Solving One Dimensional Ground Water Recharge Through Porous Media Using Elzaki Transform

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ABSTRACT

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In this paper, we have discussed the application of Elzaki Transform for finding the solution of One-Dimensional Ground Water Recharge through porous media. In this work, we present a reliable combination of Elzaki transform and Adomian Decomposition method. The proposed method introduces Adomian polynomials and the nonlinear terms can be handled by the use of this polynomials easily. The Elzaki Decomposition Method is used to solve the particular problem. The purpose of this method is to extend the application of Elzaki Decomposition Method. The proposed method worked perfectly to find the One-Dimensional Ground Water Recharge through porous media problem. We obtain numerical solution and graphical representation.

Keywords : Elzaki Transform, Decomposition Method, Porous Media.

I. INTRODUCTION

The nonlinear partial differential equations appear in many applications of sciences and engineering such as, fluid dynamics, plasma physics, hydrodynamics, soil state physics and other disciplines. The ground water flow plays an important role in fluid mechanics, ground water hydrology and other fields of fluids. Such problems are also useful to measure moisture content of water in One-Dimensional Ground Water Recharge and dispersion of any fluid in porous media.

The mathematical formulation of the phenomena leads to governing equation, which is nonlinear partial differential equation. This type of equations can be solved by Elzaki Transform [1] or combined methods with Elzaki Transform [2,3,4,5]. This phenomenon has been discussed by many researches taking different viewpoints. Here are some of them, One-Dimensional Ground Water Recharge through porous media is solved by RDTM by P.S. Patel [9], Elzaki transform method is used to solve One-Dimensional Ground Water Recharge through

spreading in [10], Priti A. Tandel and P.H. Bhathawala [11] obtained solution using Crank Nikolson method. Kunjn Shah and Twinkle Singh [12] have studied the problem using q- Homotopy Analysis method. Verma [13] has used Laplace transform technique. A.K. Parikh, M.N. Mehta, V.H. Pradhan [14] have obtained transcendental solution of Ground Water Recharge problem.

Elzaki Transform is incapable to handle nonlinear equations easily because of the difficulties that are caused by the nonlinear terms. Various ways have been proposed to deal with nonlinear terms, one of these combinations is Elzaki decomposition Method, which is studied in this paper.

II. METHODOLOGY

2.1 Elzaki Transform

A New integral transform called Elzaki transform defined for functions of exponential order is proclaimed. We consider functions in the set A defined by

$$A = \left\{ f(t) : \exists M, K_1, K_2 > 0, |f(t)| < M e^{\frac{|t|}{K_1}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

Definition: If $f(t)$ is function defined for all $t \geq 0$, its Elzaki transform is the integral of $f(t)$ times $e^{-\frac{t}{v}}$ from $t = 0$ to ∞ . It is a function of v and is defined as

$$E[f(t)] = T(v) = v \int_0^\infty f(t) e^{-\frac{t}{v}} dt, \quad v \in (K_1, K_2)$$

$$\text{Or equivalently} \quad T(v) = v^2 \int_0^\infty f(vt) e^{-t} dt, \quad K_1, K_2 > 0$$

Properties of Elzaki transform

1. $E(1) = v^2$
2. $E(t^n) = n! v^{n+2}$
3. $E(t) = v^3$
4. $E^{-1}(v^{n+2}) = \frac{t^n}{n!}$
5. $E[u^{(m)}(t)] = \frac{1}{v^m} T(x, v) - \sum_{k=0}^{m-1} v^{2-m-k} u^{(k)}(0), \quad m \geq 1$

2.2. Elzaki Decomposition Method

Consider a general nonlinear differential equation

$$LS_w(x, T) = NS_w(x, T) + g(x, T) \quad \dots\dots(i)$$

Where, L is the 1st order linear differential operator. N is the nonlinear differential operator, S_w is the dependent variable, x and T are independent variables, $g(x, T)$ is the source term.

