Optimizing of Linear Problems Subjected to Sugeno-Weber FRI

Amin Ghodousian^{1*}, Tarane Azarnejad²

¹Faculty of Engineering Science, College of Engineering, University of Tehran, Iran

²Department of Engineering Science, College of Engineering, University of Tehran, Iran

Received: 09 June, 2018; **Accepted:** 21 June, 2018; **Published:** 25 June, 2018

***Corresponding Author:** Ghodousian A, Faculty of Engineering Science, College of Engineering, University of Tehran, Iran. E-mail: a.ghodousian@ut.ac.ir

Abstract

In this paper, optimization of a linear objective function with fuzzy relational inequality constraints is investigated where the feasible region is formed as the intersection of two inequality fuzzy systems and Sugeno-Weber family of t-norms is considered as fuzzy composition. Sugeno-Weber family of t-norms and t-conorms is one of the most applied one in various fuzzy modeling problems. This family of t-norms and t-conorms was suggested by Weber for modeling intersection and union of fuzzy sets. Also, the t-conorms were suggested as addition rules by Sugeno for so-called λ -fuzzy measures. The resolution of the feasible region of the problem is firstly investigated when it is defined with max-Sugeno-Weber composition. A necessary and sufficient condition and three other necessary conditions are derived for determining the feasibility. Moreover, in order to simplify the problem, some procedures are presented. Also, it is proved that the optimal solution of the problem is always resulted from the unique maximum solution and a minimal solution of the feasible region. A method is proposed to generate random feasible max-Sugeno-Weber fuzzy relational inequalities and an algorithm is presented to solve the problem. Finally, an example is described to illustrate these algorithms.

Keywords: Fuzzy Relation; Fuzzy Relational Inequality; Linear Optimization; Fuzzy Compositions and t-norms

Introduction

In this paper, we study the following linear problem in which the constraints are formed as the intersection of two fuzzy

systems of relational inequalities defined by Sugeno-Weber family of t-norms:

$$\begin{aligned}
 \min \quad & Z = c^T x \\
 & A\varphi x \leq b^1 \\
 & D\varphi x \geq b^2 \\
 & x \in [0,1]^n
 \end{aligned} \tag{1}$$

where, $I_1 = \{1, 2, \dots, m_1\}$, $I_2 = \{m_1 + 1, m_1 + 2, \dots, m_1 + m_2\}$ and $J = \{1, 2, \dots, n\}$. $A = (a_{ij})_{m_1 \times n}$ and $D = (d_{ij})_{m_2 \times n}$ are fuzzy matrices such that $0 \leq d_{ij} \leq 1$ ($\forall i \in I_2$ and $\forall j \in J$) and $b^1 = (b_i^1)_{m_1 \times 1}$ is an m_1 -dimensional fuzzy vector in $[0,1]^{m_1}$ (i.e., $0 \leq b_i^1 \leq 1, \forall i \in I_1$), $b^2 = (b_i^2)_{m_2 \times 1}$ is an m_2 -dimensional fuzzy vector in $[0,1]^{m_2}$ (i.e., $0 \leq b_i^2 \leq 1, \forall i \in I_2$), and c is a vector in R^n .

Moreover, " φ " is the max-Sugeno-Weber composition, that is,

$$\varphi(x, y) = T_{SW}^\lambda(x, y) = \max \left\{ \frac{x + y - 1 + \lambda xy}{1 + \lambda}, 0 \right\}$$

in which $\lambda > -1$. By these notations, problem (1) can be also expressed as follows:

$$\begin{aligned}
 \min \quad & Z = c^T x \\
 & \max_{j \in J} \{T_{SW}^\lambda(a_{ij}, x_j)\} \leq b_i^1, \quad i \in I_1 \\
 & \max_{j \in J} \{T_{SW}^\lambda(d_{ij}, x_j)\} \geq b_i^2, \quad i \in I_2 \\
 & x \in [0,1]^n
 \end{aligned} \tag{2}$$

Especially, by setting $A = D$ and $b^1 = b^2$, the above problem is converted to max-Sugeno-Weber fuzzy relational equations. The members of the family $\{T_{SW}^\lambda\}$ are increasing functions of the parameter λ . It can be easily shown that Sugeno-Weber t-norm $T_{SW}^\lambda(x, y)$ converges to the product fuzzy intersection xy as λ goes to infinity and converges to Drastic product t-norm as λ approaches -1

[7]. Also, it is interesting to note that $T_{SW}^0(x, y) = \max\{x + y - 1, 0\}$, that is, the Sugeno-Weber t-norm is converted to Lukasiewicz t-norm if $\lambda = 0$.

The theory of fuzzy relational equations (FRE) as a generalized version of Boolean relation equations was firstly proposed by Sanchez and applied in problems of the medical diagnosis [1]. Nowadays, it is well known that many issues associated with a body knowledge can be treated as FRE problems [2]. In addition to the preceding applications, FRE theory has been applied in many fields, including fuzzy control, discrete dynamic systems, prediction of fuzzy systems, fuzzy decision making, fuzzy pattern recognition, fuzzy clustering, image compression and reconstruction, fuzzy information retrieval, and so on. Generally, when inference rules and their consequences are known, the problem of determining antecedents is reduced to solving an FRE [3]. We refer the reader to [4] in which the authors provided a good overview of FRE and classified basic FREs by investigating the relationship among operators used in the definition of fuzzy relational equations.

The solvability determination and the finding of solutions set are the primary (and the most fundamental) subject concerning with FRE problems [3,5-8]. It is well-known that the solution set of FRE defined by continuous max-t-norm composition is often a non-convex set that is completely determined by one maximum solution and a finite number of minimal solutions [9]. Over the last decades, the solvability of FRE defined with different max-t compositions have been investigated by many researchers [10-18]. Moreover, some researchers introduced and improved theoretical aspects and applications of fuzzy relational inequalities (FRI)[19-23]. [22] studied a FRI with addition-min composition and presented an algorithm to search for

minimal solutions. Ghodousian and Khorram [19] focused on the algebraic structure of two fuzzy relational inequalities and , and studied a mixed fuzzy system formed by the two preceding FRIs, where is an operator with (closed) convex solutions. [20] investigated a kind of FRI problems and the relationship between minimal solutions and FRI paths.

The problem of optimization subject to FRE and FRI is one of the most interesting and on-going research topic among the problems related to FRE and FRI theory [23-34,19-21,4,17,22]. [44] converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by branch and bound method using jump-tracking technique. In [39] an application of optimizing the linear objective with max-min composition was employed for the streaming media provider seeking a minimum cost while fulfilling the requirements assumed by a three-tier framework. [45] improved the method used by Fang and Li, by decreasing the search domain. [24] presented new theoretical results concerning the linear optimization problem constrained by fuzzy max-min relation equations. The topic of the linear optimization problem was also investigated with max-product operation [2,32,36]. Loetamonphong and Fang defined two sub-problems by separating negative and non-negative coefficients in the objective function and then obtained the optimal solution by combining those of the two sub-problems [2]. Also, in [36] and [32] some necessary conditions of the feasibility and simplification techniques were presented for solving FRE with max-product composition. Moreover, some generalizations of the linear optimization with respect to FRE have been studied with the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [38, 17], max-star composition [34] and max-t-norm composition [30,

37,31,43]. For example [40] solved the linear optimization problem subjected to a system of sup-t equations by reducing it to a 0-1 integer optimization problem. In [37] a method was presented for solving linear optimization problems with the max-Archimedean t-norm fuzzy relation equation constraint. In [43], the authors solved the same problem with continuous Archimedean t-norm and used the covering problem rather than the branch-and-bound methods for obtaining some optimal variables.

Recently, many interesting generalizations of the linear programming subject to a system of fuzzy relations have been introduced and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [46-49,21,41]. For example, [49] represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition. [50] generalized the fuzzy linear optimization problem by considering fuzzy coefficients. [50] studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [46]. The linear optimization of bipolar FRE was studied by some researchers where FRE defined with max-min composition [47] and max-Lukasiewicz composition [48,41]. In [41], the authors presented an algorithm without translating the original problem into a 0-1 integer linear problem.

The optimization problem subjected to various versions of FRI could be found in the literature as well [26-28,31,33,19-21,51,23]. [51] applied the pseudo-minimal index algorithm for solving the minimization of linear objective function subject to FRI with addition-min composition. [23] introduced the latticized linear programming problem subject to max-product fuzzy

relation inequalities with application in the optimization management model of wireless communication emission base stations. [29,33] introduced a system of fuzzy relational inequalities with fuzzy constraints (FRI-FC) in which the constraints were defined with max-min composition.

The remainder of the paper is organized as follows. In section 2, some preliminary notions and definitions and three necessary conditions for the feasibility of problem (1) are presented. In section 3, the feasible region of problem (1) is determined as a union of the finite number of closed convex intervals. Two simplification operations are introduced to accelerate the resolution of the problem. Moreover, a necessary and sufficient condition based on the simplification operations is presented to realize the feasibility of the problem. Problem (1) is resolved by optimization of the linear objective function considered in section 4. In addition, the existence of an optimal solution is proved if problem (1) is not empty. The preceding results are summarized as an algorithm and, finally in section 5 an example is described to illustrate. Additionally, in section 5, a method is proposed to generate feasible test problems for problem (1).

Basic properties of max-Sugeno-Weber FRI

This section describes the basic definitions and structural properties concerning problem (1) that are used throughout the paper. For the sake of simplicity, let $S_{T_{SW}^\lambda}(A, b^1)$ and $S_{T_{SW}^\lambda}(D, b^2)$ denote the feasible solutions sets of inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, respectively, that is, $S_{T_{SW}^\lambda}(A, b^1) = \{x \in [0, 1]^n : A\varphi x \leq b^1\}$ and $S_{T_{SW}^\lambda}(D, b^2) = \{x \in [0, 1]^n : D\varphi x \geq b^2\}$.

Also, let $S_{T_{SW}^\lambda}(A, D, b^1, b^2)$ denote the feasible solutions set of problem (1). Based on the foregoing notations, it is clear that $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(A, b^1) \cap S_{T_{SW}^\lambda}(D, b^2)$

Definition (1)

For each $i \in I_1$ and each $j \in J$, we define $S_{T_{SW}^\lambda}(a_{ij}, b_i^1) = \{x \in [0, 1] : T_{SW}^\lambda(a_{ij}, x) \leq b_i^1\}$. Similarly, for each $i \in I_2$ and each $j \in J$, $S_{T_{SW}^\lambda}(d_{ij}, b_i^2) = \{x \in [0, 1] : T_{SW}^\lambda(d_{ij}, x) \geq b_i^2\}$. Furthermore, the notations $J_i^1 = \{j \in J : S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \neq \emptyset\}$, $\forall i \in I_1$, and $J_i^2 = \{j \in J : S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset\}$, $\forall i \in I_2$, are used in the text.

From the least-upper-bound property of $\mathbb{R}$, it is clear that $\inf_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\}$ and $\sup_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\}$ exist, if $S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \neq \emptyset$

Moreover, since T_{SW}^λ is a t-norm, its monotonicity property implies that $S_{T_{SW}^\lambda}(a_{ij}, b_i^1)$ is actually a connected subset of $[0, 1]$. Additionally, due to the continuity of T_{SW}^λ , we must have

$$\inf_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\} = \min_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\} \text{ and } \sup_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\} = \max_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\}.$$

Therefore,

$$S_{T_{SW}^\lambda}(a_{ij}, b_i^1) = \left[\min_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\}, \max_{x \in [0, 1]} \{S_{T_{SW}^\lambda}(a_{ij}, b_i^1)\} \right],$$

i.e., $S_{T_{SW}^\lambda}(a_{ij}, b_i^1)$ is a closed sub-interval of $[0, 1]$. By the similar argument, if

$S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$, then we have

$$S_{T_{SW}^\lambda}(d_{ij}, b_i^2) = \left[\min_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \right\}, \max_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \right\} \right] \subseteq [0, 1].$$

From Definition 1 and the above statements, the following two corollaries are easily resulted.

