A THEORY FOR THE LONGITUDINAL VIBRATIONS OF ANISOTROPIC ELASTIC BARS BASED ON THE RAYLEIGH-BISHOP MODEL

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Abstract. A Rayleigh-Bishop type theory of the longitudinal vibrations is developed for anisotropic bars. The equation of motion and the associated boundary conditions are derived by the use of Hamilton's principle. For an orthotropic bar the dependence of the eigenfrequencies from the position of the material coordinate system is analyzed. The numerical results obtained by the application of approximate analytical solution for the eigenfrequencies are compared with a FEM solution.

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1. INTRODUCTION

Modeling of vibration problems of elastic structural components is of great importance in engineering. The longitudinal vibration of thin isotropic elastic bar has been studied by many researchers over a long time owing to its wide applications in engineering [1–6]. The classical approximate theory of longitudinal vibration of isotropic bar was developed during the 18th century by J. D'Alembert, D. Bernoulli, L. Euler and J. Lagrange. This theory, which is called the simple theory of longitudinal vibration, is based on the analysis of the one-dimensional wave equation and is applicable for long and relatively thin bars vibrating at low frequency. The lateral effects and the corresponding lateral and axial shear modes are fully neglected in the frame of this simple theory of longitudinal vibration [1, 6]. J. Rayleigh [2] was the first who introduced the inertia of the lateral motions by which the cross sections are extended or contracted in their own planes. The lateral inertia effects are, however, important if the bar is thick. R. Bishop [7] further modified the Rayleigh theory in 1952 by taking into account the lateral shear effects. This theory is referred to as the Rayleigh-Bishop theory. From the point of view of the engineering applications this theory is slightly more complicated than the Rayleigh theory. The simple theory, the Rayleigh theory and Rayleigh-Bishop theory of longitudinal vibrations of isotropic bars are based on a fundamental assumption which says that the dynamics of the bar is described by a single unknown function and hence, a single partial differential equation containing the axial and time coordinates describes the longitudinal vibrations of isotropic

elastic bars. Another important assumption of the above mentioned three theories is that the axial displacements do not depend on the cross-sectional coordinates, they depend only on the axial and time coordinates. The stress boundary conditions on the cylindrical boundary surface of the bar are fully ignored in these theories.

In a paper by Shatalov *et al.* [8] the simple theory, the Rayleigh, Rayleigh-Bishop, Herrmann and multimode models of the longitudinal vibrations of isotropic bars are compared with the exact Pochhammer-Chree solutions of the axisymmetric vibration of an isotropic cylinder. Application of two kinds of eigenfunction orthogonality in the Rayleigh and Rayleigh-Bishop theories of longitudinal vibrations of thick isotropic elastic bars is presented by Fedotov *et al.* [9], which deals with the analytical solution of vibration problem based on Green's function. A paper by Shatalov *et al.* [10] presents the multimode theories of the longitudinal vibration of isotropic elastic bars. The multimode models formulated in [10] can be considered as generalizations of the classical Rayleigh and Rayleigh-Bishop theories.

In this paper an approximate theory of the longitudinal vibration of anisotropic bars is presented. Equations of motion and boundary conditions are derived from Hamilton's principle [6, 11]. In the derived model the inertia of the lateral motions by which the cross-sections are extended or contracted in their own planes is considered. The contribution of more than one stiffness coefficients (elastic moduli) of the general anisotropic material to the strain energy is introduced. From the assumed form of the displacement field a purely longitudinal (axial) vibration can be derived, so that the couplings between the longitudinal, torsional and flexural vibrations cannot be described in this approximate theory.

The connection between the position of material coordinate system, which is given by an angle coordinate, and the eigenfrequencies are analyzed for orthotropic bars. The numerical results obtained for the eigenfrequencies are checked by finite element computations.

