HYPER Singular BOUNDARY INTEGRAL EQUATIONS FOR PLANE ORTHOTROPIC ELASTICITY IN TERMS OF FIRST-ORDER STRESS FUNCTIONS

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Dedicated to Professor Barna Szabó on the occasion of his 85th birthday

Abstract. This paper is intended to present an implementation of the hypersingular boundary integral equations in terms of first-order stress functions for stress computations in plane orthotropic elasticity. In general, the traditional computational technique of the boundary element method used for computing the stress distribution on the boundary and close to it is not as accurate as it should be. In contrast, the accuracy of stress computations on the boundary is greatly increased by applying the hypersingular integral equations. Contrary to the method in which the solution is based on an approximation of displacement field, here the first-order stress functions and the rigid body rotation are the fundamental variables. An advantage of this approach is that the stress components can be obtained directly from the stress functions, there is, therefore, no need for Hooke's law, which should be used when they are computed from displacements. In addition, the computational work can be reduced when the stress distribution is computed at an arbitrary point on the boundary. The numerical examples presented prove the efficiency of this technique.

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1. INTRODUCTION

The theory of elasticity for orthotropic bodies is well established in the boundary element method. Paper [1] studies orthotropic plane problems and presents the most important relations for inner regions by the boundary element method, in which the displacements and the stresses are the unknowns on the boundary and the numerical solution is based on a constant approximation of the displacements and the stresses over the boundary elements. One of the most important advantages of using first-order stress functions is the fact that calculating stresses requires determining first derivatives only, whereas with stress functions of order two, the stresses are obtained in terms of the second derivatives. First derivatives are generally more convenient in

boundary element applications, although an additional equation is needed for symmetrizing the stress tensor. If the body considered is isotropic the boundary integral equations of the direct method for plane problems in terms of first-order stress functions are established in paper [2]. For orthotropic bodies [3] presents the two fundamental solutions of the basic equation and the Somigliana relations for inner and exterior regions, provided that the first-order stress functions are the basic variables and constant strains and stresses are assumed at infinity. The authors of paper [4] modified the Somigliana formulas for plane orthotropic bodies and exterior regions by assuming that the strains are constant and accordingly the displacements are linear at infinity.

When using the direct boundary element method, weakly and strongly singular integrals have to be computed. This is the main reason for obtaining partly inaccurate results for the stresses at those points located to the region's boundary. However, the accuracy of the numerical solutions could be increased by using an appropriately chosen hypersingular formulation, because in this way the terms containing singularities can be computed in semi-analytical forms. [5] came to the conclusion that a deep analysis of the hypersingular formulation provides an opportunity to increase the accuracy of the stress components computed at interior points close to the contour and on the contour itself. The theory of the hypersingular boundary integral equations, in which the displacements are the fundamental variables, is developed further for 2D and 3D problems [6–14] and there are a number of issues (see for instance [15–18]) or [19] for plane orthotropic elasticity) to which they can be applied successfully.

The goal of this paper is to utilize the integral equations in terms of first-order stress functions with hypersingular kernels [20] for the problems of plane orthotropic elasticity in order to make the stress computations more accurate on the boundary. Section 2 outlines the most important preliminaries, i.e., it presents the governing equations of the plane strain problem in terms of first-order stress functions for the orthotropic case, details the corresponding fundamental solutions, and clarifies what the boundary conditions are for the first-order stress functions. The concept of the strain boundary conditions is also introduced. Section 3 is devoted to the problem of how to improve the hypersingular integral formulation for boundary value problems under plane strain conditions. The necessary equations are set up in a suitable form in Section 4, which is devoted to some further important computational issues so that an appropriate algorithm can be developed for the stress computations. The formulas for calculating the stresses are derived in semi-analytical forms. Section 5 presents the applicability of the algorithm for two simple problems and a more difficult one. The last section is the conclusions. Some longer formulas are presented in the Appendix.

2. PRELIMINARIES

Throughout this paper a two-dimensional rectangular Cartesian coordinate system is used in which $x(x_1, x_2)$ and $y(y_1, y_2)$ are the source and field points. Greek subscripts are assumed to have the range (1, 2). The summation convention is applied to dummy indices. In accordance with the notations introduced: $\delta_{\rho\lambda}$ is the Kronecker symbol,

$\epsilon_{\rho\pi 3}$ is the permutation symbol and derivatives with respect to y_ρ are denoted by a Greek subscript preceded by a comma.

Consider a simply connected inner region $\mathcal{A}_I$ and its supplementary exterior region $\mathcal{A}_E$. We stipulate that the contour $\mathcal{L}_o$, which is divided into two parts denoted by $\mathcal{L}_t$ and $\mathcal{L}_u$, admits a nonsingular parametrization in terms of its arc length s . The positive direction on $\mathcal{L}_o$ is the one which leaves the region on the left side. The outward unit normal and the unit tangent on the contour are denoted by n_λ and τ_π .

The quantities that are prescribed on the boundary are denoted by hatted letters. [Stress functions $\mathbf{u}_\lambda(s)$ (obtainable from the prescribed tractions)] {Strain boundary conditions, i.e., the derivatives $\mathbf{t}_\lambda(s) = -d\hat{\mathbf{u}}_\lambda/ds$ } are prescribed on $[\mathcal{L}_t]$ $\{\mathcal{L}_u\}$. It is assumed that there are no body forces. The components of the elastic compliance tensor are denoted by s_{11} , $s_{12} = s_{21}$, s_{22} and s_{66} .

According to the Tonti scheme [21], stress functions are the fundamental variables, and the stresses and strains constitute the intermediate variables of the first and second kind in the dual system of elasticity. A prescribed incompatibility, which is in general zero, is the source variable. For the sake of a formal similarity to the boundary integral equations written in the usual forms, the opposite of the displacement derivative $\mathbf{t}_\lambda = -d\hat{\mathbf{u}}_\lambda/ds$ and the stress functions of order one $\mathbf{u}_\lambda$ are further notations applied in our boundary element formalism. However, highlighting the difference between the method based on an approximation of displacements and the presented one, these variables and the related quantities are typeset in calligraphic fonts. In this way, the plane strain problem in terms of the first-order stress functions for the orthotropic case is governed by the dual kinematic equations

$$\begin{aligned} t_{11} &= \mathbf{u}_{1,2} , & t_{12} &= \mathbf{u}_{2,2} , \\ t_{21} &= -\mathbf{u}_{1,1} , & t_{22} &= -\mathbf{u}_{2,1} , \end{aligned} \quad (2.1)$$

the inverse Hooke's law

$$\begin{aligned} e_{11} &= s_{11}t_{11} + s_{12}t_{22} , \\ e_{12} = e_{21} &= \frac{s_{66}}{4} (t_{12} + t_{21}) , \\ e_{22} &= s_{21}t_{11} + s_{22}t_{22} , \end{aligned} \quad (2.2)$$

the compatibility conditions

$$e_{11,2} - e_{12,1} + \varphi_{3,1} = 0 , \quad e_{21,2} - e_{22,1} + \varphi_{3,2} = 0 , \quad (2.3)$$

and the symmetry condition (equation of rotational equilibrium)

$$t_{12} = t_{21} . \quad (2.4)$$

With the prescribed traction $\hat{t}_\lambda(s)$, ($s \in \mathcal{L}_t$) the first-order stress functions on the boundary are given by

$$\mathbf{u}_\lambda(s) - \underbrace{\mathbf{u}_\lambda(P)}_{C_\lambda} = \int_P^s \hat{t}_\lambda(\sigma) d\sigma \quad s \in \mathcal{L}_t , \quad (2.5)$$

where P is the starting point of the arc $\mathcal{L}_t$ and C_λ is an undetermined integration constant. Consequently, equation

$$\mathbf{u}_\lambda(s) = \hat{\mathbf{u}}_\lambda(s) + C_\lambda \quad s \in \mathcal{L}_t \quad (2.6)$$

is the boundary condition imposed on the stress functions. Since no stresses belong to constant stress functions, C_λ can be set to zero.

The strain boundary conditions on $\mathcal{L}_u$ have the following form:

$$\mathbf{t}_\lambda(s) = -\frac{d\hat{u}_\lambda(s)}{ds} = -n_\rho[\epsilon_{\rho\pi 3}e_{\pi\lambda} - \delta_{\rho\lambda}\varphi_3] \quad s \in \mathcal{L}_u, \quad (2.7)$$

where $\hat{u}_\lambda(s)$ is the vector of the prescribed displacements on $\mathcal{L}_u$.