Corollary (1)

For each $i \in I_1$ and each $j \in J$,

$$S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \neq \emptyset. \quad \text{Also,}$$

$$S_{T_{SW}^\lambda}(a_{ij}, b_i^1) = \left[0, \max_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \right\} \right].$$

Corollary (2)

If $S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$ for some $i \in I_2$ and $j \in J$

$$, \text{ then } S_{T_{SW}^\lambda}(d_{ij}, b_i^2) = \left[\min_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \right\}, 1 \right].$$

Remark (1)

Corollary 1 together with Definition 1 implies $J_i^1 = J$, $\forall i \in I_1$.

Definition (2)

For each $i \in I_1$ and each $j \in J$, we define

$$U_{ij} = \begin{cases} 1 & a_{ij} < b_i^1 \\ \frac{(1 + \lambda)b_i^1 + 1 - a_{ij}}{1 + \lambda a_{ij}} & a_{ij} \geq b_i^1 \end{cases}$$

Also, for each $i \in I_2$ and each $j \in J$, we set

$$L_{ij} = \begin{cases} +\infty & d_{ij} < b_i^2 \\ 0 & b_i^2 = 0, d_{ij} \geq b_i^2 \\ \frac{(1 + \lambda)b_i^2 + 1 - d_{ij}}{1 + \lambda d_{ij}} & b_i^2 \neq 0, d_{ij} \geq b_i^2 \end{cases}$$

From Definition 2, if $a_{ij} = b_i^1$, then $U_{ij} = 1$.

Also, we have $L_{ij} = 1$, if $d_{ij} = b_i^2 \neq 0$.

Lemma 1 below shows that U_{ij} and L_{ij} stated in Definition 2, determine the maximum and minimum solutions of sets $S_{T_{SW}^\lambda}(a_{ij}, b_i^1)$ ($i \in I_1$) and $S_{T_{SW}^\lambda}(d_{ij}, b_i^2)$ ($i \in I_2$), respectively.

Lemma (1)

$$(a) \quad U_{ij} = \max_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \right\}, \quad \forall i \in I_1 \text{ and}$$

$$\forall j \in J.$$

$$(b) \quad \text{If } S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset \text{ for some } i \in I_2 \text{ and } j \in J, \text{ then } L_{ij} = \min_{x \in [0,1]} \left\{ S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \right\}.$$

Proof

The proof is similar to the proof of Lemma 1 in [30]. $\square$

Corollary (3)

(a) For each $i \in I_1$ and $j \in J$,

$$S_{T_{SW}^\lambda}(a_{ij}, b_i^1) = [0, U_{ij}].$$

(b) If $S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$ for some $i \in I_2$ and

$$j \in J, \text{ then } S_{T_{SW}^\lambda}(d_{ij}, b_i^2) = [L_{ij}, 1].$$

Definition (3)

For each $i \in I_1$, let

$$S_{T_{SW}^\lambda}(a_i, b_i^1) = \left\{ x \in [0,1]^n : \max_{j=1}^n \left\{ T_{SW}^\lambda(a_{ij}, x_j) \right\} \leq b_i^1 \right\}$$

Similarly, for each $i \in I_2$, we define

$$S_{T_{SW}^\lambda}(d_i, b_i^2) = \left\{ x \in [0,1]^n : \max_{j=1}^n \left\{ T_{SW}^\lambda(d_{ij}, x_j) \right\} \geq b_i^2 \right\}$$

According to Definition 3 and the constraints stated in (2), sets $S_{T_{SW}^\lambda}(a_i, b_i^1)$ and

$S_{T_{SW}^\lambda}(d_i, b_i^2)$ actually denote the feasible

solutions sets of the i 'th inequality

$$\max_{j \in J} \left\{ T_{SW}^\lambda(a_{ij}, x_j) \right\} \leq b_i^1 \quad (i \in I_1) \text{ and}$$

$\max_{j \in J} \{T_{SW}^\lambda(d_{ij}, x_j)\} \geq b_i^2 (i \in I_2)$ of problem (1), respectively.

Based on (2) and Definitions 1 and 3, it can be easily concluded that for a fixed $i \in I_1$,

$S_{T_{SW}^\lambda}(a_i, b_i^1) \neq \emptyset$ iff $S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \neq \emptyset$, $\forall j \in J$. On the other hand, by Corollary 1

we know that $S_{T_{SW}^\lambda}(a_{ij}, b_i^1) \neq \emptyset$, $\forall i \in I_1$ and $\forall j \in J$. As a result, $S_{T_{SW}^\lambda}(a_i, b_i^1) \neq \emptyset$ for each $i \in I_1$. However, in contrast to $S_{T_{SW}^\lambda}(a_i, b_i^1)$,

set $S_{T_{SW}^\lambda}(d_i, b_i^2)$ may be empty. Actually, for a fixed $i \in I_2$, $S_{T_{SW}^\lambda}(d_i, b_i^2)$ is nonempty if and only if $S_{T_{SW}^\lambda}(d_{ij}, b_i^2)$ is nonempty for at least some $j \in J$. Additionally, for each $i \in I_2$ and

$j \in J$ we have $S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$ if and only if $d_{ij} \geq b_i^2$. These results have been summarized in the following lemma. Part (b) of the lemma gives a necessary and sufficient condition for the feasibility of set $S_{T_{SW}^\lambda}(d_i, b_i^2)$ ($\forall i \in I_2$). It is to be noted that the lemma 2 (part (b)) also provides a necessary condition for problem (1).

Lemma (2)

(a) $S_{T_{SW}^\lambda}(a_i, b_i^1) \neq \emptyset$, $\forall i \in I_1$.

(b) For a fixed $i \in I_2$, $S_{T_{SW}^\lambda}(d_i, b_i^2) \neq \emptyset$ iff

$\bigcup_{j=1}^n S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$. Additionally, for each

$i \in I_2$ and $j \in J$, $S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset$ iff $d_{ij} \geq b_i^2$.

Definition (4)

For each $i \in I_2$ and $j \in J_i^2$, we define

$S_{T_{SW}^\lambda}(d_i, b_i^2, j) = [0, 1] \times \dots \times [0, 1] \times [L_{ij}, 1] \times [0, 1] \times \dots \times [0, 1]$, where $[L_{ij}, 1]$ is in the j 'th position.

In the following lemma, the feasible solutions set of the i 'th fuzzy relational inequality is characterized.

Lemma (3)

(a) $S_{T_{SW}^\lambda}(a_i, b_i^1) = [0, U_{i1}] \times [0, U_{i2}] \times \dots \times [0, U_{in}]$, $\forall i \in I_1$.

(b) $S_{T_{SW}^\lambda}(d_i, b_i^2) = \bigcup_{j \in J_i^2} S_{T_{SW}^\lambda}(d_i, b_i^2, j)$, $\forall i \in I_2$.

Proof

For a more general case, see Lemma 2.3 in [19]. $\square$

Definition (5)

Let $\bar{X}(i) = [U_{i1}, U_{i2}, \dots, U_{in}]$, $\forall i \in I_1$. Also, let $\underline{X}(i, j) = [\underline{X}(i, j)_1, \underline{X}(i, j)_2, \dots, \underline{X}(i, j)_n]$, $\forall i \in I_2$ and $\forall j \in J_i^2$, where

$$\underline{X}(i, j)_k = \begin{cases} L_{ij} & k = j \\ 0 & k \neq j \end{cases}$$

Lemma 3 together with Definitions 4 and 5, results in Theorem 1, which completely determines the feasible region for the i 'th relational inequality.

Theorem (1)

(a) $S_{T_{SW}^\lambda}(a_i, b_i^1) = [\mathbf{0}, \bar{X}(i)]$, $\forall i \in I_1$. (b)

$S_{T_{SW}^\lambda}(d_i, b_i^2) = \bigcup_{j \in J_i^2} [\underline{X}(i, j), \mathbf{1}]$, $\forall i \in I_2$,

where $\mathbf{0}$ and $\mathbf{1}$ are n -dimensional vectors with each component equal to zero and one, respectively. Theorem 1 gives the upper and lower bounds for the feasible solutions set of the i 'th relational inequality. Actually, for each $i \in I_1$, vectors $\mathbf{0}$ and $\bar{X}(i)$ are the

unique minimum and the unique maximum of set $S_{T_{SW}}^\lambda(a_i, b_i^1)$. In addition, for each $i \in I_2$, set $S_{T_{SW}}^\lambda(d_i, b_i^2)$ has the unique maximum (i.e., vector $\mathbf{1}$), but the finite number of minimal solutions $\underline{X}(i, j)$ ($\forall j \in J_i^2$). Furthermore, part (b) of Theorem 1 presents another feasible necessary condition for problem (1) as stated in the following corollary.

Corollary (4)

If $S_{T_{SW}}^\lambda(A, D, b^1, b^2) \neq \emptyset$, then $\mathbf{1} \in S_{T_{SW}}^\lambda(d_i, b_i^2)$, $\forall i \in I_2$ (i.e., $\mathbf{1} \in \bigcap_{i \in I_2} S_{T_{SW}}^\lambda(d_i, b_i^2) = S_{T_{SW}}^\lambda(D, b^2)$).

Proof

Let $S_{T_{SW}}^\lambda(A, D, b^1, b^2) \neq \emptyset$. Then, $S_{T_{SW}}^\lambda(D, b^2) \neq \emptyset$, and therefore, $S_{T_{SW}}^\lambda(d_i, b_i^2) \neq \emptyset$, $\forall i \in I_2$. Now, Theorem 1 (part (b)) implies $\mathbf{1} \in S_{T_{SW}}^\lambda(d_i, b_i^2)$, $\forall i \in I_2$. $\square$

Lemma 4 describes the shape of the feasible solutions set for the fuzzy relational inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$, separately.

Lemma(4)

$$(a) S_{T_{SW}}^\lambda(A, b^1) = \bigcap_{i \in I_1} [0, U_{i1}] \times \bigcap_{i \in I_1} [0, U_{i2}] \times \dots \times \bigcap_{i \in I_1} [0, U_{in}].$$

$$(b) S_{T_{SW}}^\lambda(D, b^2) = \bigcap_{i \in I_2} \bigcup_{j \in J_i^2} S_{T_{SW}}^\lambda(d_i, b_i^2, j)$$

Proof

The proof is obtained from Lemma 3 and equations $S_{T_{SW}}^\lambda(A, b^1) = \bigcap_{i \in I_1} S_{T_{SW}}^\lambda(a_i, b_i^1)$ and

$$S_{T_{SW}}^\lambda(D, b^2) = \bigcap_{i \in I_2} S_{T_{SW}}^\lambda(d_i, b_i^2). \square$$

Definition (6)

Let $e: I_2 \rightarrow J_i^2$ so that $e(i) = j \in J_i^2$, $\forall i \in I_2$, and let E_D be the set of all vectors e . For the sake of convenience, we represent each $e \in E_D$ as an m_2 -dimensional vector $e = [j_1, j_2, \dots, j_{m_2}]$ in which $j_k = e(k)$, $k = 1, 2, \dots, m_2$.