2. MECHANICAL MODEL

The considered anisotropic elastic bar with uniform cross section is shown in Figure 1. The governing equations are formulated in the Cartesian coordinate system $Oxyz$. The position of the left end cross section is given by $z = 0$, and the position of the right-end cross section is determined by $z = L$, where L is the length of the bar. The z axis is the centerline of the bar while the x and y axes are the principal axes of the left-end cross section. Hence we have

$$\int_A x \, dA = \int_A y \, dA = 0, \quad \int_A xy \, dA = 0. \quad (2.1)$$

The material of the bar is a general anisotropic material referred to as triclinic material. In this case the generalized Hooke's law can be formulated as [12, 13]

$$\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}, \quad (2.2)$$

where

$$\boldsymbol{\sigma}^T = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{yz} & \tau_{xz} & \tau_{xy} \end{bmatrix}, \quad \boldsymbol{\varepsilon}^T = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{yz} & \gamma_{xz} & \gamma_{xy} \end{bmatrix},$$

Figure 1. Anisotropic elastic bar with uniform cross section

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} = C^T$$

while σ_x , σ_y and σ_z are the normal stresses, τ_{yz} , τ_{xz} and τ_{xy} are the shear stresses, ε_x , ε_y and ε_z are the longitudinal strains, γ_{yz} , γ_{xz} and γ_{xy} are the shear strains, c_{ij} ($i, j = 1, \dots, 6$) are the elastic moduli of the considered anisotropic elastic material. The compliance matrix S is the inverse of stiffness matrix C :

$$\varepsilon = S\sigma, \quad S = C^{-1}, \quad (2.3)$$

where $S = \{s_{ij}\}$ and $s_{ij} = s_{ji}$ ($i, j = 1, \dots, 6$).

Let the anisotropic bar be loaded at its end cross sections by uniform normal stresses, whose stress resultant is the axial force P . The displacement components associated with this axial force can be given in the following forms [12, 13]:

$$u = \frac{P}{A} \left(s_{13}x + \frac{1}{2}s_{36}y \right), \quad (2.4)$$

$$v = \frac{P}{A} \left(\frac{1}{2}s_{36}x + s_{23}y \right), \quad (2.5)$$

$$w = \frac{P}{A} (s_{35}x + s_{34}y + s_{33}z) \quad (2.6)$$

where u , v and w are the displacement components parallel to the axes x , y and z ; A is the area of the bar cross section; s_{13} , s_{23} , s_{33} , s_{34} , s_{35} and s_{36} are the flexibility coefficients. It can be seen by substituting P/A from the derivative of equation (2.6) with respect to z into equations (2.4) and (2.5) that

$$u = \left(\frac{s_{13}}{s_{33}}x + \frac{1}{2}\frac{s_{36}}{s_{33}}y \right) \frac{\partial w}{\partial z}, \quad (2.7)$$

$$v = \left(\frac{1}{2} \frac{s_{36}}{s_{33}} x + \frac{s_{23}}{s_{33}} y \right) \frac{\partial w}{\partial z}. \quad (2.8)$$

Equations (2.7) and (2.8) give the lateral displacements in the anisotropic bar in terms of axial displacement w . In the sequel we shall assume that for longitudinal vibrations the displacements in the directions x and y directions are given by equations (2.7) and (2.8) and the axial displacement w depends only on the axial coordinate z and the time coordinate t . All results of this paper are based on the above-mentioned assumptions. The equation of motion and the boundary conditions are derived from Hamilton's principle, which can be stated as

$$\delta \int_{t_1}^{t_2} (T - \Pi + W) dt = 0, \quad (2.9)$$

where T is the kinetic energy of the bar, Π is the strain energy and the work done by the external force $f = f(z, t)$ is denoted by W (Figure 1). Of all of the possible varied paths only those are considered that coincide with the true path at the two instants t_1 and t_2 [4, 6, 11].

First we determine the kinetic energy starting from the following equation

$$\begin{aligned} T &= \frac{1}{2} \int_B \rho \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right] dB \\ &= \frac{1}{2} \int_B \rho \left\{ \left[\left(\frac{s_{13}}{s_{23}} x + \frac{1}{2} \frac{s_{36}}{s_{33}} y \right)^2 + \left(\frac{1}{2} \frac{s_{36}}{s_{33}} x + \frac{s_{23}}{s_{33}} y \right)^2 \right] \left(\frac{\partial^2 w}{\partial z \partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right\} dB. \end{aligned} \quad (2.10)$$