Apply Elzaki Transform to the equation (i)

$$E(LS_w(x, T)) = E(NS_w(x, T)) + E(g(x, T))$$

Using differentiation property of the Elzaki Transform, we have

$$\frac{E(S_w(x, T))}{v} - vS_{w0}(x, T) = E(NS_w(x, T)) + E(g(x, T))$$

$$E(S_w(x, T)) = v^2 S_{w0}(x, T) + vE(NS_w(x, T)) + vE(g(x, T)) \quad \dots\dots(ii)$$

Using inverse Elzaki Transform on both side of the equation (ii), we get

$$S_w(x, T) = G(x, T) + E^{-1}[vE(NS_w(x, T))] \quad \dots\dots(iii)$$

Where, $G(x, T)$ represents the term arising from the source term and the prescribed initial conditions.

The representation of the solution (iii) as an infinite series is given below,

$$S_w(x, T) = \sum_{n=0}^{\infty} S_{wn}(x, T) \quad \text{.....(iv)}$$

The non-linear term has been decomposed as:

$$N S_w(x, T) = A_n(S_{w0}, S_{w1}, S_{w2}, \dots, S_{wn}) \quad \text{.....(v)}$$

Where A_n are the Adomian polynomials of functions $S_{w0}, S_{w1}, S_{w2}, \dots, S_{wn}$ and can be calculated by formula given as:

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [\sum_{k=0}^{\infty} \lambda^k S_{wk}]_{\lambda=0} \quad n = 0, 1, 2, \dots$$

Substituting (iv) and (v) into (iii),

$$S_{wn}(x, T) = G(x, T) + E^{-1}[v E(A_n)]$$

Where, $S_{w0}(x, T) = G(x, T)$

$$S_{wn+1}(x, T) = E^{-1}[v E(A_n)] \quad n = 0, 1, 2, \dots \quad \text{.....(vi)}$$

Where, the Elzaki Transform and the inverse Elzaki Transform are applied on (vi) respectively, the iteration $S_{w0}, S_{w1}, S_{w2}, \dots, S_{wn}$ were obtained, which in turn gave the general solution as,

$$S_w(x, T) = S_{w0}(x, T) + S_{w1}(x, T) + S_{w2}(x, T) + \dots$$

III. Problem: One Dimensional Ground Water Recharge Through Porous Media with Parabolic Permeability

3.1 Statement of the Problem

In the investigated mathematical model, we consider that the groundwater recharge takes place over a large basin of such geological location that sides are limited by rigid boundaries and the bottom by a thick layer of water table. In this case, take flow is assumed vertically downwards through unsaturated porous media. It is assumed that diffusivity coefficient is equivalent to its average value over the whole range of moisture content, and the permeability of the media is continuous linear function of the problem yield a nonlinear partial differential equation for the moisture content.

3.2 Formulation of the Problem

We may write fundamental equation as below. The equation of continuity for unsaturated medium is given by

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla M$$

Where ρ_s is bulk density of the medium, θ is the moisture content on a dry weight basis, and M is the mass flux of moisture.

From Darcy's law for the motion of water in a porous media we get,

$$V = -k \nabla \phi$$

Where $\nabla \phi$ represent the gradient of whole moisture potential, V the volume flux of moisture potential, k the coefficient of aqueous conductivity. Combining above two equations we obtain,

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla(\rho k \nabla \phi)$$

Where, ρ is the fluid density. Since in the present case we consider that the flow takes place only in the vertical direction, equation (3) reduces to,

$$\rho_x \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(\rho k \frac{\partial \phi}{\partial z} \right) - \frac{\partial}{\partial z}(\rho k g)$$

Where ϕ is the capillary pressure potential, g is the gravitational constant and $\phi = \phi - gz$.

The positive direction of the z -axis is the same as that of the gravity.

Considering θ and ϕ to be conducted by single valued function, we may write (4) as,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial \theta}{\partial z} \right) - \frac{\rho}{\rho_s} g \frac{\partial k}{\partial z}$$

Where $D = \frac{\rho}{\rho_s} k \frac{\partial \phi}{\partial \theta}$ and is called diffusivity coefficient.