Definition (7)

Let $e = [j_1, j_2, \dots, j_{m_2}] \in E_D$. We define $\overline{X} = \min_{i \in I_1} \{\overline{X}(i)\}$, that is, $\overline{X}_j = \min_{i \in I_1} \{\overline{X}(i)_j\}$, $\forall j \in J$. Moreover, let $\underline{X}(e) = [\underline{X}(e)_1, \underline{X}(e)_2, \dots, \underline{X}(e)_n]$, where $\underline{X}(e)_j = \max_{i \in I_2} \{\underline{X}(i, e(i))_j\} = \max_{i \in I_2} \{\underline{X}(i, j_i)_j\}$, $\forall j \in J$. Based on Theorem 1 and the above definition, we have the following theorem characterizing the feasible regions of the general inequalities $A\varphi x \leq b^1$ and $D\varphi x \geq b^2$ in the most familiar way.

Theorem (2)

$$(a) S_{T_{SW}}^\lambda(A, b^1) = [\mathbf{0}, \overline{X}], \forall i \in I_1.$$

$$(b) S_{T_{SW}}^\lambda(D, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}].$$

Proof

See Theorem 2.2 and Remark 2.5 in [19]. $\square$

Corollary (5)

Assume that $S_{T_{SW}}^\lambda(A, D, b^1, b^2) \neq \emptyset$. Then, there exists some $e \in E_D$ such that $[\mathbf{0}, \overline{X}] \cap [\underline{X}(e), \mathbf{1}] \neq \emptyset$.

Corollary (6)

Assume that $S_{T_{SW}}^\lambda(A, D, b^1, b^2) \neq \emptyset$. Then, $\overline{X} \in S_{T_{SW}}^\lambda(D, b^2)$.

Proof

Let $S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$. By Corollary 5,

$$[\mathbf{0}, \bar{X}] \cap [\underline{X}(e'), \mathbf{1}] \neq \emptyset \text{ for some } e' \in E_D.$$

Thus, $\bar{X} \in [\underline{X}(e'), \mathbf{1}]$ that means

$$\bar{X} \in \bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}]. \text{ Therefore, from}$$

Theorem 2 (part (b)), $\bar{X} \in S_{T_{SW}^\lambda}(D, b^2)$. $\square$

Feasible solutions set and simplification operations

In this section, two operations are presented to simplify the matrices A and D , and a necessary and sufficient condition is derived to determine the feasibility of the main problem. At first, we give a theorem in which the bounds of the feasible solutions set of problem (1) are attained. As is shown in the following theorem, by using these bounds, the feasible region is completely found.

Theorem (3)

Suppose that $S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$. Then

$$S_{T_{SW}^\lambda}(A, D, b^1, b^2) = \bigcup_{e \in E_D} [\underline{X}(e), \bar{X}].$$

Proof

Since $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(A, b^1) \cap S_{T_{SW}^\lambda}(D, b^2)$,

then by Theorem 2,

$$S_{T_{SW}^\lambda}(A, D, b^1, b^2) = [\mathbf{0}, \bar{X}] \cap \left(\bigcup_{e \in E_D} [\underline{X}(e), \mathbf{1}] \right)$$

and the statement is established. $\square$

In practice, there are often some components of matrices A and D , which have no effect on the solutions to problem (1). Therefore, we can simplify the problem by changing the values of these components to zeros. We refer the interesting reader to [19] where a brief review of such these processes is given. Here, we present two

simplification techniques based on the Sugeno-Weber family of t-norms.

Definition (8)

If a value changing in an element, say a_{ij} , of a given fuzzy relation matrix A has no effect on the solutions of problem (1), this value changing is said to be an equivalence operation.

Corollary (7)

Suppose that $i \in I_1$ and $T_{SW}^\lambda(a_{ij_0}, x_{j_0}) < b_i$,

$\forall x \in S_{T_{SW}^\lambda}(A, b^1)$. In this case, it is obvious

that $\max_{j=1}^n \{T_{SW}^\lambda(a_{ij}, x_j)\} \leq b_i^1$ is equivalent to

$$\max_{\substack{j=1 \\ j \neq j_0}}^n \{T_{SW}^\lambda(a_{ij}, x_j)\} \leq b_i^1, \text{ that is, "resetting}$$

a_{ij_0} to zero" has no effect on the solutions of

problem (1) (since component a_{ij_0} only

appears in the i 'th constraint of problem

(1)). Therefore, if $T_{SW}^\lambda(a_{ij_0}, x_{j_0}) < b_i^1$,

$\forall x \in S_{T_{SW}^\lambda}(A, b^1)$, then "resetting a_{ij_0} to

zero" is an equivalence operation.

Lemma (5) (simplification of matrix A)

Suppose that matrix $\tilde{A} = (\tilde{a}_{ij})_{m_1 \times n}$ is

resulted from matrix A as follows:

$$\tilde{a}_{ij} = \begin{cases} 0 & a_{ij} < b_i^1 \\ a_{ij} & a_{ij} \geq b_i^1 \end{cases}$$

for each $i \in I_1$ and $j \in J$. Then,

$$S_{T_{SW}^\lambda}(A, b^1) = S_{T_{SW}^\lambda}(\tilde{A}, b^1).$$

Proof

From corollary 7, it is sufficient to show that

$$T_{SW}^\lambda(a_{ij_0}, x_{j_0}) < b_i^1, \forall x \in S_{T_{SW}^\lambda}(A, b^1). \text{ But,}$$

from the monotonicity and identity laws of T_{SW}^λ , we have

$$T_{SW}^\lambda(a_{ij_0}, x_{j_0}) \leq T_{SW}^\lambda(a_{ij_0}, 1) = a_{ij_0} < b_i^1,$$

$$\forall x_{j_0} \in [0, 1]. \text{ Thus, } T_{SW}^\lambda(a_{ij_0}, x_{j_0}) < b_i^1,$$

$$\forall x \in S_{T_{SW}^\lambda}(A, b^1). \square$$

Lemma 5 gives a condition to reduce the matrix A . In this lemma, $\tilde{A}$ denote the simplified matrix resulted from A after applying the simplification process. Based on this notation, we define

$$\tilde{J}_i^1 = \left\{ j \in J : S_{T_{SW}^\lambda}(\tilde{a}_{ij}, b_i^1) \neq \emptyset \right\} \quad (\forall i \in I_1)$$

where $\tilde{a}_{ij}$ denotes (i, j) 'th component of matrix $\tilde{A}$. So, from Corollary 1 and Remark 1, it is clear that $\tilde{J}_i^1 = J_i^1 = J$. Moreover, since $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(A, b^1) \cap S_{T_{SW}^\lambda}(D, b^2)$, from Lemma 5 we can also conclude that $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(\tilde{A}, D, b^1, b^2)$. By considering a fixed vector $e \in E_D$ in Theorem 3, interval $[\underline{X}(e), \bar{X}]$ is meaningful iff $\underline{X}(e) \leq \bar{X}$. Therefore, by deleting infeasible intervals $[\underline{X}(e), \bar{X}]$ in which $\underline{X}(e) \not\leq \bar{X}$, the feasible solutions set of problem (1) stays unchanged. In order to remove such infeasible intervals from the feasible region, it is sufficient to neglect vectors e generating infeasible solutions $\underline{X}(e)$ (i.e., solutions $\underline{X}(e)$ such that $\underline{X}(e) \not\leq \bar{X}$). These considerations lead us to introduce a new set

$E'_D = \{e \in E_D : \underline{X}(e) \leq \bar{X}\}$ to strengthen Theorem 3. By this new set, Theorem 3 can be written as $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = \bigcup_{e \in E'_D} [\underline{X}(e), \bar{X}]$, if $S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$.

Lemma (6)

Let $I_j(e) = \{i \in I_2 : e(i) = j\}$, $\forall e \in E_D$.

$$\text{Then, } \underline{X}(e)_j = \begin{cases} \max_{i \in I_j(e)} \{L_{ie(i)}\} & j \in J(e) \\ 0 & j \notin J(e) \end{cases}$$

Proof

From Definition 7, $\underline{X}(e)_j = \max_{i \in I_2} \{X(i, e(i))_j\}$, $\forall j \in J$. On the other hand, by Definition 5, we have

$$X(i, e(i))_j = \begin{cases} L_{ie(i)} & j = e(i) \\ 0 & j \neq e(i) \end{cases}$$

Now, the result follows by combining these two equations. $\square$

Corollary (8)

$e \in E'_D$ if and only if $L_{ie(i)} \leq \bar{X}_{e(i)}$, $\forall i \in I_2$.

Proof

Firstly, from the definition of set E'_D , we note that $e \in E'_D$ if and only if $\underline{X}(e)_j \leq \bar{X}_j$, $\forall j \in J$. Now, let $e \in E'_D$ and by contradiction, suppose that $L_{i_0 e(i_0)} > \bar{X}_{e(i_0)}$ for some $i_0 \in I_2$. So, by setting $e(i_0) = j_0$, we have $j_0 \in J(e)$, and therefore lemma 6 implies $\underline{X}(e)_{j_0} = \max_{i \in I_{j_0}(e)} \{L_{ie(i)}\} \geq L_{i_0 e(i_0)} > \bar{X}_{e(i_0)}$. Thus, $\underline{X}(e)_{j_0} > \bar{X}_{e(i_0)}$ that contradicts $e \in E'_D$. The converse statement is easily proved by Lemma 6. $\square$

As mentioned before, to accelerate identification of the meaningful solutions $\underline{X}(e)$, we reduce our search to set E'_D instead of set E_D . As a result from Corollary 8, we can confine set J_i^2 by removing each

$j \in J_i^2$ such that $L_{ij} > \bar{X}_j$ before selecting the vectors e to construct solutions $\underline{X}(e)$. However, lemma 7 below shows that this purpose can be accomplished by resetting some components of matrix D to zeros. Before formally presenting the lemma, some useful notations are introduced.

Definition (9) (simplification of matrix D)

Let $\tilde{D} = (\tilde{d}_{ij})_{m_2 \times n}$ denote a matrix resulted from D as follows:

$$\tilde{d}_{ij} = \begin{cases} 0 & j \in J_i^2 \text{ and } L_{ij} > \bar{X}_j \\ d_{ij} & \text{otherwise} \end{cases}$$

Also, similar to Definition 1, assume that $\tilde{J}_i^2 = \{j \in J : S_{T_{SW}^\lambda}(\tilde{d}_{ij}, b_i^2) \neq \emptyset\}$ ($\forall i \in I_2$) where $\tilde{d}_{ij}$ denotes (i, j) 'th components of matrix $\tilde{D}$.

According to the above definition, it is easy to verify that $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$. Furthermore, the following lemma demonstrates that the infeasible solutions $\underline{X}(e)$ are not generated, if we only consider those vectors e generated by the components of the matrix $\tilde{D}$, or equivalently vectors e generated based on the set $\tilde{J}_i^2$ instead of J_i^2 .

Lemma (7)

$E_{\tilde{D}} = E'_D$, where $E_{\tilde{D}}$ is the set of all functions $e: I_2 \rightarrow \tilde{J}_i^2$ so that $e(i) = j \in \tilde{J}_i^2$, $\forall i \in I_2$.

Proof

Let $e \in E'_D$. Then, by Corollary 8, $L_{ie(i)} \leq \bar{X}_{e(i)}$, $\forall i \in I_2$. Therefore, we have $\tilde{d}_{ie(i)} = d_{ie(i)}$, $\forall i \in I_2$, that necessitates $\tilde{J}_i^2 = J_i^2$, $\forall i \in I_2$.

