non-linear](linear) the difference [varies](is constant).

Quadratic polynomials are fitted onto the computational results. Their graphs are drawn using continuous lines in the figures.

The beam behaves as if it were a fixed-pinned beam if $b \rightarrow 0$. Hence the results obtained should be the same as those valid for fixed-pinned beams. A comparison of the present results to those published in [18] – see Section 8.17.2 – proves that there is a very good agreement.

Table 2.

		b tends to zero			
		$\sqrt{\mathcal{N}_{\text{crit}}}/\pi = 1.4304, \tilde{\lambda}_1 = 1.5622$			
		Compression		Tension	
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	$\omega_1^2/\tilde{\omega}_1^2$ ($\lambda_1/\tilde{\lambda}_1$)	Difference	$\omega_1^2/\tilde{\omega}_1^2$ ($\lambda_1/\tilde{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.902010	0.0979900	1.097670	0.097670
3	0.20	0.803642	-0.098369	1.195005	0.097335
4	0.30	0.704879	-0.098763	1.292038	0.097032
5	0.40	0.605692	-0.099186	1.388785	0.096747
6	0.50	0.506050	-0.099642	1.485264	0.096478
7	0.60	0.405916	-0.100134	1.581488	0.096224
8	0.70	0.305247	-0.100668	1.677471	0.095983
9	0.80	0.203999	-0.101248	1.773227	0.095755
10	0.90	0.102117	-0.101881	1.868766	0.095538

Figure 5.

The quadratic polynomials fitted onto the computational results both for compression and tension are given by the following equations:

$$\frac{\omega_1^2}{\tilde{\omega}_1^2} = \frac{\lambda_1}{\tilde{\lambda}_1} = 0.9999 - 0.9756 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 2.4061 \times 10^{-2} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2, \quad (5.4a)$$

$$\frac{\omega_1^2}{\tilde{\omega}_1^2} = \frac{\lambda_1}{\tilde{\lambda}_1} = 1.0000 + 0.9770 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 1.3205 \times 10^{-2} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2 \quad (5.4b)$$

5.3. **Numerical results if $b = 0.1$.** Table 3 and Figure 6 represent the results obtained.

Table 3.

$b = 0.1$					
$\sqrt{\mathcal{N}_{\text{crit}}/\pi} = 1.5341, \check{\lambda}_1 = 1.8098$					
		Compression		Tension	
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.898254	-0.098720	1.093780	0.093780
3	0.20	0.800038	-0.098215	1.191130	0.097350
4	0.30	0.701491	-0.098547	1.288230	0.097099
5	0.40	0.602586	-0.098905	1.385094	0.096863
6	0.50	0.503293	-0.099292	1.481736	0.096642
7	0.60	0.403580	-0.099712	1.578170	0.096434
8	0.70	0.303411	-0.100169	1.674408	0.096237
9	0.80	0.202743	-0.100667	1.770460	0.096051
10	0.90	0.101529	-0.101214	1.866336	0.095876

Figure 6.

The quadratic polynomials fitted onto the numerical results are as follows:

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.998\,400 - 0.985\,245 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 1.174\,606 \times 10^{-2} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2, \quad (5.5a)$$

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.998\,592 + 0.966\,543 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 2.109\,042 \times 10^{-3} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2. \quad (5.5b)$$

5.4. **Numerical results if $b = 0.2$.** Table 4 and Figure 6 represent the results obtained.

Table 4.

$b = 0.2$					
$\sqrt{\mathcal{N}_{\text{crit}}}/\pi = 1.6557, \tilde{\lambda}_1 = 2.1610$					
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	Compression		Tension	
		$\omega_1^2/\tilde{\omega}_1^2$ ($\lambda_1/\tilde{\lambda}_1$)	Difference	$\omega_1^2/\tilde{\omega}_1^2$ ($\lambda_1/\tilde{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.904873	-0.099724	1.104529	0.099631
3	0.20	0.804869	-0.100004	1.204192	0.099662
4	0.30	0.704740	-0.100128	1.303750	0.099558
5	0.40	0.604481	-0.100259	1.403208	0.099458
6	0.50	0.504084	-0.100396	1.502570	0.099362
7	0.60	0.403543	-0.100541	1.601840	0.099269
8	0.70	0.302849	-0.100693	1.701020	0.099180
9	0.80	0.201995	-0.100854	1.800115	0.099094
10	0.90	0.100970	-0.101025	1.899127	0.099011

Figure 7.

Equations

$$\frac{\omega_1^2}{\tilde{\omega}_1^2} = \frac{\lambda_1}{\tilde{\lambda}_1} = 0.999596 - 0.987083 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 1.214649 \times 10^{-2} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2, \quad (5.6a)$$

$$\frac{\omega_1^2}{\tilde{\omega}_1^2} = \frac{\lambda_1}{\tilde{\lambda}_1} = 0.999633 + 0.987212 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 6.295185 \times 10^{-3} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2. \quad (5.6b)$$

are the quadratic polynomials fitted onto the computational results.

5.5. **Numerical results if $b = 0.3$.** Table 5, Figure 8 and equations (5.7) represent the results obtained.

Table 5.