Here, B is the space domain occupied by the cylindrical bar, ρ is the mass density. A detailed computation, in which equation (2.1) has been used, yields

$$T = \frac{1}{2} \int_0^L \left[A \rho \left(\frac{\partial w}{\partial t} \right)^2 + \rho I \left(\frac{\partial^2 w}{\partial z \partial t} \right)^2 \right] dz, \quad (2.11)$$

where

$$\begin{aligned} I &= \left[\left(\frac{s_{13}}{s_{33}} \right)^2 + \frac{1}{4} \left(\frac{s_{36}}{s_{33}} \right)^2 \right] I_y + \left[\frac{1}{4} \left(\frac{s_{36}}{s_{33}} \right)^2 + \left(\frac{s_{23}}{s_{33}} \right)^2 \right] I_x \\ &= \left\{ \left[\left(\frac{s_{13}}{s_{33}} \right)^2 + \frac{1}{4} \left(\frac{s_{36}}{s_{33}} \right)^2 \right] i_y^2 + \left[\frac{1}{4} \left(\frac{s_{36}}{s_{33}} \right)^2 + \left(\frac{s_{23}}{s_{33}} \right)^2 \right] i_x^2 \right\} A = i^2 A \end{aligned} \quad (2.12)$$

$$I_x = \int_A y^2 dA = i_x^2 A, \quad I_y = \int_A x^2 dA = i_y^2 A. \quad (2.13)$$

For an anisotropic linearly elastic material the strain energy density as a function of strain field is obtained as [12, 13]

$$\begin{aligned}
a(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}) = & \frac{1}{2} (c_{11}\varepsilon_x^2 + 2c_{12}\varepsilon_x\varepsilon_y + 2c_{13}\varepsilon_x\varepsilon_z + 2c_{14}\varepsilon_x\gamma_{yz} + \\
& 2c_{15}\varepsilon_x\gamma_{xz} + 2c_{16}\varepsilon_x\gamma_{xy} + c_{22}\varepsilon_y^2 + 2c_{23}\varepsilon_y\varepsilon_z + 2c_{24}\varepsilon_y\gamma_{yz} + 2c_{25}\varepsilon_y\gamma_{xz} + 2c_{26}\varepsilon_y\gamma_{xy} \\
& + c_{33}\varepsilon_z^2 + 2c_{34}\varepsilon_z\gamma_{yz} + 2c_{35}\varepsilon_z\gamma_{xz} + 2c_{36}\varepsilon_z\gamma_{xy} \\
& + c_{44}\gamma_{yz}^2 + 2c_{45}\gamma_{yz}\gamma_{xz} + 2c_{46}\gamma_{yz}\gamma_{xy} + 2c_{55}\gamma_{xz}^2 + 2c_{56}\gamma_{xz}\gamma_{xy} + c_{66}\gamma_{xy}^2), \quad (2.14)
\end{aligned}$$

in which

$$\begin{aligned}
\varepsilon_x &= \frac{s_{13}}{s_{33}} \frac{\partial w}{\partial z}, & \varepsilon_y &= \frac{s_{23}}{s_{33}} \frac{\partial w}{\partial z}, & \varepsilon_z &= \frac{\partial w}{\partial z}, \quad (2.15) \\
\gamma_{yz} &= \left(\frac{1}{2} \frac{s_{36}}{s_{33}} x + \frac{s_{23}}{s_{33}} y \right) \frac{\partial^2 w}{\partial z^2}, & \gamma_{xz} &= \left(\frac{s_{13}}{s_{33}} x + \frac{1}{2} \frac{s_{36}}{s_{33}} y \right) \frac{\partial^2 w}{\partial z^2}, & \gamma_{xy} &= \frac{s_{36}}{s_{33}} \frac{\partial w}{\partial z}. \quad (2.16)
\end{aligned}$$

The whole strain energy of the anisotropic cylindrical bar is calculated from the following equation

$$\Pi = \int_B a(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}) \, dB. \quad (2.17)$$

