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Schumpeterian evolution of firms' capital-labor ratio distribution¹

Abstract. According to J. Schumpeter innovation and imitation are two key drivers of economic growth. A quantitative realization of this idea using the formalism of kinetic equations was described in a number of papers. In most of these studies only one firm efficiency factor, the total factor productivity, was considered. In general, a description of economic evolution should include more efficiency factors such as, e.g., total factor productivity (TFP) and capital-labor ratio. The present study makes a preliminary step in the direction of two factor model development by considering central planner's problem of endogenous growth driven by the capital-labor ratio. The model describes an evolution of a distribution of firms on an odal developmentefficient path by considering a difference-differential analogue of the Burgers' type equation operating at a set of discrete capital-labor ratio levels. It is shown that if investment efficiency does not depend on the investment size, and production is characterised by decreasing returns to scale then firms concentrate at a certain level of capital-labor ratio. In the case of decreasing efficiency of investment with respect to its size, one observes widening of the distribution of firms in the capital-labor ratio. In addition, it is shown that the latter result holds in the case of increasing returns to scale.

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1. Introduction

A description of endogenous economic growth driven by innovations and imitations proposed by Joseph Schumpeter in his famous work (Schumpeter, 1961) triggered one of the most important directions of research on the topic (Aghion, Howitt, 1998). This idea is that there exist two different possibilities for firms to maximize their performance. The first is to invest in R&D and benefit from the competitive advantage of leadership. The second is not to invest much funds into research, but rather to imitate, or, in other words, to profit from following the leader. While an effectiveness of the first strategy mostly depends on the characteristics of the specific firm, that of the second one is in addition influenced by the firm's "neighbourhood", i.e. by a set of technologies available for imitation. Therefore, the model of knowledge spillovers over a network of firms is probably the best representation of the technology diffusion process. It should also be noted that in practice different technologies are likely to spread over the network in different ways, in particular because of being used in the particular sector. Therefore, a general description of economic growth requires a multi-sectoral approach.

¹ We are very grateful to V.M. Polterovich for discussions that had a profound influence on shaping this research. We also thank an anonymous reviewer for valuable comments.

However, development of such multidimensional picture seems very difficult – in particular, because a number of questions related to simplified one-sector models is still not clarified. In the present study we restrict our analysis to the one-sector case, i.e. assume that all the technologies are specific for the sector and, therefore, spread only within it. In such a setting a single-product model of endogenous growth can be interpreted as a model of the evolution of this particular sector. In this case the natural starting point for the analysis is to use some kind of the “mean-field approximation» for the single-product economy in which one studies the evolution of the distribution of firms by some characteristics of their knowledge stock, e.g., the total factor productivity (TFP). The evolution of such a system can be described in the framework of kinetic equations formalism (see (Polterovich, 2017) and references therein)). In particular, in (Polterovich, Henkin, 1990; Henkin, Polterovich, 1991, 1999; Henkin, 2012) it was shown that evolution of distribution of firms with respect to TFP can be described by the difference-differential analogue of the Burgers' type equation. Namely, in such a case the dynamics of firms yields at a particular TFP level can be described by the travelling wave solution. As shown in (König et al., 2016), the travelling wave provides a good description of the observed time evolution of the distribution of firms by TFP. However, the above-listed models can be characterised as being “absolutely mechanistic”, because they do not assume any strategic behaviour (optimisation) of firms. This element was taken into account in a series of recent papers (Lucas, Moll, 2014; Luttmer, 2012; König et al., 2016; Acemoglu, Cao, 2015) in which evolution of firms distribution by TFP is also determined by some kinetic equation, but some parameters in its right-hand side (RHS) are determined through agent's optimising behaviour. However, as noted in (Polterovich, 2017), all these models consider only one growth factor (the TFP), whereas it is clear that a general description should be multidimensional². In particular, in (Polterovich, 2017) a two-factor model of firms' efficiency was proposed, the second factor being the capital-labor ratio³. This model links the ideas proposed in (Lucas, Moll, 2014) and in (Henkin, Polterovich, 1991). Namely, similar to (Lucas, Moll, 2014), the model belongs to the class of dynamic models of optimal control and, as in (Henkin, Polterovich, 1991), it includes two factors of productivity growth. The quantitative analysis of the model described in (Polterovich, 2017) has so far not been done.