Replacing D by its average value D_a and assuming $k = k_0 \theta$, $k_0 = 0.232$, we have

$$\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_0 \frac{\partial \theta}{\partial z}$$

Considering water table to be situated at a depth L , and putting,

$$\frac{z}{L} = \xi, \quad \frac{tD_a}{L^2} = T, \quad \beta_0 = \frac{\rho}{\rho_s} k_0 \frac{\partial \theta}{\partial z}$$

One-dimensional Ground water recharge through porous media with linear permeability is,

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta_0 \frac{\partial \theta}{\partial \xi}$$

Where ξ = penetration depth (dimensionless)

T = time (dimensionless)

β_0 = flow parameter (cm^2)

Set of appropriate boundary conditions are $\theta(0, T) = \theta_0$, $\theta(1, T) = 1$, $\theta(\xi, 0) = 0$

But here we assumed, the permeability of the moisture content to have a parabolic distribution, i.e. $k = k_i + k_0 \theta^2$ ($k_0 = 0.232$) where k_i and k_0 are constants. Then equation (5) becomes,

$$\frac{\partial \theta}{\partial t} = D_a \frac{\partial^2 \theta}{\partial z^2} - \frac{\rho}{\rho_s} k_0 \frac{\partial \theta}{\partial z}$$

Considering the water table to be situated at a depth L , and put,

$$\frac{z}{L} = \xi, \quad \frac{tD_a}{L^2} = T, \quad \beta_0 = \frac{\rho}{\rho_s} \frac{k_0}{D_a}$$

We may write the problem as below from the boundary value

$$\frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} - \beta_0 \theta \frac{\partial \theta}{\partial \xi}$$

It may be mentioned for definiteness that a set of appropriate boundary conditions are

$$\theta(0, T) = \theta_0, \quad \theta(1, T) = 1, \quad \theta(\xi, 0) = 0$$

Where the moisture content throughout the region is zero initially, at the layer $z=0$ it is θ_0 and at the water table $z=L$ it is assumed to remain 100% throughout the process of investigation. It may be remarked that the effect of capillary action at the stationary ground water level, being small is neglected.

We can write equation (7) as $\theta_T = \theta_{\xi\xi} - \beta_0 \theta \theta_\xi$

This problem is solved by Elzaki Decomposition Method and numerical solution is obtained. The graph indicates the moisture content corresponding to various time periods.

3.3 Solution using Elzaki Decomposition Method

$$\theta_T = \theta_{\xi\xi} - \beta_0 \theta \theta_\xi \quad (1)$$

$$\text{Taking initial condition } \theta(\xi, 0) = \theta_0 = f(\xi) = \frac{e^\xi - 1}{e - 1} \quad (2)$$

This equation is nonlinear differential equation, which is solved by Elzaki Decomposition Method.

Apply Elzaki transform on (1)

$$E(\theta) = v^2 \theta_0 + vE[\theta_{\xi\xi} - \beta_0 \theta \theta_\xi]$$

Now, applying inverse Elzaki transform,

$$\begin{aligned} \theta &= \theta_0 + E^{-1}[vE(\theta_{\xi\xi} - \beta_0 \theta \theta_\xi)] \\ \therefore \theta &= \theta_0 + E^{-1}[vE(\phi_1 - \beta_0 \phi_2)] \end{aligned}$$

Where $\phi_1 = \sum_{n=0}^{\infty} A_n$ and $\phi_2 = \sum_{n=0}^{\infty} B_n$

$$\therefore \theta = \theta_0 + E^{-1}[vE(\sum_{n=0}^{\infty} (A_n - \beta_0 B_n))] \quad (3)$$

The Adomian polynomials A_n and B_n are calculated as

$$A_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [\phi_1(\sum_{k=0}^{\infty} \lambda^k \theta_k)]_{\lambda=0} \quad n = 0, 1, 2, \dots$$

$$B_n = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [\phi_2(\sum_{k=0}^{\infty} \lambda^k \theta_k)]_{\lambda=0} \quad n = 0, 1, 2, \dots$$