A detailed calculation in which equation (2.1) is also used leads to the result

$$\Pi = \frac{1}{2} \int_0^L \left[C_1 \left(\frac{\partial w}{\partial z} \right)^2 + C_2 \left(\frac{\partial^2 w}{\partial z^2} \right)^2 \right] dz, \quad (2.18)$$

where

$$\begin{aligned}
C_1 &= \left[c_{11} \left(\frac{s_{13}}{s_{33}} \right)^2 + c_{22} \left(\frac{s_{23}}{s_{33}} \right)^2 + c_{33} + c_{66} \left(\frac{s_{36}}{s_{33}} \right)^2 \right. \\
&\quad \left. + 2c_{12} \frac{s_{13}s_{23}}{s_{33}^2} + 2c_{13} \frac{s_{13}}{s_{33}} + 2c_{23} \frac{s_{23}}{s_{33}} + 2c_{16} \frac{s_{13}s_{36}}{s_{33}^2} + 2c_{26} \frac{s_{23}s_{36}}{s_{33}^2} + 2c_{36} \frac{s_{36}}{s_{33}} \right] A \\
&= \left[\frac{c_{11}}{c_{33}} \left(\frac{s_{13}}{s_{33}} \right)^2 + \frac{c_{22}}{c_{33}} \left(\frac{s_{23}}{s_{33}} \right)^2 + 1 + \frac{c_{66}}{c_{33}} \left(\frac{s_{36}}{s_{33}} \right)^2 + 2 \frac{c_{12}}{c_{33}} \frac{s_{13}s_{23}}{s_{33}^2} + \right. \\
&\quad \left. 2 \frac{c_{13}}{c_{33}} \frac{s_{13}}{s_{33}} + 2 \frac{c_{23}}{c_{33}} \frac{s_{23}}{s_{33}} + 2 \frac{c_{16}}{c_{33}} \frac{s_{13}s_{36}}{s_{33}^2} + 2 \frac{c_{26}}{c_{33}} \frac{s_{23}s_{36}}{s_{33}^2} + 2 \frac{c_{36}}{c_{33}} \frac{s_{36}}{s_{33}} \right] c_{33} A = c_1 c_{33} A, \quad (2.19)
\end{aligned}$$

$$\begin{aligned}
C_2 &= \left[\frac{1}{4} c_{44} \left(\frac{s_{36}}{s_{33}} \right)^2 + c_{45} \frac{s_{13}s_{36}}{s_{33}^2} + c_{55} \left(\frac{s_{13}}{s_{33}} \right)^2 \right] I_y \\
&\quad + \left[c_{44} \left(\frac{s_{23}}{s_{33}} \right)^2 + c_{45} \frac{s_{23}s_{36}}{s_{33}^2} + c_{55} \frac{1}{4} \left(\frac{s_{36}}{s_{33}} \right)^2 \right] I_x \\
&= \left\{ \frac{1}{4} \frac{c_{44}}{c_{33}} \left(\frac{s_{36}}{s_{33}} \right)^2 + \frac{c_{45}}{c_{33}} \frac{s_{13}s_{36}}{s_{33}^2} + \frac{c_{55}}{c_{33}} \left(\frac{s_{13}}{s_{33}} \right)^2 \right\} i_y^2 \\
&\quad + \left\{ \frac{c_{44}}{c_{33}} \left(\frac{s_{23}}{s_{33}} \right)^2 + \frac{c_{45}}{c_{33}} \frac{s_{23}s_{36}}{s_{33}^2} + \frac{1}{4} \frac{c_{55}}{c_{33}} \left(\frac{s_{36}}{s_{33}} \right)^2 \right\} i_x^2 \Big\} c_{33} A = c_2 c_{33} A. \quad (2.20)
\end{aligned}$$

The work W done by the external force $f = f(z, t)$ on the axial displacement $w = w(z, t)$ is as follows

$$W = \int_0^L f(z, t) w(z, t) dz. \quad (2.21)$$

The substitution of equations (2.11), (2.18) and (2.21) into equation (2.9) yields the equation of motion and the associated boundary conditions:

$$C_2 \frac{\partial^4 w}{\partial z^4} - \rho I \frac{\partial^4 w}{\partial z^2 \partial t^2} - C_1 \frac{\partial^2 w}{\partial z^2} + \rho A \frac{\partial^2 w}{\partial t^2} = f, \quad (2.22)$$

$$\left[C_1 \frac{\partial w}{\partial z} + \rho I \frac{\partial^3 w}{\partial z \partial t^2} - C_2 \frac{\partial^3 w}{\partial z^3} \right] \delta w \Big|_0^L + \left(C_2 \frac{\partial^2 w}{\partial z^2} \right) \delta \left(\frac{\partial w}{\partial z} \right) \Big|_0^L = 0. \quad (2.23)$$