The present study focuses on a model with a single factor of production efficiency. However, in contrast to the studies of (Lucas, Moll, 2014; Luttmer, 2012; König et al., 2016; Acemoglu, Cao, 2015), this factor is the capital-labor ratio. This model can be viewed as a preliminary step in the analysis of the two-factor model of (Polterovich, 2017). Some motivation for the further analysis of the two-factor model is given in Section 3, where exogenous TFP growth is taken into account.

As shown in (Leonardi, 2007), a remarkable property of the firms' distribution in the capital-labor ratio is its tendency to widen with time.

² See also discussion in (Sargent, Rodriguez, 2000) on growth factors alternative to TFP.

³ See also paragraph 3.2 in (Aghion, Howitt, 1998) for discussion on the topic of relationship between physical/human capital accumulation and R&D.

This property contradicts the traditional view assuming existence of the efficient growth path, because the latter property implies convergence of the firms' capital-labor ratio to the same optimal value. The main goal of the analysis in the present paper is to show that firms heterogeneity can result from economically efficient allocation. The corresponding argumentation is presented using the framework of a social planner's problem. We show that a possible explanation of this phenomenon may lie in a heterogeneity of investment efficiency considering its size. Such kind of investment heterogeneity was previously discussed in (Blanchard et al., 1989), where decreasing effectiveness was associated with increasing expenses on new capital launch. Interestingly, introduction of such a property makes it possible to analyse models with increasing returns to scale. The latter fact is of special interest, because increasing returns to scale could result from a learning process (Arrow, 1962). To support this argument, in his famous paper (Arrow, 1962) K. Arrow describes an example from (Wright, 1936), where it was noted that the time necessary for constructing an airframe is a decreasing function of the total number of the same airframes previously produced which can naturally be interpreted as resulting from learning. In addition, the quality of capital utilised in production increases with time resulting in the growth of the unit capital costs efficiency. In Section 4.3 (Example 3c) we analyse the setting with increasing returns to scale and decreasing investment effectiveness and show that firms' distribution with respect to capital-labor ratio levels widens with time. Moreover, this widening might be greater than in the case of decreasing returns.

The overall structure of the paper is the following. The Section 2 presents analysis of the baseline model. A model with exogenous TFP growth is described in the Section 3. The Section 4 presents the model modifications incorporating investment efficiency heterogeneities of two types: the one considering capital-labor ratio level (Paragraph 4.1) and the one considering investment size (Paragraphs 4.2 and 4.3). Some additional comments are provided in the Conclusion.

2. The Model

Let us start with listing the assumptions of the model discussed in the present study. These assumptions are taken directly from (Polterovich, 2017). They read:

- the economy consists of a set of firms and consumers;
- the number of consumers is the same as the number of workers;
- all consumers are equivalent, so the purpose of the central planner is to maximise the welfare of a representative consumer;
- each firm employs only one worker, therefore the capital-labor ratio equals the total capital of a firm;
- the two production efficiency factors in the economy are the capital-labor ratio and TFP;
- the levels of TFP are discrete ($m \in \mathbb{N}$);

- the firms' capital-labor ratio levels are discrete ($n \in \mathbb{N}$);
- firm's output ($g_{mn}(t)$) is completely determined by its levels of capital-labor ratio $n \in \mathbb{N}$ and TFP $m \in \mathbb{N}$;
- the firms' output can be either consumed immediately or invested into growth of one of the production efficiency factors;
- the problem to be solved is the central planner's one⁴.

The only assumption different from formulated in (Polterovich, 2017) is the one on productivity assumed to be fully determined by the capital-labor ratio level $n \in \mathbb{N}$. In our analysis the firm's output is represented by the discrete production function $g_n(t)$, $n \in \mathbb{N}$.

2.1. Problem formulation

Let us introduce the following notations to formulated the problem under study:

- $N_n(t)$, $n \in \mathbb{N}$ is the number of firms at the level n at the moment t ,
- N is the total number of firms (consumers) in the economy,
- $f_n(t) = N_n(t) / N$, $n \in \mathbb{N}$ is the fraction of firms at the level n at the moment t ,
- $F_n = \sum_{k=1}^n f_k$ is the fraction of firms at and below the level n at the moment t ,
- $c(t)$ is the consumption of a representative consumer.