The outer region $\mathcal{A}_E$ is understood as the part of the coordinate plane outside the closed curve $\mathcal{L}_o$. The stresses at infinity denoted by $t_{11}(\infty)$, $t_{12}(\infty) = t_{21}(\infty)$, $t_{22}(\infty)$ are all constants. It is assumed that the rigid body rotation vanishes at infinity:

$$\varphi_3(\infty) = 0. \quad (2.8)$$

Let $c_\kappa(\infty)$ be a constant. The stress functions

$$\tilde{\mathbf{u}}_\kappa(y_\alpha) = \epsilon_{\alpha 3 \rho} y_\alpha t_{\kappa \rho}(\infty) + c_\kappa(\infty) \quad (2.9)$$

result in a constant stress state $t_{11}(\infty)$, $t_{12}(\infty) = t_{21}(\infty)$, $t_{22}(\infty)$ at the point y . It is obvious that no stresses belong to the stress function $c_\kappa(\infty)$.

After eliminating the intermediate variables, equation system (2.1)-(2.4) yields the fundamental equations for the unknowns $\mathbf{u}_\lambda$ and φ_3 . The fundamental solutions of the dual system can be given in a relatively simple form if on the basis of [3] we introduce the following notations:

$$D_1 = \frac{2s_{21} + s_{66}}{2s_{11}}, \quad D = \left[\frac{2s_{21} + s_{66}}{2s_{11}} \right]^2 - \frac{s_{22}}{s_{11}}, \quad (2.10a)$$

$$b_1 = \sqrt{D_1} - \sqrt{D}, \quad b_2 = \sqrt{D_1} + \sqrt{D}, \quad (2.10b)$$

$$\beta_1^2 = -\frac{2s_{21} + s_{66}}{2s_{11}} + \sqrt{\left(\frac{2s_{21} + s_{66}}{2s_{11}} \right)^2 - \frac{s_{22}}{s_{11}}}, \quad (2.10c)$$

$$\beta_2^2 = -\frac{2s_{21} + s_{66}}{2s_{11}} - \sqrt{\left(\frac{2s_{21} + s_{66}}{2s_{11}} \right)^2 - \frac{s_{22}}{s_{11}}}, \quad (2.10d)$$

$$d_1 = - \begin{vmatrix} 1 & \tilde{\beta}_1 & \tilde{\beta}_1^2 \\ 1 & \beta_2 & \beta_2^2 \\ 1 & \tilde{\beta}_2 & \tilde{\beta}_2^2 \end{vmatrix}, \quad d_2 = - \begin{vmatrix} 1 & \beta_1 & \beta_1^2 \\ 1 & \tilde{\beta}_1 & \tilde{\beta}_1^2 \\ 1 & \tilde{\beta}_2 & \tilde{\beta}_2^2 \end{vmatrix}, \quad (2.10e)$$

$$I_\alpha = 4\pi \frac{d_\alpha}{b_\alpha + 1} [b_\alpha^3 s_{11} + b_\alpha^2 (s_{21} + s_{66}) - b_\alpha s_{12} - s_{22}], \quad K = -\frac{1}{I_1 + I_2}, \quad (2.10f)$$

where $\tilde{\beta}_\alpha$ is the complex conjugate of β_α , and

$$\rho_\alpha = r_1 + \beta_\alpha r_2, \quad (2.10g)$$

in which $r_\rho = x_\rho - y_\rho$.

The fundamental solution of order one $\mathfrak{U}_{\kappa\lambda}(x, y)$ in component form is given by the following equations:

$$\mathfrak{U}_{11}(x, y) = -K \operatorname{Im} \sum_{\alpha=1}^2 d_\alpha (2 \ln \rho_\alpha + 3) \beta_\alpha^2, \quad (2.11a)$$

$$\mathfrak{U}_{12}(x, y) = K \operatorname{Im} \sum_{\alpha=1}^2 d_{\alpha} (2 \ln \rho_{\alpha} + 3) \beta_{\alpha} = \mathfrak{U}_{21}(x, y), \quad (2.11b)$$

$$\mathfrak{U}_{22}(x, y) = -K \operatorname{Im} \sum_{\alpha=1}^2 d_{\alpha} (2 \ln \rho_{\alpha} + 3). \quad (2.11c)$$

As regards the physical meaning of the fundamental solutions of order one the product $e_{\kappa}(x)\mathfrak{U}_{\kappa\lambda}(x, y)$ is the dual displacement $\mathfrak{u}_{\lambda}(y)$ due to a unit incompatibility $e_{\kappa}(x)$.

Let now y be a point on the contour. The fundamental solution of order two $\mathfrak{T}_{\kappa\lambda}(x, y)$ has the following components:

$$\begin{aligned} \mathfrak{T}_{11}(x, y) = K \operatorname{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}} \left[n_2 (-\beta_{\alpha}^3 s_{11} - s_{12} \beta_{\alpha}) - n_1 \frac{s_{66}}{2} \beta_{\alpha}^2 - \right. \\ \left. - n_1 (s_{22} + (s_{21} + \frac{s_{66}}{2}) \beta_{\alpha}^2) \right], \quad (2.12a) \end{aligned}$$

$$\begin{aligned} \mathfrak{T}_{12}(x, y) = K \operatorname{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}} \left[\frac{n_2}{2} s_{66} \beta_{\alpha}^2 + n_1 (s_{21} \beta_{\alpha}^3 + s_{22} \beta_{\alpha}) - \right. \\ \left. - n_2 (s_{22} + (s_{21} + \frac{s_{66}}{2}) \beta_{\alpha}^2) \right], \quad (2.12b) \end{aligned}$$

$$\begin{aligned} \mathfrak{T}_{21}(x, y) = K \operatorname{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}} \left[n_2 (\beta_{\alpha}^2 s_{11} + s_{12}) + n_1 \frac{s_{66}}{2} \beta_{\alpha} - \right. \\ \left. - n_1 \beta_{\alpha} ((s_{12} + \frac{s_{66}}{2}) + s_{11} \beta_{\alpha}^2) \right], \quad (2.12c) \end{aligned}$$

$$\begin{aligned} \mathfrak{T}_{22}(x, y) = K \operatorname{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}} \left[-\frac{n_2}{2} s_{66} \beta_{\alpha} - n_1 (s_{21} \beta_{\alpha}^2 + s_{22}) - \right. \\ \left. - n_2 \beta_{\alpha} ((s_{21} + \frac{s_{66}}{2}) + s_{11} \beta_{\alpha}^2) \right], \quad (2.12d) \end{aligned}$$

where the outward unit normal n_p is taken at y . The product $e_{\kappa}(x)\mathfrak{T}_{\kappa\lambda}(x, y)$ is the dual stress $\mathfrak{t}_{\lambda}(y)$ due to a unit incompatibility $e_{\kappa}(x)$.

It is important to note here that $\mathfrak{U}_{\kappa\lambda}(x, y)$ is weakly singular and $\mathfrak{T}_{\kappa\lambda}(x, y)$ is strongly singular. The elements of the elastic compliance tensor can be expressed with Young's moduli E_1, E_2 , shear modulus μ_{12} and Poisson's ratios ν_{12}, ν_{21}

$$s_{11} = \frac{1}{E_1}, \quad s_{12} = -\frac{\nu_{21}}{E_2}, \quad s_{21} = -\frac{\nu_{12}}{E_1}, \quad s_{22} = \frac{1}{E_2}, \quad s_{66} = \frac{1}{\mu_{12}}. \quad (2.13)$$

For the inner region A_I the Somigliana formulas in the dual system of orthotropic plane elasticity are of the form

$$\mathfrak{u}_{\kappa}(y) = \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y) \mathfrak{t}_{\lambda}(x) \, ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y) \mathfrak{u}_{\lambda}(x) \, ds_x \quad y \in A_I, \quad (2.14a)$$

$$c_{\kappa\lambda}(y)u_\lambda(y) = \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y)t_\lambda(x) ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y)u_\lambda(x) ds_x \quad y \in \mathcal{L}_o, \quad (2.14b)$$

$$0 = \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y)t_\lambda(x) ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y)u_\lambda(x) ds_x \quad y \notin \mathcal{A}_I \cup \mathcal{L}_o, \quad (2.14c)$$

in which the matrix $c_{\kappa\lambda}(y)$ depends on the angle formed by the tangents to the contour at y ($c_{\kappa\lambda}(y) = \delta_{\kappa\lambda}/2$ if the contour is smooth at y).