The first three components of these polynomials are

$$A_0 = (\theta_0)_{\xi\xi}$$

$$A_1 = (\theta_1)_{\xi\xi}$$

$$A_2 = (\theta_2)_{\xi\xi}$$

$$B_0 = \theta_0(\theta_0)_{\xi}$$

$$B_1 = \theta_0(\theta_1)_{\xi} + \theta_1(\theta_0)_{\xi}$$

$$B_2 = \theta_0(\theta_2)_{\xi} + \theta_1(\theta_1)_{\xi} + \theta_2(\theta_0)_{\xi}$$

Other polynomials can be generated in like this manner, substituting this decomposition series in (3) and taking $\beta_0 = 1$

$$\begin{aligned} \theta_0 &= \theta_0 = \frac{e^{\xi}-1}{e-1} \\ \theta_1 &= E^{-1}[vE(A_0 - B_0)] \\ &= E^{-1}[vE((\theta_0)_{\xi\xi} - \theta_0(\theta_0)_{\xi})] \\ &= \frac{e^{\xi+1}-e^{2\xi}}{(e-1)^2} T^1 \\ \theta_2 &= E^{-1}[vE(A_1 - B_1)] \\ &= E^{-1}[vE((\theta_1)_{\xi\xi} - \theta_0(\theta_1)_{\xi} - \theta_1(\theta_0)_{\xi})] \\ &= \frac{3e^{3\xi}-6e^{2\xi+1}+2e^{2\xi}+e^{\xi+2}}{2(e-1)^3} T^2 \\ \theta_3 &= E^{-1}[vE(A_2 - B_2)] \\ &= E^{-1}[vE((\theta_2)_{\xi\xi} - \theta_0(\theta_2)_{\xi} - \theta_1(\theta_1)_{\xi} - \theta_2(\theta_0)_{\xi})] \\ &= \frac{-14e^{4\xi}+48e^{3\xi+1}-24e^{3\xi}-27e^{2(\xi+1)}+20e^{2\xi+1}-4e^{2\xi}+e^{\xi+3}}{6(e-1)^4} T^3 \end{aligned}$$

Thus, the final solution is given as

$$\begin{aligned} \theta(\xi, T) &= \sum_{n=0}^{\infty} \theta_n(\xi, T) \\ &= \frac{e^{\xi}-1}{e-1} + \frac{e^{\xi+1}-e^{2\xi}}{(e-1)^2} T^1 + \frac{3e^{3\xi}-6e^{2\xi+1}+2e^{2\xi}+e^{\xi+2}}{2(e-1)^3} T^2 + \\ &\quad \frac{-14e^{4\xi}+48e^{3\xi+1}-24e^{3\xi}-27e^{2(\xi+1)}+20e^{2\xi+1}-4e^{2\xi}+e^{\xi+3}}{6(e-1)^4} T^3 + \dots \end{aligned}$$

IV. NUMERICAL SOLUTION

The following table shows the approximate moisture content of liquid for different values of depth at different time using Elzaki decomposition method.

ξ	T = 0	T = 0.1	T = 0.2	T = 0.3	T = 0.4	T = 0.5	T = 0.6
0	0	0	0	0	0	0	0
0.1	0.06121	0.12159	0.18197	0.24235	0.30273	0.36312	0.4235
0.2	0.12885	0.19077	0.2527	0.31462	0.37655	0.43847	0.50039
0.3	0.20361	0.26617	0.32874	0.3913	0.45386	0.51643	0.57899
0.4	0.28623	0.3482	0.41017	0.47214	0.53411	0.59608	0.65805
0.5	0.37754	0.43727	0.49699	0.55672	0.61644	0.67617	0.7359
0.6	0.47845	0.53376	0.58907	0.64437	0.69968	0.75499	0.81029
0.7	0.58998	0.63803	0.68609	0.73414	0.78219	0.83024	0.8783
0.8	0.71324	0.75038	0.78752	0.82466	0.8618	0.89895	0.93609
0.9	0.84946	0.871	0.89255	0.9141	0.93565	0.9572	0.97875
1	1	1	1	1	1	1	1

V. MOISTURE CONTENT AT DIFFERENT TIME

VI. Conclusion

In the graph X-axis represents moisture content and Y-axis represents depth. It's shows that as time T increases, the moisture content also increases at each point of the basin and after some time, it become constant.

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