Let us note also that $c(t) = C(t) / N$, where $C(t)$ is total consumption.

- $U(c)$ is the representative consumer's utility function,
- $I(t)$ is the size of investment made at the moment t . To simplify the formulae it is convenient to introduce the following notation for the normalised investment – $i(t) = I(t) / N$.

The expression for the total capital stock in the economy $K(t)$ reads $K(t) = \sum_{n=1}^{\infty} n N_n(t)$. Let us also introduce the mean total capital of a firm:

$$k(t) \equiv K(t) / N = \sum_{n=1}^{\infty} n f_n(t). \quad (1)$$

As the number of firms N is constant in time, the dynamics of $k(t)$ is the same as that of $K(t)$, so in what follows we shall study only the evolution of $k(t)$.

The central planner's problem is to maximise the total welfare of a representative consumer over the fixed time interval $[t_0, T]$, (see eq. (2)) taking into account the equations for the balance constraint (3), and evolution of capital (4) and distribution (5):

$$W = \int_{t_0}^T e^{-\rho(t-t_0)} U(c(t)) dt \rightarrow \max_{s_n(\cdot), n \in \mathbb{N}}, \quad (2)$$

$$c(t) = \sum_{n=1}^{\infty} g_n(t) f_n(t) - i(t), \quad (3)$$

$$i(t) = dk / dt, \quad (4)$$

$$df_n(t) / dt = -\phi(F_n, s_n) f_n + \phi(F_{n-1}, s_{n-1}) f_{n-1}, \quad n \in \mathbb{N}, \quad (5)$$

⁴ In a more realistic description agents act in the market and the central planner is absent. In general, the results in a model with direct interaction of firms and consumers may significantly differ from the ones obtained in a model with central planner. Therefore, to interpret the results correctly some kind of market efficiency should also be assumed. However, as mentioned in the Section 1, the main goal of the current paper is to show that firms' heterogeneity can correspond to the optimal allocation. In this analysis the particular setting of central planners' problem is used.

$$f_n(0) = \text{fix}, \quad n \in \mathbb{N}, \quad (6)$$

where ρ is the discount factor, s_n , $n \in \mathbb{N}$ is the control determining the cost of transition to the next level and $\phi(F_n, s_n)$ is some function (in the simplest case $\phi(\cdot)$ decreases in F_n (Polterovich, 2017).

Let us stress that an assumption on the form of the distribution evolution equation (5) is of key importance for the properties of the model. In this study, following the model proposed in (Polterovich, 2017), the difference-differential analogue of the Burgers' type equation is used. This means that only one-step upstream transitions of the type $n \rightarrow n+1$ are allowed. The evolution equation (5) describes the process of capital-labor ratio growth, therefore one has the following restriction on the set of control values⁵:

$$\phi(F_n, s_n) \geq 0, \quad n \in \mathbb{N}. \quad (7)$$

In what follows for simplicity we assume that an output of a firm at the level n is constant in time: $g_n(t) \equiv g_n$, $n \in \mathbb{N}$.

The dynamics of the overall capital-labor ratio $k = \sum_{n=1}^{\infty} n f_n$ is described by the following equation:

$$\frac{dk}{dt} = \sum_{n=1}^{\infty} \phi(F_n, s_n) f_n. \quad (8)$$

We see that the product $\phi(F_n, s_n) f_n$ stands for the volume of normalized investment into the firms at the level n .