For the outer region $\mathcal{A}_E$ equations (2.14) are slightly modified by taking the stress state at infinity into account:

$$u_\kappa(y) = \tilde{u}_\kappa(y) + \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y)t_\lambda(x) ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y)u_\lambda(x) ds_x \quad y \in \mathcal{A}_E, \quad (2.15a)$$

$$c_{\kappa\lambda}(y)u_\lambda(y) = \tilde{u}_\kappa(y) + \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y)t_\lambda(x) ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y)u_\lambda(x) ds_x \quad y \in \mathcal{L}_o, \quad (2.15b)$$

$$0 = \tilde{u}_\kappa(y) + \oint_{\mathcal{L}_o} \mathfrak{U}_{\kappa\lambda}(x, y)t_\lambda(x) ds_x - \oint_{\mathcal{L}_o} \mathfrak{T}_{\kappa\lambda}(x, y)u_\lambda(x) ds_x \quad y \notin \mathcal{A}_E \cup \mathcal{L}_o. \quad (2.15c)$$

The dual Somigliana formulas (2.14), (2.15) assume the same form as those established for the isotropic case [2] with the difference being in the fundamental solutions.

3. HYPERSINGULAR FORMULATION FOR STRESS EVALUATIONS

In this section, the hypersingular formulation is applied to determine the stresses in the vicinity of the boundary with high accuracy. After solving integral equations (2.14b and (2.15b) of the direct method one obtains the values of $u_\lambda(x)$ and $t_\lambda(x)$ at every nodal point. The traditional approach to calculating the stress components at an arbitrary nodal point is the use of an equation system based partly on equations (2.1), (2.2) and on the polynomial approximation of u_λ and t_λ on the element that contains the nodal point selected [3]. In many cases the stresses obtained in this way are, however, not as accurate as they should be. For this reason we shall present a hypersingular integral equation formulation which significantly increases the accuracy of the stresses computed.

The line of thought is based on a technique developed by [5] in the primal system of plane elasticity. The partial derivatives of equation (2.15c) with respect to the coordinates y_ρ of the source point y should be determined taking equation (2.10g) into account, similarly to the case of isotropic materials [10], [20]. For this purpose, a small neighborhood of the source point y should be removed from the region of integration in order to obtain appropriate formulas. This way, singular integrals can be avoided – the kernels are singular at the point $x = y$. For convenience, the small circular domain $\mathcal{A}_\epsilon$ with the source point as its center is removed from the original region $\mathcal{A}_E$. Consequently, the new contour consists of two arcs $\mathcal{L}_\epsilon$ left from $\mathcal{L}_o$ after the removal of the small circle with radius ϵ and the circular arc s_ϵ .

Figure 1. Outer region A_E bounded by the boundary curve $\mathcal{L}_\varepsilon \cup s_\varepsilon$.

Under this condition equation (2.15c) can be written in the form

$$0 = \lim_{\varepsilon \rightarrow 0} \left\{ \tilde{\mathbf{u}}_\kappa(y) + \int_{\mathcal{L}_\varepsilon} [\mathfrak{U}_{\kappa\lambda}(x, y) \mathbf{t}_\lambda(x) - \mathfrak{T}_{\kappa\lambda}(x, y) \mathbf{u}_\lambda(x)] ds_x + \int_{s_\varepsilon} [\mathfrak{U}_{\kappa\lambda}(x, y) \mathbf{t}_\lambda(x) - \mathfrak{T}_{\kappa\lambda}(x, y) \mathbf{u}_\lambda(x)] ds_x \right\}. \quad (3.1)$$

After deriving equation (3.1) with respect to y_ρ , we arrive at

$$\lim_{\varepsilon \rightarrow 0} \left\{ \tilde{\mathbf{u}}_{\kappa,\rho}(y) + \int_{\mathcal{L}_\varepsilon} [\mathfrak{U}_{\kappa\lambda,\rho}(x, y) \mathbf{t}_\lambda(x) - \mathfrak{T}_{\kappa\lambda,\rho}(x, y) \mathbf{u}_\lambda(x)] ds_x + \int_{s_\varepsilon} \mathfrak{U}_{\kappa\lambda,\rho}(x, y) \mathbf{t}_\lambda(x) - \mathfrak{T}_{\kappa\lambda,\rho}(x, y) \mathbf{u}_\lambda(x) ds_x \right\} = 0, \quad (3.2)$$

where the derivatives $\mathfrak{U}_{\kappa\lambda,\rho}$ and $\mathfrak{T}_{\kappa\lambda,\rho}$ are presented in the Appendix – see equations (A.1) and (A.2). Due to the derivation, the order of singularity is increased, i.e., $\mathcal{D}_{\kappa\lambda\rho} = \mathfrak{U}_{\kappa\lambda,\rho}$ has a strong singularity of $O(\rho_\alpha^{-1})$, whereas $\mathcal{S}_{\kappa\lambda\rho} = \mathfrak{T}_{\kappa\lambda,\rho}$ is hypersingular of $O(\rho_\alpha^{-2})$.

Assume further that the stress functions of order one are continuous and differentiable in the neighborhood of the point y . Then it holds with good accuracy that

$$\mathbf{u}_\lambda(x) = \mathbf{u}_\lambda(y) + \mathbf{u}_{\lambda,\delta}(y)(x_\delta - y_\delta) + O(r^{1+\alpha}). \quad (3.3)$$

It is also not too difficult to check that the opposite of the displacement derivative $\mathbf{t}_\lambda$ in the vicinity of the source point assumes the form

$$\mathbf{t}_\lambda(x) = -\frac{\partial \mathbf{u}_\lambda}{\partial s} \cong -u_{\lambda,\pi}(y) \tau_\pi(x) = -u_{\lambda,\pi}(y) \epsilon_{3\rho\pi} n_\rho(x). \quad (3.4)$$

Making use of these approximations, we can rewrite equation (3.2) into the following form:

$$\begin{aligned}
& \tilde{\mathbf{u}}_{\kappa,\rho}(y) + \\
& \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{L}_\varepsilon} [\mathcal{D}_{\kappa\lambda\rho} \mathbf{t}_\lambda(x) - \mathcal{S}_{\kappa\lambda\rho} \mathbf{u}_\lambda(x)] \, ds_x + \int_{S_\varepsilon} \mathcal{D}_{\kappa\lambda\rho} [\mathbf{t}_\lambda(x) + u_{\lambda,\pi}(y) \tau_\pi(x)] \, ds_x - \right. \\
& \quad - \int_{S_\varepsilon} \mathcal{S}_{\kappa\lambda\rho} [\mathbf{u}_\lambda(x) - \mathbf{u}_{\lambda,\delta}(y)(x_\delta - y_\delta) - \mathbf{u}_\lambda(y)] \, ds_x - \int_{S_\varepsilon} \mathcal{S}_{\kappa\lambda\rho} ds_x \mathbf{u}_\lambda(y) - \\
& \quad \left. - \int_{S_\varepsilon} \mathcal{D}_{\kappa\lambda\rho} u_{\lambda,\pi}(y) \tau_\pi(x) ds_x - \int_{S_\varepsilon} \mathcal{S}_{\kappa\lambda\rho} \mathbf{u}_{\lambda,\delta}(y)(x_\delta - y_\delta) ds_x \right\} = 0. \quad (3.5)
\end{aligned}$$

Observe that the second integral in the second line and the first integral in the third line are equal to zero if their limits are taken. The value of the last integral in the third line can be manipulated into the form

$$\lim_{\varepsilon \rightarrow 0} \left\{ \int_{S_\varepsilon} \mathcal{S}_{\kappa\lambda\rho} ds_x \mathbf{u}_\lambda(y) \right\} = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{b_{\kappa\lambda\rho}(y)}{\varepsilon} \mathbf{u}_\lambda(y) \right\}, \quad (3.6)$$

in which the coefficient $b_{\kappa\lambda\rho}(y)$ depends on the local geometry of S_ε at the source point y throughout the limiting process [10]. This formula clearly shows that the limit of the above integral – due to the fact that the kernel is singular – is unbounded and depends also on the value of $b_{\kappa\lambda\rho}(y)$. However, if the source point y is an internal point, then $b_{\kappa\lambda\rho}(y) = 0$ – the proof is omitted here but can be done in the same way as in paper [5]. Hence taking the limit of the last two integrals in (3.5) yields

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \left\{ \int_{S_\varepsilon} \mathcal{D}_{\kappa\lambda\rho} u_{\lambda,\pi}(y) \tau_\pi(x) ds_x + \int_{S_\varepsilon} \mathcal{S}_{\kappa\lambda\rho} \mathbf{u}_{\lambda,\delta}(y)(x_\delta - y_\delta) ds_x \right\} = \\
= d_{\kappa\alpha\beta\rho}(y) \mathbf{u}_{\alpha,\beta}(y), \quad (3.7)
\end{aligned}$$

where $d_{\kappa\alpha\beta\rho}(y) \mathbf{u}_{\alpha,\beta} = 0.5 \mathbf{u}_{\kappa,\rho}$ if the contour $\mathcal{L}_o$ is smooth at the source point y . Finally we have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{L}_\varepsilon} [\mathcal{D}_{\kappa\lambda\rho} \mathbf{t}_\lambda(x) - \mathcal{S}_{\kappa\lambda\rho} \mathbf{u}_\lambda(x)] \, ds_x \right\} = \\
= \lim_{\varepsilon \rightarrow 0} \left\{ \frac{b_{\kappa\lambda\rho}}{\varepsilon} \mathbf{u}_\lambda(y) \right\} + d_{\kappa\alpha\beta\rho}(y) \mathbf{u}_{\alpha,\beta}(y). \quad (3.8)
\end{aligned}$$