Hence, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$, and then $e \in E_{\tilde{D}}$. Conversely, let $e \in E_{\tilde{D}}$. Therefore, $e(i) \in \tilde{J}_i^2$, $\forall i \in I_2$. Since $\tilde{J}_i^2 \subseteq J_i^2$, $\forall i \in I_2$, then $e(i) \in J_i^2$, $\forall i \in I_2$, and therefore $e \in E_D$. By contradiction, suppose that $e \notin E'_D$. So, by Corollary 8, there is some $i_0 \in I_2$ such that $L_{i_0 e(i_0)} > \bar{X}_{e(i_0)}$. Hence, $\tilde{d}_{i_0 e(i_0)} = 0$ (since $e(i_0) \in J_{i_0}^2$ and $L_{i_0 e(i_0)} > \bar{X}_{e(i_0)}$) and $L_{i_0 e(i_0)} > 0$. The latter inequality together with Definition 2 and Remark 3 implies $b_{i_0}^2 > 0$. But in this case, $T_{SW}^\lambda(\tilde{d}_{i_0 e(i_0)}, x) = T_{SW}^\lambda(0, x) = 0 < b_{i_0}^2$, $\forall x \in [0, 1]$, that contradicts $e(i_0) \in J_{i_0}^2$. $\square$

By Lemma 7, we always have $\underline{X}(e) \leq \bar{X}$ for each vector e , which is selected based on the components of matrix $\tilde{D}$. Actually, matrix $\tilde{D}$ as a reduced version of matrix D , removes all the infeasible intervals from the feasible region by neglecting those vectors e generating the infeasible solutions $\underline{X}(e)$. Also, similar to Lemma 5 we have $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(A, \tilde{D}, b^1, b^2)$. This result and Lemma 5 can be summarized by $S_{T_{SW}^\lambda}(A, D, b^1, b^2) = S_{T_{SW}^\lambda}(\tilde{A}, \tilde{D}, b^1, b^2)$.

Definition (10)

Let $L = (L_{ij})_{m_2 \times n}$ be a matrix whose (i, j) 'th component is equal to L_{ij} . We define the modified matrix $L^* = (L_{ij}^*)_{m_2 \times n}$ from the matrix L as follows:

$$L_{ij}^* = \begin{cases} +\infty & L_{ij} > \bar{X}_j \\ L_{ij} & \text{otherwise} \end{cases}$$

As will be shown in the following theorem, matrix L^* is useful for deriving a necessary and sufficient condition for the feasibility of problem (1) and accelerating identification of the set $S_{T_{SW}^\lambda}(A, D, b^1, b^2)$

Theorem (4)

$S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$ iff there exists

at least some $j \in J_i^2$ such that $L_{ij}^* \neq +\infty$, $\forall i \in I_2$.

Proof

Let $x \in S_{T_{SW}^\lambda}(A, D, b^1, b^2)$. Then, from Corollary 5, there exists some $e' \in E_D$ such that $[\underline{X}(e'), \overline{X}] \neq \emptyset$. Therefore, $\underline{X}(e') \leq \overline{X}$ that implies $e' \in E'_D$. Now, by Corollary 8, we have $L_{ie'(i)} \leq \overline{X}_{e'(i)}$, $\forall i \in I_2$. Hence, by considering Definition 10, $L_{ie'(i)}^* \neq +\infty$, $\forall i \in I_2$. Conversely, suppose that $L_{ij_i}^* \neq +\infty$ for some $j_i \in J_i^2$, $\forall i \in I_2$. Then, from Definition 10 we have

$$L_{ij_i} \leq \overline{X}_{j_i}, \quad \forall i \in I_2 \quad (3)$$

Consider vector $e' = [j_1, j_2, \dots, j_m] \in E_D$. So, by noting Lemma 6, $\underline{X}(e')_{j_i} = \max_{i \in I_j(e')} \{L_{ie'(i)}\} = \max_{i \in I_j(e')} \{L_{ij_i}\}$, $\forall i \in I_2$, and $\underline{X}(e')_j = 0$ for each $j \in J - \{j_1, j_2, \dots, j_m\}$. These equations together with (3) imply $\underline{X}(e') \leq \overline{X}$ that means. $[\underline{X}(e'), \overline{X}] \neq \emptyset$ Now, the result follows from Corollary 5. $\square$

Optimization of the problem

According to the well-known schemes used for optimization of linear problems such

as (1) [44,19,20,22], problem (1) is converted to the following two sub-problems:

$$(4): \min Z_1 = \sum_{j=1}^n c_j^+ x_j$$

$$A\varphi x \leq b^1$$

$$D\varphi x \geq b^2$$

$$x \in [0, 1]^n \quad (4)$$

And

$$(5): \min Z_2 = \sum_{j=1}^n c_j^- x_j$$

$$A\varphi x \leq b^1$$

$$D\varphi x \geq b^2$$

$$x \in [0, 1]^n \quad (5)$$

where $c_j^+ = \max\{c_j, 0\}$ and $c_j^- = \min\{c_j, 0\}$ for $j = 1, 2, \dots, n$. It is easy to prove that $\overline{X}$ is the optimal solution of (5), and the optimal solution of (4) is $\underline{X}(e')$ for some $e' \in E'_D \setminus$

Theorem (5)

Suppose that $S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$, and $\overline{X}$ and $\underline{X}(e^*)$ are the optimal solutions of sub-problems (5) and (4), respectively. Then $c^T x^*$ is the lower bound of the optimal objective function in (1), where $x^* = [x_1^*, x_2^*, \dots, x_n^*]$ is defined as follows:

$$x_j^* = \begin{cases} \overline{X}_j & c_j < 0 \\ \underline{X}(e^*)_j & c_j \geq 0 \end{cases} \quad (6)$$

for $j = 1, 2, \dots, n$.

Proof

See Corollary 4.1 in [19]. $\square$

Corollary (9)

Suppose that $S_{T_{SW}^\lambda}(A, D, b^1, b^2) \neq \emptyset$. Then, $x^* = [x_1^*, x_2^*, \dots, x_n^*]$ as defined in (6), is the optimal solution of problem (1).

Proof

As in the poof of Theorem 5, $c^T x^*$ is the lower bound of the optimal objective function. According to the definition of vector x^* , we have $\underline{X}(e^*)_j \leq x_j^* \leq \overline{X}_j$, $\forall j \in J$, which implies $x^* \in \bigcup_{e \in E_D} [\underline{X}(e), \overline{X}] = S_{T_{SW}^\lambda}(A, D, b^1, b^2)$. $\square$

We now summarize the preceding discussion as an algorithm.

Algorithm 1 (solution of problem (1))

Given problem (1):

1. Compute U_{ij} ($\forall i \in I_1$ and $\forall j \in J$) and L_{ij} ($\forall i \in I_2$ and $\forall j \in J$) by Definition 2.
2. If $1 \in S_{T_{SW}^\lambda}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 4).
3. Compute vectors $\overline{X}(i)$ ($\forall i \in I_1$) from Definition 5, and then vector $\overline{X}$ from Definition 7.
4. If $\overline{X} \in S_{T_{SW}^\lambda}(D, b^2)$, then continue; otherwise, stop, the problem is infeasible (Corollary 6).
5. Compute simplified matrices $\tilde{A}$ and $\tilde{D}$ from Lemma 5 and Definition 9, respectively.
6. Compute modified matrix L^* from Definition 10.

7. For each $i \in I_2$, if there exists at least some $j \in J_i^2$ such that $L_{ij}^* \neq +\infty$, then continue; otherwise, stop, the problem is infeasible (Theorem 4).

8. Find the optimal solution $\underline{X}(e^*)$ for the sub-problem (4) by considering vectors $e \in E_D$ and set $\tilde{J}_i^2$, $\forall i \in I_2$ (Lemma 7).

9. Find the optimal solution $x^* = [x_1^*, x_2^*, \dots, x_n^*]$ for the problem (1) by (6) (Corollary 9).

It should be noted that there is no polynomial time algorithm for complete solution of FRIs with the expectation $N \neq NP$. Hence, the problem of solving FRIs is an NP-hard problem in terms of computational complexity [5].

Construction of test problems and numerical example

In this section, we present a method to generate random feasible regions formed as the intersection of two fuzzy inequalities with Sugeno-Weber family of t-norms. In section 5.1, we prove that the max-Sugeno-Weber fuzzy relational inequalities constructed by the introduced method are actually feasible. In section 5.2, the method is used to generate a random test problem for problem (1), and then the test problem is solved by Algorithm 1 presented in section 4.

Construction of test problems

There are several ways to generate a feasible FRI defined with max-Sugeno-Weber composition. In what follows, we present a procedure to generate random feasible max-Sugeno-Weber fuzzy relational inequalities:

Algorithm 2 (construction of feasible Max-Sugeno-Weber FRI)

1. Generate random scalars $a_{ij} \in [0,1]$, $i = 1,2,\dots,m_1$ and $j = 1,2,\dots,n$, and $b_i^1 \in [0,1]$, $i = 1,2,\dots,m_1$.
2. Compute $\bar{X}$ by Definition 7.
2. Randomly select m_2 columns $\{j_1, j_2, \dots, j_{m_2}\}$ from $J = \{1, 2, \dots, n\}$.
2. For $i \in \{1, 2, \dots, m_2\}$, assign a random number from $[0, \bar{X}_{j_i}]$ to b_i^2 .
3. For $i \in \{1, 2, \dots, m_2\}$, if $b_i^2 \neq 0$, then
 Assign a random number from interval $\left[\max \left\{ b_i^2, \frac{(1+\lambda)b_i^2 + 1 - \bar{X}_{j_i}}{1 + \lambda \bar{X}_{j_i}} \right\}, 1 \right]$ to d_{ij_i} .
 End
4. For $i \in \{1, 2, \dots, m_2\}$
 For each $k \in \{1, 2, \dots, m_2\} - \{i\}$
 Assign a random number from $[0, 1]$ to d_{kj_i} .
 End
- End
5. For each $i \in \{1, 2, \dots, m_2\}$ and each $j \notin \{j_1, j_2, \dots, j_{m_2}\}$
 Assign a random number from $[0,1]$ to d_{ij} .
 End

By the following theorem, it is proved that Algorithm 2 always generates random feasible max-Sugeno-Weber fuzzy relational inequalities.

Theorem (6)

Problem (1) with feasible region constructed by Algorithm (2) has the nonempty feasible solutions set (i.e., $S_{TSW}^\lambda(A, D, b^1, b^2) \neq \emptyset$).