Taking into account (8) one can rewrite the balance equation (3) in the following form:

$$c(t) = \sum_{n=1}^{\infty} (g_n - \phi(F_n, s_n)) f_n \quad (9)$$

Let us denote the problem specified by the equations (2), (9), (5), (6) by $\mathcal{P}$. To solve it we shall use the Pontryagin's maximum principle. The Hamiltonian for $\mathcal{P}$ reads:

$$\mathcal{H} = e^{-\rho(t-t_0)} U(c(t)) + \sum_{n=1}^{\infty} \lambda_n (-\phi(F_n, s_n) f_n + \phi(F_{n-1}, s_{n-1}) f_{n-1}), \quad (10)$$

where $\lambda = (\lambda_n, n \in \mathbb{N})$ is the vector of costate variables. The evolution of state ($\mathbf{f} = (f_n, n \in \mathbb{N})$) and costate variables is described by the following equations:

$$\dot{f}_n^n = \partial \mathcal{H}(\mathbf{f}, \lambda, \hat{\mathbf{s}}) / \partial \lambda^n, \quad (11)$$

$$\dot{\lambda}_n = - \frac{\partial \mathcal{H}(\mathbf{f}, \lambda, \hat{\mathbf{s}})}{\partial f_n} - \sum_{n=1}^{\infty} \frac{\partial \mathcal{H}(\mathbf{f}, \lambda, \mathbf{s})}{\partial s_n} \bigg|_{s_n = \hat{s}_n} \frac{\partial \hat{s}_n}{\partial f_n}, \quad (12)$$

$$f_n(0) = \text{fix}, \quad n \in \mathbb{N}, \quad (13)$$

$$\lambda_n(T) = 0, \quad n \in \mathcal{N}, \quad (14)$$

where $\hat{\mathbf{s}} = \hat{\mathbf{s}}(\mathbf{f}, \lambda, t)$ is the optimal control and (14) is the transversality condition.

2.2. Solution methodology

Let us introduce the notation $u_n = \phi(F_n, s_n)$, $n \in \mathbb{N}$ and note that the control s_n appears in the Hamiltonian and constraints of the problem $\mathcal{P}$ only as an argument of the function $\phi(\cdot)$. Let $\mathcal{P}'$ be the problem obtained

⁵ This follows from the difference-differential analogue of the Burgers' type equation describing the time evolution of $F_n(t)$. The equation (5) can be rewritten in the following form: $dF_n/dt = \phi(F_n, s_n)(F_{n-1} - F_n)$, or, equivalently, $dF_n/dt = -\phi(F_n, s_n)F_n$. dF_n/dt should be non-positive, because the transitions from the level $n+1$ to n are forbidden. Therefore $\phi(F_n, s_n)$ should be non-negative.

by replacement in the initial problem $\mathcal{P}$ of $\phi(F_n, s_n)$ by u_n :

$$W = \int_{t_0}^T e^{-\rho(t-t_0)} U(c(t)) dt \rightarrow \max_{u_n(\cdot), n \in \mathbb{N}}, \quad (2')$$

$$c(t) = \sum_{n=1}^{\infty} (g_n - u_n(t)) f_n(t), \quad (9')$$

$$df_n(t)/dt = -u_n f_n + u_{n-1} f_{n-1}, \quad n \in \mathbb{N}, \quad (5')$$

$$f_n(0) = \text{fix}, \quad n \in \mathbb{N}, \quad (6')$$

$$u_n \geq 0. \quad (7')$$

Let us assume that the function $\phi(F_n, s_n)$ is monotone with respect to s_n and $\forall F_n \in [0, 1], u \in [0, \infty) \exists s_n : \phi(F_n, s_n) = u$. Then (for details see Appendix A.1) if $\tilde{u}_n$ ($n \in \mathbb{N}$) is the optimal control for the problem $\mathcal{P}'$, then $\tilde{s}_n = \phi^{-1}(\tilde{u}_n | F_n)$ ($n \in \mathbb{N}$) is the optimal control for the problem $\mathcal{P}$, where $\phi^{-1}(\cdot | F_n)$ is the inverse of $\phi(F_n, \cdot)$ for fixed F_n .

The evolution of the overall capital-labor ratio is then described by the following expression:

$$\frac{dk}{dt} = \sum_{n=1}^{\infty} u_n f_n. \quad (8')$$

Despite of appearing technically simple, the above-described possibility of changing variables in the problem $\mathcal{P}$ is, nevertheless, rather interesting. In the fully “kinetic” models, like e.g. those of (Polterovich, Henkin, 1989; Henkin, Polterovich, 1991, 1999; Henkin, 2012; Aleksandrova, 2015), the properties of a solution critically depend on the properties of the function $\phi(F_n)$ ⁶. Introduction of the control parameter s_n , $n \in \mathbb{N}$, leads to the fact that non-linearities described by the function $\phi(F_n, s_n)$ now do not play any role. Qualitatively, this results from the linear relation between the evolution of the overall capital-labor ratio and normalised investment size defined in (4). The methodology of introduction of essential non-linearities into the distribution evolution equation is discussed in the Paragraph 4.2.