After performing the limiting process and assuming that the boundary curve is smooth, the Somigliana stress identity takes the following form at all boundary points y with the exception of ‘non-smooth boundary points’:

$$\tilde{\mathbf{u}}_{\kappa,\rho}(y) + \frac{1}{2} \mathbf{u}_{\kappa,\rho}(y) + \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{L}_o} [\mathcal{S}_{\kappa\lambda\rho} \mathbf{u}_\lambda(x) - \mathcal{D}_{\kappa\lambda\rho} \mathbf{t}_\lambda(x)] \, ds_x + \frac{b_{\kappa\lambda\rho}}{\varepsilon} \mathbf{u}_\lambda(y) \right\} = 0. \quad (3.9)$$

Assume further that contour $\mathcal{L}_o$ is divided into n_{BE} boundary elements $\mathcal{L}_e$ ($e = 1, \dots, n_{BE}$), which have three nodes: the two end points and the middle point numbered locally as $a = 1, \dots, 3$ in the direction s . Thus, the usual isoparametric approximation is employed with quadratic boundary elements, which may be discontinuous if necessary. With the previous notations and the Somigliana stress identity (3.9), the elements of the stress tensor $t_{\psi\kappa}$ at the boundary point y can be obtained in the following form:

$$\begin{aligned}
 t_{\psi\kappa}(y) = \epsilon_{\psi 3\rho} \mathbf{u}_{\kappa,\rho}(y) = 2\epsilon_{\psi 3\rho} \left\{ -\tilde{\mathbf{u}}_{\kappa,\rho}(y) + \right. \\
 + \sum_{e=1, e \neq m}^{n_{BE}} \int_{\mathcal{L}_e} \mathcal{D}_{\kappa\lambda\rho}(x, y) \mathbf{t}_{\lambda}(x) ds_x + \int_{\mathcal{L}_m} \mathcal{D}_{\kappa\lambda\rho}(x, y) \mathbf{t}_{\lambda}(x) ds_x - \\
 \left. - \left[\sum_{e=1, e \neq m}^{n_{BE}} \int_{\mathcal{L}_e} \mathcal{S}_{\kappa\lambda\rho}(x, y) \mathbf{u}_{\lambda}^e(x) ds_x + \int_{\mathcal{L}_m} \mathcal{S}_{\kappa\lambda\rho}(x, y) \mathbf{u}_{\lambda}(x) ds_x \right] \right\}. \quad (3.10)
 \end{aligned}$$

The superscript e identifies the e -th boundary elements, while the element containing the source point y is denoted by the index m . Therefore, the line integrals with singular kernels on the m -th boundary element can be separately treated in a semi-analytical form, while the other integrals can be determined numerically by using the standard Gauss integration technique.

4. SEMI-ANALYTICAL TREATMENT OF INTEGRALS CONTAINING SINGULARITIES

Once the boundary fields $\mathbf{u}_{\lambda}(x)$ and $\mathbf{t}_{\lambda}(x)$ in terms of nodal values are known from the solution of the direct boundary integral equation method, the nodal values $\mathbf{u}_{\lambda}^a$ and $\mathbf{t}_{\lambda}^a$ ($a = 1, \dots, 3$), which belong to boundary element e and local node number a , are set up in matrix form

$$[\mathbf{u}_j^e]^T = [u_1^1 \quad u_1^2 \quad u_1^3 \quad u_2^1 \quad u_2^2 \quad u_2^3] \quad (4.1)$$

and

$$[\mathbf{t}_j^e]^T = [t_1^1 \quad t_1^2 \quad t_1^3 \quad t_2^1 \quad t_2^2 \quad t_2^3] \quad (4.2)$$

on the e -th element ($j = 1, \dots, 6$). It is obvious that the approximations of $\mathbf{u}_{\lambda}$ and $\mathbf{t}_{\lambda}$ over the e -th element are of the form:

$$\mathbf{u}_{\lambda} = N_{\lambda j}(\xi) \mathbf{u}_j^e, \quad \mathbf{t}_{\lambda} = N_{\lambda j}(\xi) \mathbf{t}_j^e, \quad (4.3)$$

in which

$$[N_{\lambda j}(\xi)] = \begin{bmatrix} N^1(\xi) & 0 & N^2(\xi) & 0 & N^3(\xi) & 0 \\ 0 & N^1(\xi) & 0 & N^2(\xi) & 0 & N^3(\xi) \end{bmatrix} \quad (4.4)$$

is the shape function matrix constituted by the quadratic Lagrange polynomials

$$N^1(\xi) = \frac{1}{2}\xi(\xi - 1), \quad N^2(\xi) = 1 - \xi^2, \quad N^3(\xi) = \frac{1}{2}\xi(\xi + 1), \quad (4.5)$$

which map the element onto the interval $\xi \in [-1, 1]$ – isoparametric approximation. The line integral with the strong singularity $O(\rho_{\alpha}^{-1})$ taken over the m -th element in (3.10) is of the form

$$\tilde{I}_{\kappa j\rho}^m = \int_{\mathcal{R}^m} \mathcal{D}_{\kappa\lambda\rho}(\xi, \eta) N_{\lambda j}(\xi) J(\xi) d\xi = \int_{\mathcal{R}^m} G_{\kappa j\rho}^m(\xi, \eta) d\xi. \quad (4.6)$$

The line integral with the hypersingularity $O(\rho_\alpha^{-2})$ in (3.10) can be expressed in the following form:

$$I_{\kappa j \rho}^m = \int_{\mathcal{R}^m} \mathcal{S}_{\kappa \lambda \rho}(\xi, \eta) N_{\lambda j}(\xi) J(\xi) d\xi = \int_{\mathcal{R}^m} F_{\kappa j \rho}^m(\xi, \eta) d\xi. \quad (4.7)$$

Making use of the notations introduced the stress tensor $t_{\psi \kappa}$ at the boundary point y is given by

$$t_{\psi \kappa}(y) = 2\epsilon_{\psi 3 \rho} \left\{ -\tilde{u}_{\kappa, \rho}(y) + \sum_{e=1, e \neq m}^{n_{BE}} \int_{\mathcal{R}^e} \mathcal{D}_{\kappa \lambda \rho}(\xi, \eta) N_{\lambda j}(\xi) J(\xi) d\xi t_j^e + \tilde{I}_{\kappa j \rho}^m t_j^m - \left[\sum_{e=1, e \neq m}^{n_{BE}} \int_{\mathcal{R}^e} \mathcal{S}_{\kappa \lambda \rho}(\xi, \eta) N_{\lambda j}(\xi) J(\xi) d\xi u_j^e + I_{\kappa j \rho}^m u_j^m \right] \right\}. \quad (4.8)$$

Here $J(\xi)$ is the Jacobian, $\mathcal{R}^e$ is the mapping of $\mathcal{L}_e$ onto the interval $[-1, 1]$. The meaning of $\mathcal{R}_\epsilon^m$ can be read off Figure 2, which also shows the mapping of the boundary element $\mathcal{L}_m$ onto the axis ξ . The length of $\mathcal{R}_\epsilon^m$ is denoted by $\alpha(\epsilon)$, while η is the image of the source point y on this interval.

Figure 2. Illustration of the interval $\mathcal{R}_\epsilon^m$ on the map of $\mathcal{L}_m$ on the axis ξ .