Proof

By considering the columns $\{j_1, j_2, \dots, j_{m_2}\}$ selected by Algorithm 2, let $e' = [j_1, j_2, \dots, j_{m_2}]$. We show that $e' \in E_D$ and $\underline{X}(e') \leq \bar{X}$. Then, the result follows from Corollary 5. From Algorithm 2, the following inequalities are resulted for each $i \in I_2$: (I) $b_i^2 \leq \bar{X}_{j_i}$; (II) $b_i^2 \leq d_{ij_i}$; (III) $\frac{(1+\lambda)b_i^2 + 1 - \bar{X}_{j_i}}{1 + \lambda \bar{X}_{j_i}} \leq d_{ij_i}$. By (I),

we have $\frac{(1+\lambda)b_i^2 + 1 - \bar{X}_{j_i}}{1 + \lambda \bar{X}_{j_i}} \leq 1$. This inequality together with $b_i^2 \in [0,1]$, $\forall i \in I_2$, implies that the interval $\left[\max \left\{ b_i^2, \frac{(1+\lambda)b_i^2 + 1 - \bar{X}_{j_i}}{1 + \lambda \bar{X}_{j_i}} \right\}, 1 \right]$ is meaningful. Also, by (II), $e'(i) = j_i \in J_i^2$, $\forall i \in I_2$. Therefore, $e' \in E_D$. Moreover, since the columns $\{j_1, j_2, \dots, j_{m_2}\}$ are distinct, sets $I_{j_i}(e')$ ($i \in I_2$) are all singleton, i.e.,

$$I_{j_i}(e') = \{i\}, \forall i \in I_2 \quad (7)$$

As a result, we also have $J(e') = \{j_1, j_2, \dots, j_{m_2}\}$ and $I_j(e') = \emptyset$ for each $j \notin \{j_1, j_2, \dots, j_{m_2}\}$. On the other hand, from Definition 5, we have $\underline{X}(i, e'(i))_{e'(i)} = \underline{X}(i, j_i)_{j_i} = L_{ij_i}$ and

$\underline{X}(i, e'(i))_j = 0$ for each $j \notin J - \{j_i\}$. This fact together with (7) and Lemma 6 implies $\underline{X}(e')_{j_i} = L_{i_{j_i}}$, $\forall i \in I_2$, and $\underline{X}(e')_j = 0$ for $j \notin \{j_1, j_2, \dots, j_{m_2}\}$. So, in order to prove $\underline{X}(e') \leq \bar{X}$, it is sufficient to show that $\underline{X}(e')_{j_i} \leq \bar{X}_{j_i}$, $\forall i \in I_2$. But, from Definition 2 and Remark 3,

$$\underline{X}(e')_{j_i} = L_{i_{j_i}} = \begin{cases} 0 & b_i^2 = 0 \\ \frac{(1+\lambda)b_i^2 + 1 - d_{ij_i}}{1 + \lambda d_{ij_i}} & b_i^2 \neq 0 \end{cases} \quad (8)$$

$$\begin{aligned} \min \quad Z = & 6.2945 x_1 + 8.1158 x_2 - 7.4603 x_3 + 8.2675 x_4 + 2.6472 x_5 \\ & - 8.0492 x_6 - 4.4300 x_7 + 0.9376 x_8 + 9.1501 x_9 + 9.2978 x_{10} \end{aligned}$$

$$\begin{bmatrix} 0.1576 & 0.6557 & 0.7060 & 0.4387 & 0.2760 & 0.7513 & 0.8407 & 0.3517 & 0.0759 & 0.1622 \\ 0.9706 & 0.0357 & 0.0318 & 0.3816 & 0.6797 & 0.2551 & 0.2543 & 0.8308 & 0.0540 & 0.7943 \\ 0.9572 & 0.8491 & 0.2769 & 0.7655 & 0.6551 & 0.5060 & 0.8143 & 0.5853 & 0.5308 & 0.3112 \\ 0.4854 & 0.9340 & 0.0462 & 0.7952 & 0.1626 & 0.6991 & 0.2435 & 0.5497 & 0.7792 & 0.5285 \\ 0.8003 & 0.6787 & 0.0971 & 0.1869 & 0.1190 & 0.8909 & 0.9293 & 0.9172 & 0.9340 & 0.1656 \\ 0.1419 & 0.7577 & 0.8235 & 0.4898 & 0.4984 & 0.9593 & 0.3500 & 0.2858 & 0.1299 & 0.6020 \\ 0.4218 & 0.7431 & 0.6948 & 0.4456 & 0.9597 & 0.5472 & 0.1966 & 0.7572 & 0.5688 & 0.2630 \\ 0.9157 & 0.3922 & 0.3171 & 0.6463 & 0.3404 & 0.1386 & 0.2511 & 0.7537 & 0.4694 & 0.6541 \\ 0.7922 & 0.6555 & 0.9502 & 0.7094 & 0.5853 & 0.1493 & 0.6160 & 0.3804 & 0.0119 & 0.6892 \\ 0.9595 & 0.1712 & 0.0344 & 0.7547 & 0.2238 & 0.2575 & 0.4733 & 0.5678 & 0.3371 & 0.7482 \end{bmatrix} \varphi x \leq \begin{bmatrix} 0.4505 \\ 0.0838 \\ 0.2290 \\ 0.9133 \\ 0.1524 \\ 0.8258 \\ 0.5383 \\ 0.9961 \\ 0.0782 \\ 0.4427 \end{bmatrix}$$

$$\begin{bmatrix} 0.3674 & 0.4899 & 0.9730 & 0.6099 & 0.3909 & 0.4588 & 0.6797 & 0.9810 & 0.8909 & 0.0714 \\ 0.9880 & 0.1679 & 0.6490 & 0.6177 & 0.8314 & 0.7984 & 0.1366 & 0.0855 & 0.3342 & 0.5216 \\ 0.9995 & 0.9787 & 0.8003 & 0.8594 & 0.8034 & 0.9631 & 0.7212 & 0.2625 & 0.6987 & 0.0967 \\ 0.0377 & 0.7127 & 0.4538 & 0.8055 & 0.0605 & 0.5468 & 0.7982 & 0.8010 & 0.1978 & 0.8181 \\ 0.8852 & 0.5005 & 0.4324 & 0.5767 & 0.3993 & 0.5211 & 0.1068 & 0.0292 & 0.8241 & 0.8175 \\ 0.9133 & 0.4711 & 0.8253 & 0.8643 & 0.5269 & 0.2316 & 0.6538 & 0.9289 & 0.0305 & 0.7224 \\ 0.7962 & 0.9426 & 0.0835 & 0.1829 & 0.4168 & 0.4889 & 0.4942 & 0.7303 & 0.7441 & 0.1499 \\ 0.0987 & 0.0596 & 0.1332 & 0.2399 & 0.6569 & 0.6241 & 0.7791 & 0.4886 & 0.5000 & 0.7311 \\ 0.2619 & 0.6820 & 0.8822 & 0.8865 & 0.6280 & 0.6791 & 0.7150 & 0.5785 & 0.4799 & 0.6596 \\ 0.3354 & 0.0424 & 0.1734 & 0.0287 & 0.8612 & 0.3955 & 0.9037 & 0.2373 & 0.9047 & 0.5186 \end{bmatrix} \varphi x \geq \begin{bmatrix} 0.0492 \\ 0.0876 \\ 0.0864 \\ 0.0810 \\ 0.0471 \\ 0.1292 \\ 0.1511 \\ 0.0392 \\ 0.0291 \\ 0.1028 \end{bmatrix}$$

$$x \in [0, 1]^n$$

where $|I_1| = |I_2| = |J| = 10$ and $\varphi(x, y) = T_{sw}^2(x, y) = \max \left\{ \frac{x + y - 1 + 2xy}{3}, 0 \right\}$ (i.e., $\lambda = 2$). Moreover,

$$Z_1 = 6.2945 x_1 + 8.1158 x_2 + 8.2675 x_4 + 2.6472 x_5 + 0.9376 x_8 + 9.1501 x_9 + 9.2978 x_{10} \quad \text{is the}$$

Now, inequality (III) implies

$$\frac{(1+\lambda)b_i^2 + 1 - d_{ij_i}}{1 + \lambda d_{ij_i}} \leq \bar{X}_{j_i} \quad (9)$$

Therefore, by relations (8) and (9), we have $\underline{X}(e')_{j_i} \leq \bar{X}_{j_i}$, $\forall i \in I_2$. This completes the proof. $\square$

Numerical example

Consider the following linear optimization problem (1) in which the feasible region has been randomly generated by Algorithm 2 presented in section 5.1.

objective function of sub-problem (4) and $Z_2 = -7.4603 x_3 - 8.0492 x_6 - 4.4300 x_7$ is that of sub-problem (5). By Definition 2, matrices $U = (U_{ij})_{10 \times 10}$ and $L = (L_{ij})_{10 \times 10}$ are as follows:

$$U = \begin{bmatrix} 1.0000 & 0.7337 & 0.6822 & 1.0000 & 1.0000 & 0.6395 & 0.5635 & 1.0000 & 1.0000 & 1.0000 \\ 0.0955 & 1.0000 & 1.0000 & 0.4934 & 0.2423 & 0.6598 & 0.6610 & 0.1580 & 1.0000 & 0.1766 \\ 0.2504 & 0.3105 & 0.9074 & 0.3640 & 0.4466 & 0.5870 & 0.3320 & 0.5076 & 0.5608 & 0.8479 \\ 1.0000 & 0.9784 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.2526 & 0.3302 & 1.0000 & 0.9247 & 1.0000 & 0.2035 & 0.1847 & 0.1905 & 0.1824 & 0.9701 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 0.8628 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 1.0000 & 0.7529 & 0.8035 & 1.0000 & 0.5670 & 0.9873 & 1.0000 & 0.7389 & 0.9572 & 1.0000 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.1711 & 0.2506 & 0.0980 & 0.2171 & 0.2991 & 0.8357 & 0.2771 & 0.4850 & 1.0000 & 0.2293 \\ 0.4688 & 1.0000 & 1.0000 & 0.6270 & 1.0000 & 1.0000 & 0.9528 & 0.8242 & 1.0000 & 0.6329 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.4496 & 0.3322 & 0.0592 & 0.2422 & 0.4246 & 0.3591 & 0.1983 & 0.0562 & 0.0922 & 0.9415 \\ 0.0923 & 0.8195 & 0.2671 & 0.2886 & 0.1620 & 0.1788 & 0.8846 & \infty & 0.5566 & 0.3627 \\ 0.0866 & 0.0949 & 0.1765 & 0.1470 & 0.1749 & 0.1012 & 0.2203 & 0.6536 & 0.2338 & 0.9741 \\ \infty & 0.2187 & 0.4138 & 0.1676 & \infty & 0.3326 & 0.1714 & 0.1699 & 0.7490 & 0.1612 \\ 0.0924 & 0.3202 & 0.3801 & 0.2621 & 0.4125 & 0.3036 & 0.8524 & \infty & 0.1197 & 0.1228 \\ 0.1678 & 0.4718 & 0.2121 & 0.1918 & 0.4190 & 0.7900 & 0.3180 & 0.1605 & \infty & 0.2720 \\ 0.2534 & 0.1770 & \infty & 0.9300 & 0.5652 & 0.4875 & 0.4823 & 0.2938 & 0.2850 & \infty \\ 0.8508 & 0.9452 & 0.7773 & 0.5930 & 0.1991 & 0.2195 & 0.1323 & 0.3181 & 0.3087 & 0.1569 \\ 0.5417 & 0.1714 & 0.0742 & 0.0724 & 0.2036 & 0.1731 & 0.1532 & 0.2358 & 0.3099 & 0.1844 \\ 0.5824 & \infty & 0.8427 & \infty & 0.1642 & 0.5097 & 0.1441 & 0.7264 & 0.1437 & 0.3877 \end{bmatrix}$$

Therefore, by Corollary 3 we have, for example:

$$S_{T_{SW}^\lambda}(a_{21}, b_2^1) = [0, U_{21}] = [0, 0.0955]$$

$$\text{and } S_{T_{SW}^\lambda}(a_{91}, b_9^1) = [0, U_{91}] = [0, 0.1711].$$

$$S_{T_{SW}^\lambda}(d_{11}, b_1^2) = [L_{11}, 1] = [0.4496, 1]$$

$$\text{and } S_{T_{SW}^\lambda}(d_{63}, b_6^2) = [L_{63}, 1] = [0.2121, 1].$$

Also, from Definition 1,

$$J_1^2 = J_3^2 = J_8^2 = J_9^2 = J = \{1, 2, \dots, 10\},$$

$$J_2^2 = J_5^2 = J - \{8\}, \quad J_6^2 = J - \{9\},$$

$$J_7^2 = J - \{3, 10\} \text{ and } J_{10}^2 = J - \{2, 4\}.$$

By Definition 5, we have

Moreover, the only components of matrix D such that $d_{ij} < b_i^2$ are as follows:

d_{28} (in the second row), d_{41} , d_{45} (in the fourth row), d_{58} (in the fifth row), d_{69} (in the sixth row), d_{73} , $d_{7,10}$ (in the seventh row) and $d_{10,2}$, $d_{10,4}$ (in the tenth row).