The Hamiltonian of the problem $\mathcal{P}'$ and the evolution of the state and costate variables are described by the following equations:

$$\mathcal{H} = e^{-\rho(t-t_0)} U(c(\mathbf{f}, \mathbf{u})) + \sum_{n=1}^{\infty} \lambda_n (-u_n f_n + u_{n-1} f_{n-1}), \quad (10')$$

$$\dot{f}_n = -u_n f_n + u_{n-1} f_{n-1}, \quad (11')$$

$$\dot{\lambda}_n = -\left(e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u})) (g_n - u_n) + u_n (\lambda_{n+1} - \lambda_n)\right), \quad (12')$$

$$f_n(0) = \text{fix}, \quad (13')$$

$$\lambda_n(T) = 0. \quad (14')$$

To simplify the analysis we will assume that $\lim_{c \rightarrow 0} U(c) = -\infty$ (e.g. $U(c) = \ln c$) and therefore the condition $c(t) \geq 0$ is fulfilled automatically⁷.

The optimal control maximises the Hamiltonian. Therefore, in accordance with the Karush–Kuhn–Tucker theorem, the following condition should be fulfilled:

⁶ Here we have a single argument F_n as in these models there are no controls s_n .

⁷ One can check that this simplification does not significantly influence general results.

$\partial \mathcal{H} / \partial u_n = e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u}))(-f_n) + (\lambda_{n+1} - \lambda_n) f_n \leq 0 \quad \forall n \in \mathbb{N},$ (15)
where the equality holds for $u_n > 0$.

For $f_n = 0$ the condition (15) is fulfilled for an arbitrary value of u_n . Without loss of generality we will assume that in this case $u_n = 0$.

Let us analyse the case $f_n > 0$ and introduce the following notation: $\Psi(t, \mathbf{f}, \boldsymbol{\lambda}) = e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u}))$. Equation (15) can be rewritten in the following form

$$\forall n \in \mathbb{N}: f_n > 0 \rightarrow \lambda_{n+1} - \lambda_n \leq \Psi(t, \mathbf{f}, \boldsymbol{\lambda}) \Leftrightarrow \max_{n \in \mathbb{N}: f_n > 0} (\lambda_{n+1} - \lambda_n) \leq \Psi(t, \mathbf{f}, \boldsymbol{\lambda}) \quad (16)$$

where the equality holds for $u_n > 0$. We have:

- if $u_n > 0$ then $n \in \text{Argmax}_{m \in \mathbb{N}: f_m > 0} (\lambda_{m+1} - \lambda_m)$ and vice versa;
- from (12') and (16) it follows that

$$\dot{\lambda}_n = -g_n \Psi(t, \mathbf{f}, \boldsymbol{\lambda}). \quad (17)$$

From eq. (17) it immediately follows that there exists a function $\alpha(t)$ such that $\alpha(T) = 0$ and

$$\lambda_n(t) = g_n \alpha(t) \quad \forall n \in \mathbb{N}. \quad (18)$$

The corresponding technical details are provided in the Appendix A.2.

Let us list some properties of the solution (below we use the notation $\Delta_n = g_{n+1} - g_n$):

- if $n \in \text{Arg max}_{m \in \mathbb{N}: f_m > 0} \Delta_m$ then $u_n \geq 0$, else $u_n = 0$;
- the optimal value of consumption c^* can be obtained from the following expression

$$c^* = \begin{cases} U'^{-1} \left(e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}: f_m > 0} \Delta_m \right), & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}: f_m > 0} \Delta_m; \\ c_{\max}, & \text{otherwise,} \end{cases}$$

where $c_{\max} = \sum_{n=1}^{\infty} g_n f_n$;