After determining the power series representation of $F_{\kappa j \rho}^m(\xi, \eta)$, the singular terms in the integral expression (4.7) can be isolated. If we want to give the power series of $\mathcal{S}_{\kappa \lambda \rho}(\xi, \eta)$ in the vicinity of the source point η , the power series expansion

$$\frac{1}{\rho_\alpha^2} = \frac{1}{(A_1 + \beta_\alpha A_2)^2 \delta^2} - 2 \frac{B_1 + \beta_\alpha B_2}{(A_1 + \beta_\alpha A_2)^3 \delta} + O(1) \quad (4.9)$$

is needed, in which $\delta = \xi - \eta$. The constants A_λ and B_λ can be obtained from the first two terms of the Taylor series of the position vector r_λ taken in the vicinity of the source point η :

$$r_\lambda = x_\lambda(\xi) - y_\lambda(\eta) = 0 + \frac{\partial x_\lambda}{\partial \xi} \Big|_{\xi=\eta} (\xi - \eta) + \frac{1}{2!} \frac{\partial^2 x_\lambda}{\partial \xi^2} \Big|_{\xi=\eta} (\xi - \eta)^2 + \dots = A_\lambda \delta + B_\lambda \delta^2 + O(\delta^3), \quad (4.10)$$

where

$$A_\lambda = (x_\lambda^1 - 2x_\lambda^2 + x_\lambda^3)\eta + \frac{1}{2}(x_\lambda^3 - x_\lambda^1), \quad B_\lambda = \frac{1}{2}(x_\lambda^1 - 2x_\lambda^2 + x_\lambda^3) \quad (4.11)$$

are constants at η ; x_λ^1 , x_λ^2 and x_λ^3 are the nodal coordinates in the usual representation. Moreover, let $A = J(\eta) = \sqrt{\sum_{\lambda=1}^2 A_\lambda^2}$. Then the power series of the shape function matrix is of the form

$$\begin{aligned} N_{\lambda j}(\xi) &= N_{\lambda j}(\xi) \Big|_{\xi=\eta} + \frac{d}{d\xi} N_{\lambda j}(\xi) \Big|_{\xi=\eta} (\xi - \eta) + O(\delta^2) = \\ &= {}^0N_{\lambda j}(\eta) + {}^1N_{\lambda j}(\eta)\delta + O(\delta^2), \end{aligned} \quad (4.12)$$

where

$$[{}^0N_{\lambda j}(\eta)] = \begin{bmatrix} {}^0N^1 & 0 & {}^0N^2 & 0 & {}^0N^3 & 0 \\ 0 & {}^0N^1 & 0 & {}^0N^2 & 0 & {}^0N^3 \end{bmatrix}, \quad (4.13)$$

in which

$${}^0N^1(\eta) = \frac{1}{2}\eta(\eta - 1), \quad {}^0N^2(\eta) = 1 - \eta^2, \quad {}^0N^3(\eta) = \frac{1}{2}\eta(\eta + 1) \quad (4.14)$$

and

$$[{}^1N_{\lambda j}(\eta)] = \begin{bmatrix} {}^1N^1 & 0 & {}^1N^2 & 0 & {}^1N^3 & 0 \\ 0 & {}^1N^1 & 0 & {}^1N^2 & 0 & {}^1N^3 \end{bmatrix}, \quad (4.15)$$

in which

$${}^1N^1(\eta) = \frac{1}{2}(\eta - 1), \quad {}^1N^2(\eta) = -2\eta, \quad {}^1N^3(\eta) = \frac{1}{2}(\eta + 1). \quad (4.16)$$

For the product of the Jacobian and the vector n_ρ we obtain:

$$J_1(\xi) = n_1 J(\xi) = A_2 + 2B_2\delta + O(\delta^2), \quad (4.17)$$

$$J_2(\xi) = n_2 J(\xi) = -A_1 - 2B_1\delta + O(\delta^2). \quad (4.18)$$

After substituting power series (4.9)-(4.18) into integral (4.7), the integrand $F_{\kappa j \rho}^m(\xi, \eta)$ can be manipulated into the following Laurent series in the vicinity of the source point η :

$$F_{\kappa j \rho}^m(\xi, \eta) = \frac{{}^{-2}F_{\kappa j \rho}^m(\eta)}{(\xi - \eta)^2} + \frac{{}^{-1}F_{\kappa j \rho}^m(\eta)}{\xi - \eta} + O(1), \quad (4.19)$$

where ${}^{-1}F_{\kappa j \rho}^m(\eta)$ and ${}^{-2}F_{\kappa j \rho}^m(\eta)$ are constant values at a fixed $\eta \in (-1, 1)$. The details are presented in Appendix (A.3). Subtracting the first two terms of the power series of the integrand $F_{\kappa j \rho}^m(\xi, \eta)$ from equation (4.7) and investigating these terms separately, three line integrals are obtained:

$$\begin{aligned} I_{\kappa j \rho}^m &= \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{R}^m - 2\mathcal{R}_\varepsilon^m} \left(F_{\kappa j \rho}^m(\xi, \eta) - \left[\frac{{}^{-2}F_{\kappa j \rho}^m(\eta)}{(\xi - \eta)^2} + \frac{{}^{-1}F_{\kappa j \rho}^m(\eta)}{\xi - \eta} \right] \right) d\xi \right\} + \\ &+ \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{R}^m - 2\mathcal{R}_\varepsilon^m} \frac{{}^{-2}F_{\kappa j \rho}^m(\eta)}{(\xi - \eta)^2} d\xi \right\} + \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{R}^m - 2\mathcal{R}_\varepsilon^m} \frac{{}^{-1}F_{\kappa j \rho}^m(\eta)}{\xi - \eta} d\xi \right\} = \\ &= {}^0I_{\kappa j \rho}^m + {}^{-2}I_{\kappa j \rho}^m + {}^{-1}I_{\kappa j \rho}^m. \end{aligned} \quad (4.20)$$

Note that the integrand of ${}^0I_{\kappa j\rho}^m$ is a polynomial, i.e., it is not singular. Hence the limit

$${}^0I_{\kappa j\rho}^m = \int_{-1}^1 \left\{ F_{\kappa j\rho}^m(\xi, \eta) - \left[\frac{-2F_{\kappa j\rho}^m(\eta)}{(\xi - \eta)^2} + \frac{-1F_{\kappa j\rho}^m(\eta)}{\xi - \eta} \right] \right\} d\xi \quad (4.21)$$

is a regular integral, which can be computed by using the standard numerical integration techniques. We have chosen a 14-node Gauss-Legendre quadrature rule for the interval $[-1, 1]$. The second and third integrals can be calculated analytically. After dividing $\mathcal{R}^m$ into two subintervals $[-1, -\alpha]$ and $[\alpha, 1]$ separated from each other at the singular point η and making use of the series presented in the Appendix, the limit of the two last integrals in equation (4.20) can be obtained. Some paper and pencil calculations yield the following analytical result:

$$\begin{aligned} -1I_{\kappa j\rho}^m &= \lim_{\varepsilon \rightarrow 0^-} \left\{ \int_{-1}^{\eta-\alpha} \frac{-1F_{\kappa j\rho}^m(\eta)}{\xi - \eta} d\xi \right\} + \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{\eta+\alpha}^1 \frac{-1F_{\kappa j\rho}^m(\eta)}{\xi - \eta} d\xi \right\} = \\ &= \ln \left| \frac{1 - \eta}{-1 - \eta} \right|^{-1} F_{\kappa j\rho}^m(\eta) . \end{aligned} \quad (4.22)$$

A similar procedure results in the last integral in equation (4.20) in a closed form:

$$\begin{aligned} -2I_{\kappa j\rho}^m &= \lim_{\varepsilon \rightarrow 0^-} \left\{ \int_{-1}^{\eta-\alpha} \frac{-2F_{\kappa j\rho}^m(\eta)}{(\xi - \eta)^2} d\xi \right\} + \lim_{\varepsilon \rightarrow 0^+} \left\{ \int_{\eta+\alpha}^1 \frac{-2F_{\kappa j\rho}^m(\eta)}{(\xi - \eta)^2} d\xi \right\} = \\ &= - \left[\frac{1}{1 + \eta} + \frac{1}{1 - \eta} \right]^{-2} F_{\kappa j\rho}^m(\eta) . \end{aligned} \quad (4.23)$$

Consequently, formulas (4.21), (4.22) and (4.23) can be used to determine the integrals taken over the element that contains the singularity. It is obvious that the singular integrals can be given in closed form while the line integral without singularities is computed numerically. After collecting the results, the final formula for the evaluation of the hypersingular integral is presented in the following semi-analytical form:

$$\begin{aligned} I_{\kappa j\rho}^m &= \int_{-1}^1 \left(F_{\kappa j\rho}^m(\xi, \eta) - \left[\frac{-2F_{\kappa j\rho}^m(\eta)}{(\xi - \eta)^2} + \frac{-1F_{\kappa j\rho}^m(\eta)}{\xi - \eta} \right] \right) d\xi + \\ &+ \ln \left| \frac{1 - \eta}{-1 - \eta} \right|^{-1} F_{\kappa j\rho}^m(\eta) - \left[\frac{1}{1 + \eta} + \frac{1}{1 - \eta} \right]^{-2} F_{\kappa j\rho}^m(\eta) , \end{aligned} \quad (4.24)$$

where $\eta \in (-1, 1)$. When the point y coincides with the endpoints ($\eta = -1$ or $\eta = 1$), one has to apply the formula below:

$$\begin{aligned} I_{\kappa j\rho}^i &= \int_{-1}^1 \left(F_{\kappa j\rho}^i(\xi, \eta) - \left[\frac{-2F_{\kappa j\rho}^i(\eta)}{(\xi - \eta)^2} + \frac{-1F_{\kappa j\rho}^i(\eta)}{\xi - \eta} \right] \right) d\xi + \\ &+ \ln \left| \frac{2}{\vartheta_i} \right| \operatorname{sgn}(\delta)^{-1} F_{\kappa j\rho}^i(\eta) - \left[\frac{\gamma_i}{\vartheta_i^2} \operatorname{sgn}(\delta) + \frac{1}{2} \right]^{-2} F_{\kappa j\rho}^i(\eta) , \end{aligned} \quad (4.25)$$

where $i = m$ if $\eta = 1$, $i = m + 1$ if $\eta = -1$ and, in addition, $\vartheta_i(\eta) = A^{-1}$ and $\gamma_i(\eta) = -\frac{1}{A^4}(\sum_{\lambda=1}^2 A_\lambda B_\lambda)$ as proposed by [10].