It can be seen from equations (2.7) and (2.8) that

$$w = 0, \quad \frac{\partial w}{\partial z} = 0 \quad (2.24)$$

on an end cross section if it is rigidly fixed. Hence it holds there that

$$\delta w = 0, \quad \delta \left(\frac{\partial w}{\partial z} \right) = 0. \quad (2.25)$$

Since δw , and $\delta \left(\frac{\partial w}{\partial z} \right)$ are arbitrary on free end cross sections it follows that the stress boundary conditions in terms of $w = w(z, t)$ on these cross sections assume the form

$$C_1 \frac{\partial w}{\partial z} + \rho I \frac{\partial^3 w}{\partial z \partial t^2} - C_2 \frac{\partial^3 w}{\partial z^3} = 0, \quad (2.26)$$

$$C_2 \frac{\partial^2 w}{\partial z^2} = 0. \quad (2.27)$$

3. FREE VIBRATIONS

For free vibrations

$$w(z, t) = W(z) \cos \omega t, \quad f = 0 \quad (3.1)$$

is the solution of (2.22) in which $W(z)$ is the amplitude of motion and ω is the circular frequency of the vibrations. Substitution of solution (3.1) into equation (2.22) yields

$$C_2 \frac{d^4 W}{dz^4} + (\omega^2 \rho I - C_1) \frac{d^2 W}{dz^2} - \omega^2 \rho A W = 0. \quad (3.2)$$

The homogeneous boundary conditions equation (3.2) is associated with are

$$W = 0, \quad \frac{dW}{dz} = 0 \quad (3.3)$$

for fixed-end cross sections,

$$C_1 \frac{dW}{dz} - \omega^2 \rho I \frac{dW}{dz} - C_2 \frac{d^3 W}{dz^3} = 0, \quad \frac{d^2 W}{dz^2} = 0 \quad (3.4)$$

for free-end cross section and

$$W = 0, \quad \frac{d^2 W}{dz^2} = 0 \quad (3.5)$$

for loosely fixed end cross sections [6]

The general solution of equation (3.2), which contains four integration constants, can be written in the form [14, 15]

$$W(z) = K_1 \cosh qz + K_2 \sinh qz + K_3 \cos rz + K_4 \sin rz, \quad (3.6)$$

where the constants K_1 , K_2 , K_3 and K_4 to be determined from the boundary conditions of the anisotropic bar are such that one of them may have arbitrary value. As regards the parameters q and r we have

$$q = \sqrt{\frac{C_1 - \omega^2 \rho I + \sqrt{(C_1 - \omega^2 \rho I)^2 + 4\omega^2 C_2 \rho A}}{2C_2}}, \quad (3.7)$$

$$r = \sqrt{\frac{\sqrt{(C_1 - \omega^2 \rho I)^2 + 4\omega^2 C_2 \rho A} - C_1 + \omega^2 \rho I}{2C_2}}. \quad (3.8)$$

As regards the free bending vibrations of an isotropic Euler-Bernoulli-Rayleigh beam subjected to an axial force, the differential equation and the boundary conditions associated with it are similar to those which describe the longitudinal free vibrations of the anisotropic bar considered in this paper. The differential equation and associated boundary condition for the bending vibrations of an isotropic Euler-Bernoulli beam with constant axial force are as follows [16]:

- differential equation:

$$EI_x \frac{d^4 V}{dz^4} + (\omega^2 \rho J - P) \frac{d^2 V}{dz^2} - \omega^2 \rho A V = 0, \quad (3.9)$$

- boundary conditions for fixed-end cross sections:

$$V = 0, \quad \frac{dV}{dz} = 0, \quad (3.10)$$

- boundary conditions for free-end cross sections:

$$P \frac{dV}{dz} - EI_x \frac{d^3 V}{dz^3} - \rho \omega^2 J \frac{dV}{dz} = 0, \quad \frac{d^2 V}{dz^2} = 0, \quad (3.11)$$

- boundary conditions for simply supported end cross sections:

$$V = 0, \quad \frac{d^2 V}{dz^2} = 0. \quad (3.12)$$

In equations (3.9)-(3.12) $V = V(z)$ is the amplitude of the displacement of center-line in the direction of axes y , E is the Young modulus, $J = I_x + I_y$, $P > 0$ is the applied axial force. The form of equations (3.9)-(3.12) makes it possible to establish an analogy between the longitudinal vibration of anisotropic bar and the bending vibration of the isotropic Euler-Bernoulli-Rayleigh beam subjected to constant positive axial force. The analogy is shown in Table 1.