Therefore, by Lemma 2 (part (b)),

$$S_{T_{SW}^\lambda}(d_i, b_i^2) = \bigcup_{j=1}^{10} S_{T_{SW}^\lambda}(d_{ij}, b_i^2) \neq \emptyset,$$

$$\forall i \in I_2.$$

$$\begin{aligned}
 \bar{X}(1) &= [1 \quad 0.7337 \quad 0.6822 \quad 1 \quad 1 \quad 0.6395 \quad 0.5635 \quad 1 \quad 1 \quad 1] \\
 \bar{X}(2) &= [0.0955 \quad 1 \quad 1 \quad 0.4934 \quad 0.2423 \quad 0.6598 \quad 0.6610 \quad 0.1580 \quad 1 \quad 0.1766] \\
 \bar{X}(3) &= [0.2504 \quad 0.3105 \quad 0.9074 \quad 0.3640 \quad 0.4466 \quad 0.5870 \quad 0.3320 \quad 0.5076 \quad 0.5608 \quad 0.8479] \\
 \bar{X}(4) &= [1 \quad 0.9784 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \\
 \bar{X}(5) &= [0.2526 \quad 0.3302 \quad 1 \quad 0.9247 \quad 1 \quad 0.2035 \quad 0.1847 \quad 0.1905 \quad 0.1824 \quad 0.9701] \\
 \bar{X}(6) &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.8628 \quad 1 \quad 1 \quad 1 \quad 1] \\
 \bar{X}(7) &= [1 \quad 0.7529 \quad 0.8035 \quad 1 \quad 0.5670 \quad 0.9873 \quad 1 \quad 0.7389 \quad 0.9572 \quad 1] \\
 \bar{X}(8) &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \\
 \bar{X}(9) &= [0.1711 \quad 0.2506 \quad 0.0980 \quad 0.2171 \quad 0.2991 \quad 0.8357 \quad 0.2771 \quad 0.4850 \quad 1 \quad 0.2293] \\
 \bar{X}(10) &= [0.4688 \quad 1 \quad 1 \quad 0.6270 \quad 1 \quad 1 \quad 0.9528 \quad 0.8242 \quad 1 \quad 0.6329]
 \end{aligned}$$

Also, for example

$$\begin{aligned}
 \underline{X}(4,2) &= [0 \quad 0.2187 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \underline{X}(4,3) = [0 \quad 0 \quad 0.4138 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \\
 \underline{X}(4,4) &= [0 \quad 0 \quad 0 \quad 0.1676 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \underline{X}(4,6) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.3326 \quad 0 \quad 0 \quad 0 \quad 0] \\
 \underline{X}(4,7) &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1714 \quad 0 \quad 0 \quad 0], \underline{X}(4,8) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1699 \quad 0 \quad 0] \\
 \underline{X}(4,9) &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.7490 \quad 0], \underline{X}(4,10) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1612]
 \end{aligned}$$

Therefore, by Theorem 1, $S_{T_{SW}^\lambda}(a_i, b_i^1) = [\mathbf{0}, \bar{X}(i)]$, $\forall i \in I_1$, and for example

$$S_{T_{SW}^\lambda}(d_4, b_4^2) = \bigcup_{j=2}^4 [\underline{X}(4, j), \mathbf{1}] \cup \bigcup_{j=6}^{10} [\underline{X}(4, j), \mathbf{1}],$$

for the third row of matrix D (i.e., $i = 4 \in I_2$). From Corollary 4, the necessary condition holds for the feasibility of the problem. More precisely, we have

$$D\varphi\mathbf{1} = \begin{bmatrix} 0.9810 \\ 0.9880 \\ 0.9995 \\ 0.8181 \\ 0.8852 \\ 0.9289 \\ 0.9426 \\ 0.7791 \\ 0.8865 \\ 0.9047 \end{bmatrix} \geq \begin{bmatrix} 0.0492 \\ 0.0876 \\ 0.0864 \\ 0.0810 \\ 0.0471 \\ 0.1292 \\ 0.1511 \\ 0.0392 \\ 0.0291 \\ 0.1028 \end{bmatrix} = b^2$$

that means $\mathbf{1} \in S_{T_{SW}^\lambda}(D, b^2)$.

From Definition 7,

$$\bar{X} = [0.0955 \quad 0.2506 \quad 0.0980 \quad 0.2171 \quad 0.2423 \quad 0.2035 \quad 0.1847 \quad 0.1580 \quad 0.1824 \quad 0.1766]$$

which determines the feasible region of the first inequalities, i.e., $S_{T_{SW}^\lambda}(A, b^1) = [\mathbf{0}, \bar{X}]$ (Theorem 2, part (a)). Also, Therefore, we have $\bar{X} \in S_{T_{SW}^\lambda}(D, b^2)$, which satisfies the necessary feasibility condition stated in Corollary 6.

$$D\varphi\bar{X} = \begin{bmatrix} 0.1497 \\ 0.1589 \\ 0.2399 \\ 0.1241 \\ 0.1024 \\ 0.1522 \\ 0.2218 \\ 0.0838 \\ 0.1629 \\ 0.1737 \end{bmatrix} \geq \begin{bmatrix} 0.0492 \\ 0.0876 \\ 0.0864 \\ 0.0810 \\ 0.0471 \\ 0.1292 \\ 0.1511 \\ 0.0392 \\ 0.0291 \\ 0.1028 \end{bmatrix} = b^2$$

On the other hand, from Definition 6, we have $|E_D| = 3732480000$. Therefore, the number of all vectors $e \in E_D$ is equal to 3732480000. However, each solution $\underline{X}(e)$ generated by vectors $e \in E_D$ is not necessary a feasible solution. For example, for $e' = [1, 7, 2, 3, 9, 4, 2, 5, 6, 9]$, we attain from Definition 7

$$\begin{aligned} \underline{X}(e') &= \max_{i \in I_2} \left\{ \underline{X}(i, e'(i)) \right\} \\ &= \max \left\{ \underline{X}(1, 1), \underline{X}(2, 7), \underline{X}(3, 2), \underline{X}(4, 3), \underline{X}(5, 9), \underline{X}(6, 4), \underline{X}(7, 2), \underline{X}(8, 5), \underline{X}(9, 6), \underline{X}(10, 9) \right\} \end{aligned}$$

Where

$$\begin{aligned} \underline{X}(1, 1) &= [0.4496 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ \underline{X}(2, 7) &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0.8846 \ 0 \ 0 \ 0] \\ \underline{X}(3, 2) &= [0 \ 0.0949 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ \underline{X}(4, 3) &= [0 \ 0 \ 0.4138 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \\ \underline{X}(5, 9) &= [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0.1197 \ 0] \\ \underline{X}(6, 4) &= [0 \ 0 \ 0 \ 0.1918 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \end{aligned}$$

$$\tilde{A} = \begin{bmatrix} 0 & 0.6557 & 0.7060 & 0 & 0 & 0.7513 & 0.8407 & 0 & 0 & 0 \\ 0.9706 & 0 & 0 & 0.3816 & 0.6797 & 0.2551 & 0.2543 & 0.8308 & 0 & 0.7943 \\ 0.9572 & 0.8491 & 0.2769 & 0.7655 & 0.6551 & 0.5060 & 0.8143 & 0.5853 & 0.5308 & 0.3112 \\ 0 & 0.9340 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8003 & 0.6787 & 0 & 0.1869 & 0 & 0.8909 & 0.9293 & 0.9172 & 0.9340 & 0.1656 \\ 0 & 0 & 0 & 0 & 0 & 0.9593 & 0 & 0 & 0 & 0 \\ 0 & 0.7431 & 0.6948 & 0 & 0.9597 & 0.5472 & 0 & 0.7572 & 0.5688 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7922 & 0.6555 & 0.9502 & 0.7094 & 0.5853 & 0.1493 & 0.6160 & 0.3804 & 0 & 0.6892 \\ 0.9595 & 0 & 0 & 0.7547 & 0 & 0 & 0.4733 & 0.5678 & 0 & 0.7482 \end{bmatrix}$$

Also, by Definition 9, we can change the value of components $d_{11}, d_{12}, d_{14}, d_{15}, d_{16}, d_{17}, d_{1,10}$; $d_{22}, d_{23}, d_{24}, d_{27}, d_{29}, d_{2,10}$; $d_{37}, d_{38}, d_{39}, d_{3,10}$; $d_{43}, d_{46}, d_{48}, d_{49}$; $d_{52}, d_{53}, d_{54}, d_{55}, d_{56}, d_{57}$; $d_{61}, d_{62}, d_{63}, d_{65}, d_{66}, d_{67}, d_{68}, d_{6,10}$; $d_{71}, d_{74}, d_{75}, d_{76}, d_{77}, d_{78}, d_{79}$; $d_{81}, d_{82}, d_{83}, d_{84},$

$$\underline{X}(7, 2) = [0 \ 0.1770 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$\underline{X}(8, 5) = [0 \ 0 \ 0 \ 0 \ 0.1991 \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$\underline{X}(9, 6) = [0 \ 0 \ 0 \ 0 \ 0 \ 0.1731 \ 0 \ 0 \ 0 \ 0]$$

$$\underline{X}(10, 9) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0.1437 \ 0]$$

Therefore,

$$\underline{X}(e') = [0.4496 \ 0.0949 \ 0.4138 \ 0.1918 \ 0.1991 \ 0.1731 \ 0.8846 \ 0 \ 0.1437 \ 0]$$

It is obvious that $\underline{X}(e') \not\leq \bar{X}$ (actually, $\underline{X}(e')_1 > \bar{X}_1$, $\underline{X}(e')_3 > \bar{X}_3$ and $\underline{X}(e')_7 > \bar{X}_7$) which means

$\underline{X}(e') \notin S_{T_{SW}^A}(A, D, b^1, b^2)$ from Theorem 3.