The line integral with the strong singularity $O(\rho_\alpha^{-1})$ on the m -th element can be treated similarly to equation (4.7). The integrand $G_{\kappa j \rho}^m(\xi, \eta)$ in equation (4.6) is manipulated into the following series in the vicinity of the source point η :

$$G_{\kappa j \rho}^m(\xi, \eta) = \frac{{}^{-1}G_{\kappa j \rho}^m(\eta)}{\xi - \eta} + O(1). \quad (4.26)$$

The details are presented in Appendix (A.4). The corresponding line integrals can be determined in the same manner as for equation (4.20):

$$\begin{aligned} \tilde{I}_{\kappa j \rho}^m = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{R}^m - 2\mathcal{R}_\varepsilon^m} \left(G_{\kappa j \rho}^m(\xi, \eta) - \frac{{}^{-1}G_{\kappa j \rho}^m(\eta)}{\xi - \eta} \right) d\xi \right\} + \\ + \lim_{\varepsilon \rightarrow 0} \left\{ \int_{\mathcal{R}^m - 2\mathcal{R}_\varepsilon^m} \frac{{}^{-1}G_{\kappa j \rho}^m(\eta)}{\xi - \eta} d\xi \right\} = {}^0\tilde{I}_{\kappa j \rho}^m + {}^{-1}\tilde{I}_{\kappa j \rho}^m. \end{aligned} \quad (4.27)$$

After finding the limit values in question, it turns out that the term ${}^0\tilde{I}_{\kappa j \rho}^m$ is not singular. It can, therefore, be computed numerically with sufficient accuracy. As regards the term ${}^{-1}\tilde{I}_{\kappa j \rho}^m$ we can express it analytically:

$$\tilde{I}_{\kappa j \rho}^m = \int_{-1}^1 \left(G_{\kappa j \rho}^m(\xi, \eta) - \frac{{}^{-1}G_{\kappa j \rho}^m(\eta)}{\xi - \eta} \right) d\xi + \ln \left| -\frac{1 - \eta}{1 + \eta} \right| {}^{-1}G_{\kappa j \rho}^m(\eta). \quad (4.28)$$

If y coincides with an endpoint of two neighboring elements ($i = m$ for $\eta = 1$ and $i = m + 1$ for $\eta = -1$) the following formula should be applied:

$$\tilde{I}_{\kappa j \rho}^i = \int_{-1}^1 \left(G_{\kappa j \rho}^i(\xi, \eta) - \frac{{}^{-1}G_{\kappa j \rho}^i(\eta)}{\xi - \eta} \right) d\xi + \ln \left| \frac{2}{\vartheta_i} \right| \text{sgn}(\delta) {}^{-1}G_{\kappa j \rho}^i(\eta). \quad (4.29)$$

The above results show that the singularities can be integrated analytically over the m -th element. A program has been developed in Fortran 90 for finding numerical solutions.

5. NUMERICAL EXAMPLES

Problem 1. We shall consider a plane strain problem. It is assumed that there is a rigid circular inclusion in the plane of the strain with a radius $R = 10$ mm and a constant stress state is prescribed at infinity: $t_{11} = 100$ [MPa], $t_{12}(\infty) = t_{21}(\infty) = t_{22}(\infty) = 0$. The stress functions that characterize a constant stress state are given by equation (2.9). The material of the body in plane strain is birch for which $s_{11} = 8.497 \cdot 10^{-5}$ [mm²/N], $s_{12} = s_{21} = -6.11 \cdot 10^{-6}$ [mm²/N], $s_{22} = 1.699 \cdot 10^{-4}$ [mm²/N] and $s_{66} = 1.456 \cdot 10^{-3}$ [mm²/N] are the elastic coefficients [3].

It is clear that the inner boundary is fixed. Then the strain boundary condition (2.7) that can be described on the circle is of Neumann type. Determination of the stresses on the inner boundary can serve as a benchmark for proving numerically that

the use of equations (4.24) and (4.25) results in increasing accuracy concerning the stress computations. The contour is divided into only 24 uniform quadratic elements. Though there are two undetermined integration constants in equation (2.6) which have, however, no effect on the stress computations, we must to set them to a prescribed value (to zero) at such nodal points on the boundary where this can be done with ease. For Problem 1 the stress function u_1 is set to zero at point A and the stress function u_2 is set to zero at point B – see Figure 3 for details.

Figure 3. Outer region bounded by a circle for which a constant stress state is prescribed at infinity.

Table 1 contains the quotients $t_{rr}/t_{11}(\infty)$, $t_{r\varphi}/t_{11}(\infty)$ and $t_{\varphi\varphi}/t_{11}(\infty)$ obtained at various boundary points for the rigid circular inclusion using a polar coordinate system. Analytical results at given φ are obtained from Lekhnitskii's book [22].

Table 1. Results for the rigid inclusion.

φ [rad]	$\frac{t_{rr}}{t_{11}(\infty)}$			$\frac{t_{r\varphi}}{t_{11}(\infty)}$			$\frac{t_{\varphi\varphi}}{t_{11}(\infty)}$		
	HBIE	DBEM [3]	anal. [22]	HBIE	DBEM [3]	anal. [22]	HBIE	DBEM [3]	anal. [22]
0	1.1364	1.1363	1.237	0.0000	0.0000	0.000	0.0449	0.0444	0.044
$\pi/12$	1.1562	1.1558	1.156	-2.9936	-2.9999	-0.299	0.0934	0.0936	0.093
$\pi/6$	0.9370	0.9364	0.937	-0.5185	-0.5188	-0.519	0.2696	0.2701	0.270
$\pi/4$	0.6377	0.6370	0.698	-0.5987	-0.5986	-0.599	0.5154	0.5158	0.516
$\pi/3$	0.3383	0.3377	0.338	-0.5185	-0.5181	-0.519	0.6989	0.6990	0.699
$5\pi/12$	0.1192	0.1188	0.119	-0.2994	-0.2987	-0.299	0.5637	0.5627	0.564
$\pi/2$	0.0390	0.0389	0.039	0.0000	0.0000	0.000	0.0044	0.0028	0.003

The present HBIE solutions show very good agreement with the literature data [22] and are also more accurate than the solutions from the direct boundary element method (DBEM) [3] with the same coarse mesh.

Figure 4 depicts the hypersingular solutions for the stresses on the quarter $\varphi \in [0, \pi/2]$ of the circle. The numerical results (lines) show good agreement with the analytical results (marks): t_{rr} (solid line, diamonds), $t_{\varphi\varphi}$ (dot dot dashed line, boxes)

and $t_{r\varphi}$ (dashed line, circles). The stress distributions plotted from the stress values computed at arbitrarily selected points per boundary element match very well with the marks. The further advantage of using HBIE, which is also demonstrated in the following figures (Figure 4, 5, 7 and 8), is its ability to directly calculate the stress components anywhere on the boundary, contrary to the direct method [3].

Figure 4. Exact and numerical solutions for the rigid circular inclusion as a function φ .

Problem 2. We consider the same outer region including a circular hole with radius $R = 10\text{ mm}$ with the previous constant stress state prescribed at infinity. The inner boundary is, however, free; thus, the stress functions are zero on the whole boundary (Dirichlet boundary condition). Through this example, equations (4.28) and (4.29) can be separately validated for computational accuracy.

Using a polar coordinate system, Table 2 represents the numerical results for $t_{\varphi\varphi}/t_{11}(\infty)$ obtained at various boundary points.

Table 2. Results for the circular hole.