Table 1. Analogous quantities and boundary conditions.

Longitudinal vibrations of anisotropic bars	Bending vibrations of isotropic beams
$W = W(z)$ C_2 C_1 I	$V = V(z)$ EI_x $P > 0$ J
Boundary conditions	Boundary conditions
fixed end: $W = 0, \frac{dW}{dz} = 0$ free end: $(C_1 - \omega^2 \rho I) \frac{dW}{dz} - C_2 \frac{d^3 W}{dz^3} = 0, \frac{d^2 W}{dz^2} = 0$ loosely fixed end: $W = 0, \frac{d^2 W}{dz^2} = 0$	fixed end: $V = 0, \frac{dV}{dz} = 0$ free end: $(P - \omega^2 \rho J) \frac{dV}{dz} - EI_x \frac{d^3 V}{dz^3} = 0, \frac{d^2 V}{dz^2} = 0$ simply supported end: $V = 0, \frac{d^2 V}{dz^2} = 0$

4. EXAMPLE

Assume that the anisotropic bar is fixed loosely at both ends. Then the axial displacement and shear strains will be zero at each end of the bar. Hence

$$W = 0, \quad \frac{d^2 W}{dz^2} = 0, \quad \text{at } z = 0 \quad \text{and} \quad z = L. \quad (4.1)$$

It is obvious from taking the analogy formulated in Table 1 into account that the boundary conditions related to the loosely fixed end condition of the anisotropic bar are similar to those valid for the simply supported end conditions of the vibrating Euler-Bernoulli-Rayleigh beam. On the basis of this similarity we shall assume

$$W_j(z) = W_j \sin j \frac{\pi}{L} z, \quad j = 1, 2, \dots \quad (4.2)$$

The function given by equation (4.2) satisfies the boundary conditions for any value of W_j ($j = 1, 2, \dots$)

$$W(0) = W(L) = 0, \quad \frac{d^2 W}{dz^2} \Big|_{z=0} = \frac{d^2 W}{dz^2} \Big|_{z=L} = 0. \quad (4.3)$$

From equation (3.2) it follows that the natural frequencies of the longitudinal vibration of anisotropic bar fixed loosely at both ends are expressed as

$$\omega_j^2 = \frac{C_2 \left(\frac{j\pi}{L}\right)^4 + C_1 \left(\frac{j\pi}{L}\right)^2}{\rho \left(\frac{j\pi}{L}\right)^2 I + A\rho} = \frac{c_2 \left(\frac{j\pi}{L}\right)^2 + c_1}{i^2 \left(\frac{j\pi}{L}\right)^2 + 1} \Omega_j^2, \quad \Omega_j = \sqrt{\frac{c_{33}}{\rho}} \frac{j\pi}{L} \quad j = 1, 2, \dots \quad (4.4)$$

In equation (4.4) Ω_j ($j = 1, 2, 3, \dots$) is the natural frequency of the longitudinal vibration of the considered bar with fixed ends, which can be derived by the application of the following assumptions (simple theory):

- The lateral displacement u and v are neglected and $w = w(z, t)$.
- The dominant element of the stiffness matrix $\mathbf{C}$ is c_{33} , so the expression of axial force N can be written in the form

$$N = A c_{33} \frac{\partial w}{\partial z}. \quad (4.5)$$

This formulation does not introduce the effect of lateral motions of cross sections and the material anisotropy. The eigenfunction (mode shape) corresponding to the frequency ω_j is given by equation (4.2).

5. DEPENDENCE OF THE NATURAL FREQUENCIES ON THE POSITION OF THE PRINCIPAL MATERIAL COORDINATE SYSTEM

We consider an orthotropic material whose principal directions are x_1 , x_2 and $x_3 = z$ [12, 13, 17, 18]. The position of the material coordinate system $Ox_1x_2x_3$ is given by angle α as shown in Figure 2 [17, 18].