From the first simplification (Lemma 5), “resetting the following components a_{ij} to zeros” are equivalence operations: $a_{11}, a_{14}, a_{15}, a_{18}, a_{19}, a_{1,10}; a_{22}, a_{23}, a_{29}; a_{4j} (j \in J - \{2\}); a_{53}, a_{56}; a_{6j} (j \in J - \{6\}); a_{71}, a_{74}, a_{77}, a_{7,10}; a_{8j} (j \in J); a_{99}; a_{10,2}, a_{10,3}, a_{10,5}, a_{10,6}, a_{10,9}$. So, matrix $\tilde{A}$ is resulted as follows:

$d_{86}, d_{88}, d_{89}; d_{91}, d_{98}, d_{99}, d_{9,10}; d_{10,1}, d_{10,3}, d_{10,6}, d_{10,8}, d_{10,10}$ to zeros. For example, since $5 \in J_5^2$ and $L_{55} = 0.4125 > 0.2423 = \bar{X}_5$, then $\tilde{d}_{55} = 0$. Simplified matrix $\tilde{D}$ is obtained as follows:

$$\tilde{D} = \begin{bmatrix} 0 & 0 & 0.9730 & 0 & 0 & 0 & 0 & 0.9810 & 0.8909 & 0 \\ 0.9880 & 0 & 0 & 0 & 0.8314 & 0.7984 & 0 & 0.0855 & 0 & 0 \\ 0.9995 & 0.9787 & 0 & 0.8594 & 0.8034 & 0.9631 & 0 & 0 & 0 & 0 \\ 0.0377 & 0.7127 & 0 & 0.8055 & 0.0605 & 0 & 0.7982 & 0 & 0 & 0.8181 \\ 0.8852 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0292 & 0.8241 & 0.8175 \\ 0 & 0 & 0 & 0.8643 & 0 & 0 & 0 & 0 & 0.0305 & 0 \\ 0 & 0.9426 & 0.0835 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1499 \\ 0 & 0 & 0 & 0 & 0.6569 & 0 & 0.7791 & 0 & 0 & 0.7311 \\ 0 & 0.6820 & 0.8822 & 0.8865 & 0.6280 & 0.6791 & 0.7150 & 0 & 0 & 0 \\ 0 & 0.0424 & 0 & 0.0287 & 0.8612 & 0 & 0.9037 & 0 & 0.9047 & 0 \end{bmatrix}$$

Additionally, $\tilde{J}_1^2 = \{3, 8, 9\}$, $\tilde{J}_2^2 = \{1, 5, 6\}$,
 $\tilde{J}_3^2 = \{1, 2, 4, 5, 6\}$, $\tilde{J}_4^2 = \{2, 4, 7, 10\}$,
 $\tilde{J}_5^2 = \{1, 9, 10\}$, $\tilde{J}_6^2 = \{4\}$, $\tilde{J}_7^2 = \{2\}$,
 $\tilde{J}_8^2 = \{5, 7, 10\}$, $\tilde{J}_9^2 = \{2, 3, 4, 5, 6, 7\}$ and
 $\tilde{J}_{10}^2 = \{5, 7, 9\}$. Based on these results and

Lemma 7, we have $|E_{\tilde{D}}| = |E'_D| = 29160$.

Therefore, the simplification processes reduced the number of the minimal candidate solutions from 3732480000 to 29160, by removing 3732450840 infeasible points $\underline{X}(e)$. Consequently, the feasible region has 29160 minimal candidate solutions, which are feasible. In other words, for each $e \in E_{\tilde{D}}$,

we have $\underline{X}(e) \in S_{T_{SW}^{\lambda}}(A, D, b^1, b^2)$. However,

each feasible solution $\underline{X}(e)$ ($e \in E_{\tilde{D}}$) may not be a minimal solution for the problem. For example, by selecting $e' = [9, 1, 2, 4, 10, 4, 2, 5, 6, 7]$, the corresponding solution is obtained as

$$\underline{X}(e') = [0.0923 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0.1731 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0.1228]$$

. Although $\underline{X}(e')$ is feasible (because of the inequality $\underline{X}(e') \leq \bar{X}$) but it is not actually a minimal solution. To see this, let $e'' = [9, 5, 2, 4, 10, 4, 2, 7, 2, 7]$. Then,

$$\underline{X}(e'') = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0.1228]$$

. Obviously, $\underline{X}(e'') \leq \underline{X}(e')$ which shows that $\underline{X}(e')$ is not a minimal solution.

Now, we obtain the modified matrix L^* according to Definition 10:

$$L^* = \begin{bmatrix} \infty & \infty & 0.0592 & \infty & \infty & \infty & \infty & 0.0562 & 0.0922 & \infty \\ 0.0923 & \infty & \infty & \infty & 0.1620 & 0.1788 & \infty & \infty & \infty & \infty \\ 0.0866 & 0.0949 & \infty & 0.1470 & 0.1749 & 0.1012 & \infty & \infty & \infty & \infty \\ \infty & 0.2187 & \infty & 0.1676 & \infty & \infty & 0.1714 & \infty & \infty & 0.1612 \\ 0.0924 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0.1197 & 0.1228 \\ \infty & \infty & \infty & 0.1918 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0.1770 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.1991 & \infty & 0.1323 & \infty & \infty & 0.1569 \\ \infty & 0.1714 & 0.0742 & 0.0724 & 0.2036 & 0.1731 & 0.1532 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0.1642 & \infty & 0.1441 & \infty & 0.1437 & \infty \end{bmatrix}$$

As is shown in matrix L^* , for each $i \in I_2$ there exists at least some $j \in J_i^2$ such

that $L_{ij}^* \neq +\infty$. Thus, by Theorem 4 we have

$S_{T_{SW}}^\lambda(A, D, b^1, b^2) \neq \emptyset$. Finally, vector $\overline{X}$ is optimal solution of sub-problem (5). For this solution,

$$Z_2 = -7.4603 X_3 - 8.0492 X_6 - 4.4300 X_7 = -3.1877$$

. Also, $Z = c^T \overline{X} = 5.3428$. In order to find the optimal solution $\underline{X}(e^*)$ of sub-problems (4), we firstly compute all minimal solutions by making pairwise comparisons between all solutions $\underline{X}(e)$ ($\forall e \in E_{\bar{D}}$), and then we find

$\underline{X}(e^*)$ among the resulted minimal solutions. Actually, the feasible region has 38 minimal solutions reported in Appendix A. By comparison of the values of the objective function for the minimal solutions, $\underline{X}(e_1)$ is optimal in (4) (i.e., $e^* = e_1$), where $e_1 = [3, 1, 1, 4, 1, 4, 2, 7, 2, 7]$.

For this vector, we have

$$\underline{X}(e_1) = [0.0924 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0 \quad 0]$$

And

$$\begin{aligned} Z_1 &= \sum_{j=1}^n c_j \underline{X}(e_1)_j \\ &= 6.2945 \underline{X}(e_1)_1 + 8.1158 \underline{X}(e_1)_2 + 8.2675 \underline{X}(e_1)_4 + 2.6472 \underline{X}(e_1)_5 + 0.9376 \underline{X}(e_1)_8 \\ &\quad + 9.1501 \underline{X}(e_1)_9 + 9.2978 \underline{X}(e_1)_{10} \\ &= 3.6035 \end{aligned}$$

Also, $Z = c^T \underline{X}(e_1) = 2.523$. Thus, from Corollary 9,

Appendix

The minimal solutions of the feasible solutions set for the problem stated in section 5.2.

$$e_1 = [3, 1, 1, 4, 1, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_1) = [0.0924 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0 \quad 0]$$

$$e_2 = [8, 1, 1, 4, 1, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_2) = [0.0924 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0.0562 \quad 0 \quad 0]$$

$$e_3 = [3, 1, 1, 4, 1, 4, 2, 7, 2, 5]$$

$$x^* = [0.092423 \quad 0.17697 \quad 0.098021 \quad 0.19176 \quad 0 \quad 0.20355 \quad 0.18467 \quad 0 \quad 0 \quad 0]$$

and then $Z^* = c^T x^* = 0.41573$.

Conclusion

In this paper, we proposed an algorithm to find the optimal solution of linear problems subjected to two fuzzy relational inequalities with Sugeno-Weber family of t-norms. The feasible solutions set of the problem is completely resolved and a necessary and sufficient condition and three necessary conditions were presented to determine the feasibility of the problem. Moreover, depending on the max-Sugeno-Weber composition, two simplification operations were proposed to accelerate the solution of the problem. Finally, a method was introduced for generating feasible random max-Sugeno-Weber inequalities. This method was used to generate a test problem for our algorithm. The resulted test problem was then solved by the proposed algorithm. As future works, we aim at testing our algorithm in other type of linear optimization problems whose constraints are defined as FRI with other well-known t-norms.

Acknowledgment

We are very grateful to the anonymous referees and the editor in chief for their comments and suggestions, which were very helpful in improving the paper.

$$\underline{X}(e_3)=[0.0924 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0 \quad 0 \quad 0]$$

$$e_4=[8, 1, 1, 4, 1, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_4)=[0.0924 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0.0562 \quad 0 \quad 0]$$

$$e_5=[9, 6, 2, 4, 9, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_5)=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0.1441 \quad 0 \quad 0.1197 \quad 0]$$

$$e_6=[3, 1, 1, 4, 1, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_6)=[0.0924 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$e_7=[3, 6, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_7)=[0 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0.1441 \quad 0 \quad 0 \quad 0.1228]$$

$$e_8=[8, 1, 1, 4, 1, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_8)=[0.092 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0.0562 \quad 0 \quad 0]$$

$$e_9=[8, 6, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_9)=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0.1441 \quad 0.0562 \quad 0 \quad 0.1228]$$

$$e_{10}=[9, 6, 2, 4, 9, 4, 2, 7, 2, 9]$$

$$\underline{X}(e_{10})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0.1323 \quad 0 \quad 0.1437 \quad 0]$$

$$e_{11}=[9, 1, 1, 4, 1, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{11})=[0.0924 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0]$$

$$e_{12}=[9, 5, 2, 4, 9, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{12})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1441 \quad 0 \quad 0.1197 \quad 0]$$

$$e_{13}=[9, 5, 2, 4, 9, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_{13})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0 \quad 0.1197 \quad 0]$$

$$e_{14}=[3, 5, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{14})=[0 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1441 \quad 0 \quad 0 \quad 0.1228]$$

$$e_{15}=[3, 5, 2, 4, 10, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_{15})=[0 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0 \quad 0 \quad 0.1228]$$

$$e_{16}=[9, 5, 2, 4, 9, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_{16})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0 \quad 0.1197 \quad 0]$$

$$e_{17}=[8, 5, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{17})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1441 \quad 0.0562 \quad 0 \quad 0.1228]$$

$$e_{18}=[8, 5, 2, 4, 10, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_{18})=[0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0.0562 \quad 0 \quad 0.1228]$$

$$e_{19} = [3, 5, 2, 4, 10, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_{19}) = [0 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1228]$$

$$e_{20} = [9, 1, 1, 4, 9, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{20}) = [0.0923 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0.1197 \quad 0]$$

$$e_{21} = [8, 5, 2, 4, 10, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_{21}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0.0562 \quad 0 \quad 0.1228]$$

$$e_{22} = [3, 1, 1, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{22}) = [0.0923 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0 \quad 0.1228]$$

$$e_{23} = [9, 5, 2, 4, 9, 4, 2, 7, 2, 9]$$

$$\underline{X}(e_{23}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1323 \quad 0 \quad 0.1437 \quad 0]$$

$$e_{24} = [8, 1, 1, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{24}) = [0.0923 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0.0562 \quad 0 \quad 0.1228]$$

$$e_{25} = [9, 1, 1, 4, 1, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_{25}) = [0.0924 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0 \quad 0.0922 \quad 0]$$

$$e_{26} = [3, 5, 2, 4, 10, 4, 2, 10, 2, 5]$$

$$\underline{X}(e_{26}) = [0 \quad 0.1770 \quad 0.0592 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1570]$$

$$e_{27} = [9, 1, 1, 4, 9, 4, 2, 7, 2, 9]$$

$$\underline{X}(e_{27}) = [0.0923 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1323 \quad 0 \quad 0.1437 \quad 0]$$

$$e_{28} = [8, 5, 2, 4, 10, 4, 2, 10, 2, 5]$$

$$\underline{X}(e_{28}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0 \quad 0.0562 \quad 0 \quad 0.1570]$$

$$e_{29} = [9, 1, 1, 4, 1, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_{29}) = [0.0924 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0 \quad 0.0922 \quad 0]$$

$$e_{30} = [9, 6, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{30}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0.1228]$$

$$e_{31} = [9, 5, 2, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{31}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0.1228]$$

$$e_{32} = [9, 5, 2, 4, 10, 4, 2, 7, 2, 5]$$

$$\underline{X}(e_{32}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0.1323 \quad 0 \quad 0.0922 \quad 0.1228]$$

$$e_{33} = [9, 5, 2, 4, 10, 4, 2, 5, 2, 5]$$

$$\underline{X}(e_{33}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1991 \quad 0 \quad 0 \quad 0 \quad 0.0922 \quad 0.1228]$$