φ [rad]	$\frac{t_{\varphi\varphi}}{t_{11}(\infty)}$		
	HBIE	DBEM [3]	anal. [22]
0	-0.7075	-0.7074	-0.707
$\pi/12$	-0.3399	-0.3393	-0.340
$\pi/6$	0.0691	0.0695	0.069
$\pi/4$	0.4039	0.4045	0.404
$\pi/3$	0.9664	0.9661	0.966
$5\pi/12$	2.5771	2.5774	2.577
$\pi/2$	5.4526	5.4541	5.453

Figure 5 depicts the hypersingular solutions for the stress $t_{\varphi\varphi}$ on the first quarter $\varphi \in [0, \pi/2]$ of the circle: $t_{rr} = 0$, $t_{\varphi\varphi}$ (dot dot dashed line, boxes) and $t_{r\varphi} = 0$.

Figure 5. Exact and numerical solutions for the circular hole as function φ .

The results of the present HBIE solutions in comparison with the existing numerical [3] and analytical [22] solutions show a high accuracy and good fitting at various boundary point.

Problem 3. We consider an L-shape domain (see Figure 6) in order to investigate the effect of singularities arising at the reentrant corner, in the vicinity of point C. In the computations the following dimensions of the L-shape are chosen: $b_1 = h_1 = 100$ mm, $b_2 = h_2 = 50$ mm, and under the given loading ($p_1 = 5$ MPa and $p_2 = 10$ MPa) an equilibrium state is investigated. Our results are compared with those of the commercial software Abaqus, to assess the accuracy and efficiency of the method presented in this paper.

According to papers [23], [24], [25] the finite element method (FEM) mesh for Problem 3 has 5281 nodes and 16232 quadratic elements (CPE8R, 8-node biquadratic plane strain quadrilateral, reduced integration, approximate global size = 1.5 mm), and a bias ratio of 7 is used along the lines BC and CD to refine the mesh in the vicinity of point C. The components of the elastic compliance tensor are the same as in the previous examples, therefore, the elastic constants are $E_1 = 11768.86$ MPa, $E_2 = 5885.82$ MPa, $\mu_{12} = 686.81$ MPa and $\nu_{12} = 0.072$ (see equation (2.13)).

For our BEM solution, the traction boundary conditions with the zero values for C_λ at point O are, formally, given by equation (2.6), where $\hat{u}_\lambda(s)$ is either constant or linear with gradients p_1 or p_2 on the boundary. The boundary of the L-shape domain has been discretized by 160 uniform quadratic elements with a length of 2.5 mm so that the lines BC and CD are divided into 20 elements. But even in that case, the

Figure 6. Geometry and boundary conditions for L-shape domain in an equilibrium state.

Figure 7. Comparison between FEM ($\diamond$) and HBIE solution (—) along the line BC.

stress solutions using HBIE show a good agreement with the Abaqus results denoted by diamond marks. Nevertheless, there is little difference between the FEM and BEM results in the vicinity of the corner point where the local mesh refinement is used in FEM analysis for stress singularity [24]. The stress t_{11} along line BC is plotted in Figure 7 to further demonstrate the efficiency of the method. $t_{12} = t_{22} = 0$. The stress t_{22} along line BC is plotted in Figure 8, while $t_{11} = t_{12} = 0$. Corner points B,

Figure 8. Comparison between FEM ($\diamond$) and HBIE solution (—) along the line CD.

C, D, E and F on the boundary of L-shaped domain do not lead to any difficulties since discontinuous boundary elements can be used at those corner points in BEM. This is a significant advantage over most collocation schemes.

6. CONCLUSIONS

This paper is devoted to an application of first-order stress functions in the hypersingular boundary integral formulation by which the stress components on the boundary can be computed with a high accuracy both in the plane region and on its boundary for plane orthotropic elasticity. For this case the hypersingular boundary integral equation method has been developed providing that (i) the region under consideration is a simply-connected region with a smooth boundary, (ii) there are no body forces, (iii) the material is orthotropic.

It has been proven that the formulas for calculating the stresses on the boundary can be derived in semi-analytical forms. The hypersingular boundary integrals are resolved into two parts. The first one, which is completely free from singularity, should be computed numerically – this computation results in, however, a very small numerical error. The second part (terms containing singularities) can be expressed analytically in a closed form. Consequently, in contrast to the conventional stress calculation, which uses polynomial approximations for the derivatives and thus might even lead to significant errors both in the formalism and in the numerical results in the vicinity of the boundary, the stress calculation method suggested in this paper is much more reliable and accurate due to the way it handles the hypersingular integrals.

A formulation of Somigliana stress identity for plane orthotropic elasticity has been developed and implemented in a Fortran 90 code for the numerical computations. The computational results we have obtained illustrate that the accuracy of the computations is increased significantly and confirm that a semi-analytical treatment of hypersingular integrals is very effective.

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APPENDIX A.

A.1. Derivatives of the fundamental solutions of order one. Making use of equations (2.11), we can determine the derivatives $\mathcal{D}_{\kappa\lambda\rho} = \mathfrak{U}_{\kappa\lambda,\rho}$. The results are presented below:

$$\mathcal{D}_{111}(x, y) = 2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}^2, \quad \mathcal{D}_{121}(x, y) = -2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha} = \mathcal{D}_{211}(x, y), \quad (\text{A.1a})$$

$$\mathcal{D}_{211}(x, y) = -2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}, \quad \mathcal{D}_{221}(x, y) = 2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}}, \quad (\text{A.1b})$$

$$\mathcal{D}_{112}(x, y) = 2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}^3, \quad \mathcal{D}_{122}(x, y) = -2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}^2 = \mathcal{D}_{212}(x, y), \quad (\text{A.1c})$$

$$\mathcal{D}_{212}(x, y) = -2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}^2, \quad \mathcal{D}_{222}(x, y) = 2K\text{Im} \sum_{\alpha=1}^2 \frac{d_{\alpha}}{\rho_{\alpha}} \beta_{\alpha}. \quad (\text{A.1d})$$

A.2. Derivatives of the fundamental solutions of order two. Making use of equations (2.12), we can determine the derivatives $\mathcal{S}_{\kappa\lambda\rho} = \mathfrak{T}_{\kappa\lambda,\rho}$. The results are presented below:

$$\begin{aligned} \mathcal{S}_{111}(x, y) = -K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[n_2 (s_{11}\beta_{\alpha}^3 + s_{12}\beta_{\alpha}) + n_1 \frac{s_{66}}{2} \beta_{\alpha}^2 + \right. \\ \left. + n_1 (s_{22} + (s_{21} + \frac{s_{66}}{2})\beta_{\alpha}^2) \right], \quad (\text{A.2a}) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{121}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[\frac{n_2}{2} s_{66} \beta_{\alpha}^2 + n_1 (s_{21}\beta_{\alpha}^3 + s_{22}\beta_{\alpha}) - \right. \\ \left. - n_2 (s_{22} + (s_{21} + \frac{s_{66}}{2})\beta_{\alpha}^2) \right], \quad (\text{A.2b}) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{211}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[n_2 (s_{11}\beta_{\alpha}^2 + s_{12}) + n_1 \frac{s_{66}}{2} \beta_{\alpha} - \right. \\ \left. - n_1 \beta_{\alpha} ((s_{12} + \frac{s_{66}}{2}) + s_{11}\beta_{\alpha}^2) \right], \quad (\text{A.2c}) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{221}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[-\frac{n_2}{2} s_{66} \beta_{\alpha} - n_1 (s_{21}\beta_{\alpha}^2 + s_{22}) - \right. \\ \left. - n_2 \beta_{\alpha} ((s_{21} + \frac{s_{66}}{2}) + s_{11}\beta_{\alpha}^2) \right], \quad (\text{A.2d}) \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{112}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[-n_2 (\beta_{\alpha}^4 s_{11} + s_{12}\beta_{\alpha}^2) - n_1 \frac{s_{66}}{2} \beta_{\alpha}^3 - \right. \\ \left. - n_1 (s_{22}\beta_{\alpha} + (s_{21} + \frac{s_{66}}{2})\beta_{\alpha}^3) \right], \quad (\text{A.2e}) \end{aligned}$$

$$\mathcal{S}_{122}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[n_1 (s_{21}\beta_{\alpha}^4 + s_{22}\beta_{\alpha}^2) + \frac{n_2}{2} s_{66}\beta_{\alpha}^3 - \right. \\ \left. - n_2(s_{22}\beta_{\alpha} + (s_{21} + \frac{s_{66}}{2})\beta_{\alpha}^3) \right], \quad (\text{A.2f})$$

$$\mathcal{S}_{212}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[n_2 (s_{11}\beta_{\alpha}^3 + s_{12}\beta_{\alpha}) + n_1 \frac{s_{66}}{2} \beta_{\alpha}^2 - \right. \\ \left. - n_1\beta_{\alpha}^2((s_{12} + \frac{s_{66}}{2}) + s_{11}\beta_{\alpha}^2) \right], \quad (\text{A.2g})$$

$$\mathcal{S}_{222}(x, y) = K\text{Im} \sum_{\alpha=1}^2 \frac{2d_{\alpha}}{\rho_{\alpha}^2} \left[-\frac{n_2}{2} s_{66}\beta_{\alpha}^2 - n_1 (s_{21}\beta_{\alpha}^3 + s_{22}\beta_{\alpha}) - \right. \\ \left. - n_2\beta_{\alpha}^2((s_{21} + \frac{s_{66}}{2}) + s_{11}\beta_{\alpha}^2) \right], \quad (\text{A.2h})$$

where the normal n_{ρ} is taken at the point x .