Figure 2. Definition of α

For orthotropic elastic material in terms of engineering constants E_1 , E_2 , E_3 , G_{23} , G_{13} , G_{12} , ν_{12} , ν_{21} , ν_{13} , ν_{31} , ν_{23} , ν_{32} the stiffness matrix $\mathbf{C}(x_1, x_2, x_3)$ can be written in the form [17, 18]

$$\mathbf{C}(x_1, x_2, x_3) = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2 E_3 \Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2 E_3 \Delta} & 0 & 0 & 0 \\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2 E_3 \Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1 E_3 \Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1 E_3 \Delta} & 0 & 0 & 0 \\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1 E_3 \Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1 E_2 \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix}, \quad (5.1)$$

where

$$E_1 \nu_{21} = E_2 \nu_{12}, \quad E_2 \nu_{32} = E_3 \nu_{23}, \quad E_1 \nu_{31} = E_3 \nu_{13}, \quad (5.2)$$

and

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3}. \quad (5.3)$$

The inverse of $\mathbf{C}(x_1, x_2, x_3)$ gives the flexibility matrix $\mathbf{S}(x_1, x_2, x_3)$ defined in the principal material coordinate system $Ox_1x_2x_3$ [17, 18]:

$$\mathbf{S}(x_1, x_2, x_3) = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}. \quad (5.4)$$

By the use of transformation stiffness and compliance matrices [17, 18] we can derive the formulas of c_{ij} in terms of C_{ij} and the formulas of s_{ij} in terms of S_{ij} :

$$c_{11} = m^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{22}, \quad (5.5)$$

$$c_{12} = m^2 n^2 (C_{11} + C_{22} - 4C_{66}) + (m^4 + n^4) C_{12}, \quad (5.6)$$

$$c_{13} = m^2 C_{13} + n^2 C_{23}, \quad (5.7)$$

$$c_{16} = mn [m^2 (C_{11} - C_{12} - 2C_{66}) + n^2 (C_{12} - C_{22} + 2C_{66})], \quad (5.8)$$

$$c_{22} = n^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + m^4 C_{22}, \quad (5.9)$$

$$c_{23} = n^2 C_{13} + m^2 C_{23}, \quad (5.10)$$

$$c_{26} = mn [n^2 (C_{11} - C_{12} - 2C_{66}) + m^2 (C_{12} - C_{22} + 2C_{66})], \quad (5.11)$$

$$c_{33} = C_{33}, \quad (5.12)$$

$$c_{36} = mn (C_{13} - C_{23}), \quad (5.13)$$

$$c_{44} = m^2 C_{44} + n^2 C_{55}, \quad (5.14)$$

$$c_{45} = mn (C_{55} - C_{44}), \quad (5.15)$$

$$c_{55} = n^2 C_{44} + m^2 C_{55}, \quad (5.16)$$

$$c_{66} = m^2 n^2 (C_{11} - 2C_{12} + C_{22}) + (n^2 - m^2)^2 C_{66} \quad (5.17)$$

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0, \quad (5.18)$$

$$s_{11} = m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^2 S_{22}, \quad (5.19)$$

$$s_{12} = m^2 n^2 (S_{11} + S_{22} - S_{66}) + (m^4 + n^4) S_{12}, \quad (5.20)$$

$$s_{13} = m^2 S_{13} + n^2 S_{23}, \quad (5.21)$$

$$s_{16} = mn [m^2 (2S_{11} - 2S_{12} - S_{66}) + n^2 (2S_{12} - 2S_{22} + S_{66})], \quad (5.22)$$

$$s_{22} = n^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + m^4 S_{22}, \quad (5.23)$$

$$s_{23} = n^2 S_{13} + m^2 S_{23}, \quad (5.24)$$

$$s_{26} = mn [n^2 (2S_{11} - 2S_{12} - S_{66}) + m^2 (2S_{12} - 2S_{22} + S_{66})], \quad (5.25)$$