		$b = 0.3$			
		$\sqrt{\mathcal{N}_{\text{crit}}/\pi} = 1.79382, \check{\lambda}_1 = 2.6683$			
		Compression		Tension	
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.900657	-0.099075	1.098279	0.098279
3	0.20	0.801574	-0.099083	1.196842	0.098563
4	0.30	0.702290	-0.099283	1.295255	0.098412
5	0.40	0.602790	-0.099500	1.393526	0.098271
6	0.50	0.503055	-0.099734	1.491664	0.098138
7	0.60	0.403065	-0.099990	1.589678	0.098013
8	0.70	0.302795	-0.100269	1.687574	0.097895
9	0.80	0.202221	-0.100574	1.785359	0.097784
9	0.90	0.101311	-0.100910	1.883038	0.097679

Figure 8.

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 1.004689 - 0.997570 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 7.276534 \times 10^{-3} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2, \quad (5.7a)$$

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 1.004794 + 0.997875 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 4.643405 \times 10^{-3} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2. \quad (5.7b)$$

5.6. **Numerical results if $b = 0.4$.** Table 6 and Figure 9 represent the results obtained.

Table 6.

		$b = 0.4$			
		$\sqrt{\mathcal{N}_{\text{crit}}}/\pi = 1.93113, \check{\lambda}_1 = 3.3881$			
		Compression		Tension	
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.894987	-0.098455	1.091328	0.098059
3	0.20	0.796529	-0.098458	1.189232	0.097904
4	0.30	0.697862	-0.098666	1.286971	0.097739
5	0.40	0.598974	-0.098888	1.384555	0.097583
6	0.50	0.499848	-0.099125	1.481990	0.097435
7	0.60	0.400468	-0.099380	1.579285	0.097294
8	0.70	0.300814	-0.099653	1.676446	0.097160
9	0.80	0.200866	-0.099947	1.773479	0.097033
9	0.90	0.100599	-0.100266	1.870391	0.096911

Figure 9.

The two quadratic polynomials fitted onto the numerical results are presented here:

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.993\,088 - 0.980\,042 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 1.286\,556 \times 10^{-2} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2, \quad (5.8a)$$

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.993\,323 + 0.980\,873 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} - 7.074\,0713 \times 10^{-3} \left(\frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \right)^2. \quad (5.8b)$$

5.7. **Numerical results if $b = 0.5$.** Table 7 represents the results obtained.

Table 7.

		$b = 0.5$			
		$\sqrt{\mathcal{N}_{\text{crit}}}/\pi = 2.0000, \check{\lambda}_1 = 4.0000$			
		Compression		Tension	
Load step	$\mathcal{N}/\mathcal{N}_{\text{crit}}$ (p^2/p_{crit}^2)	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference	$\omega_1^2/\check{\omega}_1^2$ ($\lambda_1/\check{\lambda}_1$)	Difference
1	0.00	1.000000		1.000000	
2	0.10	0.893682	-0.099310	1.092295	0.099313
3	0.20	0.794376	-0.099306	1.191601	0.099306
4	0.30	0.695069	-0.099306	1.290908	0.099306
5	0.40	0.595763	-0.099306	1.390214	0.099306
6	0.50	0.496457	-0.099306	1.489521	0.099306
7	0.60	0.397150	-0.099306	1.588827	0.099306
8	0.70	0.297844	-0.099306	1.688133	0.099306
9	0.80	0.198538	-0.099306	1.787440	0.099306
9	0.90	0.099231	-0.099306	1.886746	0.099306

The polynomial approximations (the differences are constants)

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.993\,064 - 0.992\,989 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \approx 1.000 - \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}}, \quad (5.9a)$$

$$\frac{\omega_1^2}{\check{\omega}_1^2} = \frac{\lambda_1}{\check{\lambda}_1} = 0.993\,064 + 0.992\,989 \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}} \approx 1.000 + \frac{\mathcal{N}}{\mathcal{N}_{\text{crit}}}. \quad (5.9b)$$

show that the functions $\omega_1^2/\check{\omega}_1^2(\mathcal{N}/\mathcal{N}_{\text{crit}})$ fitted onto the computational results are linear.

5.8. **What characters do the solutions have?** If $b \neq 0.5$ the function

$$\omega_1^2/\check{\omega}_1^2(\mathcal{N}/\mathcal{N}_{\text{crit}})$$

is non-linear, but the deviation from the linearity is not significant at all. It becomes smaller when $b \rightarrow 0.5$. If $b = 0.5$ the half beam behaves as if it were a simply supported beam for which the function $\omega_1^2/\check{\omega}_1^2(\mathcal{N}/\mathcal{N}_{\text{crit}})$ is, in principal, a linear one. This fact is reflected in equations (5.9).

6. CONCLUDING REMARKS

The present paper clarifies what effect the axial load (compressive or tensile) has on the eigenfrequencies of a heterogeneous pinned-pinned beam with an intermediate roller support. This problem is, in fact, a three-point boundary value problem (eigenvalue problem) associated with homogeneous boundary conditions. After determining the Green function of the three-point boundary value problem the eigenvalue problem that provides the eigenfrequencies for the PrsP beam is transformed into an eigenvalue problem governed by a homogeneous Fredholm integral equation with the Green

function as its kernel. The later eigenvalue problem is reduced to an algebraic eigenvalue problem which we solved numerically by using an effective solution algorithm based on the boundary element method. We have derived polynomial approximations for the sought function $\omega_1^2/\tilde{\omega}_1^2(N/N_{\text{crit}})$.

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