A.3. Series expansion of the integrand $F_{\kappa j \rho}^m(\xi, \eta)$. The Laurent series of $F_{\kappa j \rho}^m(\xi, \eta)$ in accordance with equation (4.19) is given in the series expansion at a given η as follows:

$$F_{\kappa j \rho}^m(\xi, \eta) = \mathcal{S}_{\kappa \lambda \rho}(\xi, \eta) J(\xi) N_{\lambda j}(\xi) = \underbrace{\left[K\text{Im} \sum_{\alpha=1}^2 \frac{1}{(A_1 + \beta_{\alpha} A_2)^2} {}^0 a_{\kappa \lambda \rho} {}^0 N_{\lambda j}(\eta) \right]}_{-2 F_{\kappa j \rho}^m(\eta)} \delta^{-2} + \\ + \underbrace{\left[\left(K\text{Im} \sum_{\alpha=1}^2 \frac{1}{(A_1 + \beta_{\alpha} A_2)^2} {}^1 a_{\kappa \rho \lambda} - K\text{Im} \sum_{\alpha=1}^2 2 \frac{B_1 + \beta_{\alpha} B_2}{(A_1 + \beta_{\alpha} A_2)^3} {}^0 a_{\kappa \rho \lambda} \right) {}^0 N_{\lambda j}(\eta) + \right.} \\ \left. + K\text{Im} \sum_{\alpha=1}^2 \frac{1}{(A_1 + \beta_{\alpha} A_2)^2} {}^0 a_{\kappa \lambda \rho} {}^1 N_{\lambda j}(\eta) \right]}_{-1 F_{\kappa j \rho}^m(\eta)} \delta^{-1} + \\ + O(1) \quad (\text{A.3a})$$

and without entering into details, the functions are listed below:

$$\begin{aligned}
{}^0a_{111} &= 2d_\alpha (-A_2s_{22} + A_1s_{12}\beta_\alpha - A_2s_{21}\beta_\alpha^2 - A_2s_{66}\beta_\alpha^2 + A_1s_{11}\beta_\alpha^3) , \\
{}^1a_{111} &= -2d_\alpha (-2B_2s_{22} + 2B_1s_{12}\beta_\alpha - 2B_2s_{21}\beta_\alpha^2 - 2B_2s_{66}\beta_\alpha^2 + 2B_1s_{11}\beta_\alpha^3) , \\
{}^0a_{121} &= 2d_\alpha (s_{22}A_1 + s_{22}A_2\beta_\alpha + s_{21}A_1\beta_\alpha^2 + s_{21}A_2\beta_\alpha^3) , \\
{}^1a_{121} &= -2d_\alpha (2s_{22}B_1 + 2s_{22}B_2\beta_\alpha + 2s_{21}B_1\beta_\alpha^2 + 2s_{21}B_2\beta_\alpha^3) , \\
{}^0a_{211} &= 2d_\alpha (-s_{12}A_1 - s_{12}A_2\beta_\alpha - s_{11}A_1\beta_\alpha^2 - s_{11}A_2\beta_\alpha^3) , \\
{}^1a_{211} &= 2d_\alpha (-2s_{12}B_1 - 2s_{12}B_2\beta_\alpha - 2s_{11}B_1\beta_\alpha^2 - 2s_{11}B_2\beta_\alpha^3) , \\
{}^0a_{221} &= 2d_\alpha (-s_{22}A_2 + s_{21}A_1\beta_\alpha + s_{66}A_1\beta_\alpha - s_{21}A_2\beta_\alpha^2 + s_{11}A_1\beta_\alpha^3) , \\
{}^1a_{221} &= 2d_\alpha (-2s_{22}B_2 + 2s_{21}B_1\beta_\alpha + 2s_{66}B_1\beta_\alpha - 2s_{21}B_2\beta_\alpha^2 + 2s_{11}B_1\beta_\alpha^3) , \\
{}^0a_{112} &= 2d_\alpha (-s_{22}A_2\beta_\alpha + s_{12}A_1\beta_\alpha^2 - s_{21}A_2\beta_\alpha^3 - s_{66}A_2\beta_\alpha^3 + s_{11}A_1\beta_\alpha^4) , \\
{}^1a_{112} &= 2d_\alpha (-2s_{22}B_2\beta_\alpha + 2s_{12}B_1\beta_\alpha^2 - 2s_{21}B_2\beta_\alpha^3 - 2s_{66}B_2\beta_\alpha^3 + 2s_{11}B_1\beta_\alpha^4) , \\
{}^0a_{122} &= 2d_\alpha (s_{22}A_1\beta_\alpha + s_{22}A_2\beta_\alpha^2 + s_{21}A_1\beta_\alpha^3 + s_{21}A_2\beta_\alpha^4) , \\
{}^1a_{122} &= 2d_\alpha (2s_{22}B_1\beta_\alpha + 2s_{22}B_2\beta_\alpha^2 + 2s_{21}B_1\beta_\alpha^3 + 2s_{21}B_2\beta_\alpha^4) , \\
{}^0a_{212} &= 2d_\alpha (-s_{12}A_1\beta_\alpha - s_{12}A_2\beta_\alpha^2 - s_{11}A_1\beta_\alpha^3 - s_{11}A_2\beta_\alpha^4) , \\
{}^1a_{212} &= 2d_\alpha (-2s_{12}B_1\beta_\alpha - 2s_{12}B_2\beta_\alpha^2 - 2s_{11}B_1\beta_\alpha^3 - 2s_{11}B_2\beta_\alpha^4) , \\
{}^0a_{222} &= 2d_\alpha (-s_{22}A_2\beta_\alpha + s_{21}A_1\beta_\alpha^2 + s_{66}A_1\beta_\alpha^2 - s_{21}A_2\beta_\alpha^3 + s_{11}A_1\beta_\alpha^4) , \\
{}^1a_{222} &= 2d_\alpha (-2s_{22}B_2\beta_\alpha + 2s_{21}B_1\beta_\alpha^2 + 2s_{66}B_1\beta_\alpha^2 - 2s_{21}B_2\beta_\alpha^3 + 2s_{11}B_1\beta_\alpha^4) .
\end{aligned} \tag{A.3b}$$

A.4. **Series expansion of the integrand** $G_{\kappa j \rho}^m(\xi, \eta)$. The series of $G_{\kappa j \rho}^m(\xi, \eta)$ in accordance with equation (4.26) is given in the series expansion at a given η as follows:

$$\begin{aligned}
G_{\kappa j \rho}^m(\xi, \eta) &= \mathcal{D}_{\kappa \lambda \rho}(\xi, \eta) J(\xi) N_{\lambda j}(\xi) = \\
&= K \text{Im} \underbrace{\sum_{\alpha=1}^2 \frac{1}{(A_1 + \beta_\alpha A_2)} \tilde{a}_{\kappa \lambda \rho} {}^0N_{\lambda j}(\eta) \delta^{-1}}_{-1 G_{\kappa j \rho}^m(\eta)} + O(1) , \tag{A.4a}
\end{aligned}$$

in which

$$\begin{aligned}
\tilde{a}_{111} &= 2d_\alpha \beta_\alpha^2 A , & \tilde{a}_{112} &= 2d_\alpha \beta_\alpha^3 A , & \tilde{a}_{211} &= -2d_\alpha \beta_\alpha A , \\
\tilde{a}_{212} &= -2d_\alpha \beta_\alpha^2 A , & \tilde{a}_{121} &= -2d_\alpha \beta_\alpha A , & \tilde{a}_{122} &= -2d_\alpha \beta_\alpha^2 A , \\
\tilde{a}_{221} &= 2d_\alpha A , & \tilde{a}_{222} &= 2d_\alpha \beta_\alpha A .
\end{aligned} \tag{A.4b}$$