$$s_{33} = S_{33}, \quad (5.26)$$

$$s_{36} = 2mn (S_{13} - S_{23}), \quad (5.27)$$

$$s_{44} = m^2 S_{44} + n^2 S_{55}, \quad (5.28)$$

$$s_{45} = mn (S_{55} - S_{44}), \quad (5.29)$$

$$s_{55} = n^2 S_{44} + m^2 S_{55}, \quad (5.30)$$

$$s_{66} = 4m^2 n^2 (S_{11} - 2S_{12} + S_{22}) + (n^2 - m^2) S_{66}, \quad (5.31)$$

$$s_{14} = s_{15} = s_{24} = s_{25} = s_{34} = s_{35} = s_{46} = s_{56} = 0, \quad (5.32)$$

$$m = \cos \alpha, \quad n = \sin \alpha. \quad (5.33)$$

The dependence of the natural frequencies on α given by equation (4.4) for an orthotropic bar of rectangular cross section is analyzed by using the following data (Figure 3):

$$b_1 = 0.03 \text{ m}, \quad b_2 = 0.05 \text{ m}, \quad L = 1 \text{ m}, \quad \rho = 550 \frac{\text{kg}}{\text{m}^3},$$

$$E_1 = 11 \cdot 10^8 \text{ Pa}, \quad E_2 = 5.7 \cdot 10^8 \text{ Pa}, \quad E_3 = 163 \cdot 10^8 \text{ Pa},$$

$$G_{23} = 11.06 \cdot 10^8 \text{ Pa}, \quad G_{13} = 6.8 \cdot 10^8 \text{ Pa}, \quad G_{12} = 0.66 \text{ Pa},$$

$$\nu_{12} = 0.68, \quad \nu_{13} = 0.038, \quad \nu_{23} = 0.015.$$

Figure 3. Solid rectangular cross section

In the present problem $\Omega_1 = 17102.61/\text{s}$, $\Omega_2 = 34205.21/\text{s}$, $\Omega_3 = 51307.81/\text{s}$, $\Omega_4 = 68410.41/\text{s}$, which do not depend on the size or the shape of the cross section. The graphs of $\omega_j(\alpha)$ and $\omega_j(\alpha)/\Omega_j$ for $j = 1, 2, 3, 4$ and $0 \leq \alpha \leq \pi$ are shown in Figure 4.

Figure 4. α dependence of the natural frequencies for $j = 1, 2, 3, 4$

In Table 2 the α dependence of the fundamental frequency obtained from equation (4.4) is compared to the results of FEM computations for $\alpha = 0, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

Table 2. Comparison of analytical and FEM (Abaqus) solutions

$\omega_1(\alpha) \left[\frac{\text{rad}}{\text{s}} \right]$		
α	Analytical Solution	FEM Computation
0	17 007.13	17 094
$\pi/8$	17 026.93	17 094
$\pi/6$	17 044.19	17 094
$\pi/4$	17 096.92	17 093
$\pi/3$	17 165.18	17 093
$\pi/2$	17 248.79	17 092

The results of analytical solution and FEM computations for the first four natural frequencies for $\alpha = 0$ are listed in Table 3.

Table 3. Comparison of analytical and FEM (Abaqus) solutions for $\alpha = 0$

$\omega_j(0) \left[\frac{\text{rad}}{\text{s}} \right]$		
j	Analytical Solution	FEM Computation
1	17 007.13	17 094
2	33 988.48	34 130
3	50 904.12	50 912
4	67 736.28	68 191

The results obtained from FEM computations support the validity of the presented approximate analytical method.

6. CONCLUSIONS

In this paper an approximate theory of the longitudinal vibration of anisotropic elastic bars is presented. The equation of motion and the associated boundary conditions are derived by the application of Hamilton's principle. The developed theory is a Rayleigh-Bishop type model of the longitudinal vibration. An analogy is formulated between the longitudinal vibration of an anisotropic bar and the bending vibration of an isotropic Euler-Bernoulli-Rayleigh beam subjected to constant axial force.

For an orthotropic elastic bar the connection between the position of material coordinate system, which is given by an angle coordinate, and the eigenfrequencies is

analyzed. Obtained numerical results for some values of eigenfrequencies by the application of the approximate analytical method are checked by FEM (Abaqus) computations. Presented numerical results show that the eigenfrequencies of the longitudinal free vibrations slightly depend on the position of principal axes of orthotropy.

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