$$e_{34} = [9, 1, 1, 4, 10, 4, 2, 7, 2, 7]$$

$$\underline{X}(e_{34}) = [0.0923 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0 \quad 0.1441 \quad 0 \quad 0.0922 \quad 0.1228]$$

$$e_{35} = [9, 5, 2, 4, 10, 4, 2, 10, 2, 5]$$

$$\underline{X}(e_{35}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1642 \quad 0 \quad 0 \quad 0 \quad 0.0922 \quad 0.1570]$$

$$e_{36} = [9, 6, 2, 4, 9, 4, 2, 10, 2, 9]$$

$$\underline{X}(e_{36}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0 \quad 0.1788 \quad 0 \quad 0 \quad 0.1437 \quad 0.1570]$$

$$e_{37} = [9, 5, 2, 4, 9, 4, 2, 10, 2, 9]$$

$$\underline{X}(e_{37}) = [0 \quad 0.1770 \quad 0 \quad 0.1918 \quad 0.1620 \quad 0 \quad 0 \quad 0 \quad 0.1437 \quad 0.1570]$$

$$e_{38} = [9, 1, 1, 4, 9, 4, 2, 10, 2, 9]$$

References

1. Pedrycz W (2013) Granular Computing: Analysis and Design of Intelligent Systems. CRC Press, BocaRaton.
2. Loetamonphong J, Fang SC (2001) Optimization of fuzzy relation equations with max-product composition. Fuzzy Sets and Systems; 118: (3): 509-517.
3. Lin JL (2009) On the relation between fuzzy max-archimedean t-norm relational equations and the covering problem. Fuzzy Sets and Systems; 160(16): 2328-2344.
4. Guu SM, Wu YK (2002) Minimizing a linear objective function with fuzzy relation equation constraints. Fuzzy Optimization and Decision Making; 1 (4): 347-360.
5. Chen L, Wang PP (2002) Fuzzy relation equations (i): the general and specialized solving algorithms. Soft Computing; 6(5): 428-435.
6. Chen L, Wang PP (2007) Fuzzy relation equations (ii): the branch-point-solutions and the categorized minimal solutions. Soft Computing; 11(1): 33-40.
7. Lin JL, Wu YK, Guu SM (2011) On fuzzy relational equations and the covering problem. Information Sciences; 181(14): 2951-2963.
8. Markovskii AV (2005) On the relation between equations with max-product composition and the covering problem. Fuzzy Sets and Systems; 153(2): 261-273.
9. Nola A Di, Sessa S, Pedrycz W et al. (1989) Fuzzy relational Equations and their applications in knowledge engineering. Dordrecht: Kluwer Academic Press.
10. Peeva K (2013) Resolution of fuzzy relational equations-methods, algorithm and software with applications. Information Sciences; 234: 44-63.
11. Perfilieva I (2013) Finitary solvability conditions for systems of fuzzy relation equations. Information Sciences; 234:29-43.
12. Qu XB, Wang XP, Man-hua. H. Lei (2014) Conditions under which the solution sets of fuzzy relational equations over complete Brouwerian lattices form lattices. Fuzzy Sets and Systems; 234: 34-45.
13. Shieh BS (2008) Infinite fuzzy relation equations with continuous t-norms. Information Sciences; 178(8): 1961-1967.
14. Sun F, Wang XP, Qu xB (2013) Minimal join decompositions and their applications to fuzzy relation equations over complete Brouwerian lattices. Information Sciences; 224: 143-151.

15. Sun F (2012) Conditions for the existence of the least solution and minimal solutions to fuzzy relation equations over complete Brouwerian lattices. *Information Sciences*; 205: 86-92.
16. Wu YK, Guu SM(2008) An efficient procedure for solving a fuzzy relation equation with max-Archimedean t-norm composition. *IEEE Transactions on Fuzzy Systems*; 16(1): 73-84.
17. Wu YK (2007) Optimization of fuzzy relational equations with max-av composition. *Information Sciences*; 177(19): 4216-4229.
18. Yeh CT (2008) On the minimal solutions of max-min fuzzy relation equations. *Fuzzy Sets and Systems*; 159(1): 23-39.
19. Ghodousian A, Khorram E (2012) Linear optimization with an arbitrary fuzzy relational inequality. *Fuzzy Sets and Systems*; 206: 89-102.
20. Guo FF, Pang LP, Meng D et al. (2013) An algorithm for solving optimization problems with fuzzy relational inequality constraints. *Information Sciences*; 252: 20-31.
21. Guo F, Xia ZQ (2006) An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities. *Fuzzy Optimization and Decision Making*; 5 (1): 33-47.
22. Li JX, Yang SJ (2012) Fuzzy relation inequalities about the data transmission mechanism in bittorrent-like peer-to-peer file sharing systems. in: *Proceedings of the 9th International Conference on Fuzzy Systems and Knowledge discovery (FSKD 2012)*, pp. 452-456.
23. Yang XP, Zhou XG, Cao BY (2016) Latticized linear programming subject to max-product fuzzy relation inequalities with application in wireless communication. *Information Sciences*; 358-359 : 44-55.
24. Chang CW, Shieh BS (2013) Linear optimization problem constrained by fuzzy max-min relation equations. *Information Sciences*; 234: 71-79.
25. Fan YR, Huang GH, Yang AL (2013) Generalized fuzzy linear programming for decision making under uncertainty: Feasibility of fuzzy solutions and solving approach. *Information Sciences*; 241: 12-27.
26. Ghodousian A (2018) Optimization of the reducible objective functions with monotone factors subject to FRI constraints defined with continuous t-norms. *Archives of Industrial Engineering*; 1(1): 1-19.
27. Ghodousian A, Babalhavaeji A (2018) An efficient genetic algorithm for solving nonlinear optimization problems defined with fuzzy relational equations and max-Lukasiewicz composition. *Applied Soft Computing*; 69: 475-492.
28. Ghodousian A, Naeemib M, Babalhavaeji A (2018) Nonlinear optimization problem subjected to fuzzy relational equations defined by Dubois-Prade family of t-norms. *Computers & Industrial Engineering*; 119: 167-180.
29. Ghodousian A, Raeisian Parvari M (2017) A modified PSO algorithm for linear optimization problem subject to the generalized fuzzy relational inequalities with fuzzy constraints (FRI-FC). *Information Sciences*; 418-419: 317-345.
30. Ghodousian R, Zarghani (2017) Linear optimization on the intersection of two fuzzy relational inequalities defined with Yager family of t-norms. *Journal of Algorithms and Computation*; 49(1): 55-82.

31. Ghodousian A, Ahmadi A, Dehghani (2017) Solving a non-convex non-linear optimization problem constrained by fuzzy relational equations and Sugeno-Weber family of t-norms. *Journal of Algorithms and Computation*; 49(2): 63-101.
32. Ghodousian A, Khorram E (2006) An algorithm for optimizing the linear function with fuzzy relation equation constraints regarding max-prod composition. *Applied Mathematics and Computation*; 178(2):502-509.
33. Ghodousian A, Khorram E (2008) Fuzzy linear optimization in the presence of the fuzzy relation inequality constraints with max-min composition. *Information Sciences*; 178(2): 501-519.
34. Ghodousian A, Khorram E (2006) Solving a linear programming problem with the convex combination of the max-min and the max-average fuzzy relation equations. *Applied Mathematics and computation*; 180(1): 411-418.
35. Guu SM, Wu YK (2010) Minimizing a linear objective function under a max-t-norm fuzzy relational equation constraint. *Fuzzy Sets and Systems*; 161(2): 285-297.
36. Guu SM, Wu YK (2002) Minimizing a linear objective function with fuzzy relation equation constraints, *Fuzzy Optimization and Decision Making*; 12: 1568-4539.
37. Guu SM, Wu YK (2010) Minimizing an linear objective function under a max-t-norm fuzzy relational equation constraint. *Fuzzy Sets and Systems*; 161(2): 285-297.
38. Khorram E, Ghodousian A (2006) Linear objective function optimization with fuzzy relation equation constraints regarding max-av composition, *Applied Mathematics and Computation*; 173(2): 872-886.
39. Lee HC, Guu SM (2002) On the optimal three-tier multimedia streaming services. *Fuzzy Optimization and Decision Making* 2(1) (2002) 31-39.
40. Li PK, Fang SC (2008) On the resolution and optimization of a system of fuzzy relational equations with sup-t composition. *Fuzzy Optimization and Decision Making*; 7(2): 169-214.
41. Liu CC, Lur YY, Wu YK (2016) Linear optimization of bipolar fuzzy relational equations with max-Łukasiewicz composition. *Information Sciences* ; 360(C):149-162.
42. Qu XB, Wang XP (2008) Minimization of linear objective functions under the constraints expressed by a system of fuzzy relation equations. *Information Sciences*; 178(17): 3482-3490.
43. Shieh BS (2011) Minimizing a linear objective function under a fuzzy max-t-norm relation equation constraint. *Information Sciences*; 181(4): 832-841.
44. Fang SC, Li G (1999) Solving fuzzy relational equations with a linear objective function. *Fuzzy Sets and Systems*; 103(1): 107-113.
45. Wu YK, Guu SM (2005) Minimizing a linear function under a fuzzy max-min relational equation constraints. *Fuzzy Sets and Systems*; 150(1): 147-162.
46. Dubey D, Chandra S, Mehra A (2012) Fuzzy linear programming under interval uncertainty based on IFS representation. *Fuzzy Sets and Systems*; 188(1): 68-87.
47. Freson S, Baets B De, Meyer H De (2013) Linear optimization with bipolar max-min constraints. *Information Sciences*; 234: 3-15.
48. Li P, Liu Y (2014) Linear optimization with bipolar fuzzy relational equation

- constraints using lukasiewicz triangular norm. *Soft Computing*; 18(7): 1399-1404.
49. Wu YK, Guu SM, Liu JY (2008) Reducing the search space of a linear fractional programming problem under fuzzy relational equations with max-Archimedean t-norm composition. *Fuzzy Sets and Systems*; 159(24): 3347-3359.
50. Dempe S, Ruziyeva A (2012) On the calculation of a membership function for the solution of a fuzzy linear optimization problem. *Fuzzy Sets and Systems*; 188(1): 58-67.
51. Yang SJ (2014) An algorithm for minimizing a linear objective function subject to the fuzzy relation inequalities with addition-min composition. *Fuzzy Sets and Systems*; 255: 41-51.
52. Dubois D, Prade H (2000) *Fundamentals of Fuzzy Sets*. Kluwer, Boston.
53. Li P, Fang SC (2009) A survey on fuzzy relational equations, part I: classification and solvability. *Fuzzy Optimization and Decision Making*; 8(2): 179-229.
54. Mizumoto M, Zimmermann HJ (1982) Comparison of fuzzy reasoning method. *Fuzzy Sets and Systems*; 8(3): 253-283.
55. Sanchez E (1977) Solution in composite fuzzy relation equations: application to medical diagnosis in Brouwerian logic, in: M.M. Gupta. G.N. Saridis, B.R. Games (Eds.). *Fuzzy Automata and Decision Processes*, North-Holland, New York, 1977, pp. 221-234.
56. Xiong QQ, Wang XP (2012) Fuzzy relational equations on complete Brouwerian lattices. *Information Sciences*; 193: 141-152.

Citation: Ghodousian A. (2018), "Optimizing of linear problems subjected to Sugeno-Weber FRI". *Arch Ind Engg*; 1(1): 1-25.

DOI 10.31829/2637-9252/aie2018-1(1)-105

Copyright: © 2018 Ghodousian A. Azarnejad T. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original *author* and source are credited.