- the vector of optimal control $\mathbf{u}$ satisfies the following condition:

$$\mathbf{u}: \sum_{n=1}^{\infty} (g_n - u_n) f_n = c^*. \quad (19)$$

The differential equation for $\alpha(t)$ from (17) has the following form:

$$\dot{\alpha} = -\Psi(t, \mathbf{f}, \boldsymbol{\lambda}) \equiv \begin{cases} -e^{-\rho(t-t_0)} U'(c_{\max}), & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}: f_m > 0} \Delta_m; \\ -\alpha \max_{m \in \mathbb{N}: f_m > 0} \Delta_m, & \text{otherwise.} \end{cases} \quad (20)$$

The further analysis of the solution includes two cases: that of decreasing returns to scale and the general one.

2.3. Solution in the case of decreasing returns to scale

One of the most important characteristics of a continuous production function is its type in terms of return to scale. This term characterises the type of response in production induced by an increase of factors⁸. One may generalise this property to the case of discrete production function g_n , $n \in \mathbb{N}$. In the current study it will always be assumed that g_n is an increasing function of n .

⁸ For example, the concave production function is of the decreasing returns to scale type. The general definition is the following. Let $h(x): \mathbb{R}^n \rightarrow \mathbb{R}$. If $h(ax) = ah(x)$ for any $a > 0$, then $h(x)$ has constant returns to scale. If $h(ax) > ah(x)$ ($< ah(x)$) then it has increasing (decreasing) returns to scale.

Definition 1. The production function of discrete variable g_n , $n \in \mathbb{N}$ is decreasing returns to scale type if $\Delta_n < \Delta_m$ for $n > m$.

Obviously, in the case of decreasing returns to scale

$$\text{Arg max}_{m \in \mathbb{N}: f_m > 0} = \min\{m | f_m > 0\}.$$

This means that at any time point transitions may occur only from the lowest non-empty level. Let us denote this level by $m^* = m^*(t)$, i.e.

$$m^* \equiv m^*(t) = \min\{m | f_m(t) > 0\},$$

and, moreover, the value of optimal control is uniquely determined by (19).

The boundary value problem to be resolved takes the following form:

$$\dot{f}_n = \begin{cases} -u_n f_n, & \text{if } n = m^*, \\ u_{n-1} f_{n-1}, & \text{if } n = m^* + 1, \\ 0, & \text{otherwise;} \end{cases} \quad (21)$$

$$\dot{\alpha} = \begin{cases} -\alpha \Delta_{m^*}, & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \Delta_{m^*}, \\ -e^{-\rho(t-t_0)} U'(c_{\max}), & \text{otherwise;} \end{cases} \quad (22)$$

$$u_n = \begin{cases} \left[c_{\max} - U'^{-1}\left(e^{\rho(t-t_0)} \alpha \Delta_{m^*}\right) \right] / f_n, & \text{if } n = m^* \text{ \& } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \Delta_{m^*}, \\ 0, & \text{otherwise;} \end{cases} \quad (23)$$

$$m^* = \min\{m | f_m > 0\}. \quad (24)$$

Formulation of its solution in terms of a system of non-linear equations is described in the Appendix 7. In the next paragraph we provide an example of a numerical solution illustrating the dynamics in the case under consideration.

2.4. Example 1. Numerical solution in the case of decreasing returns to scale

Let us consider a particular example in which:

- the utility is logarithmic, $U(c) = \ln c$;
- initially all firms are uniformly distributed over the first five levels (Fig. 1);
- the production function g_n (and its increment Δ_n) are chosen in the form shown in Fig. 2.

The horizontal line in the Fig. 2B indicates the identity level. This level is of interest because without any additional calculation it allows to conclude that a transition from the level n to $n+1$ won't occur if $\Delta_n < 1$ as the dependence between the capital accumulation and investment

Figure 1

The initial distribution of firms in the Example 1

Figure 2

Relationship between the capital-labor ratio level n , $n \in \mathbb{N}$, and A: output (g_n) or B: its increment ($\Delta_n = g_{n+1} - g_n$) in the Example 1. The dashed horizontal line indicates the identity level

is linear but the production function is concave. Therefore, there exists a final level ($n_{\max}$) at which transitions will stop. This occurs when $\Delta_{n_{\max}-1} > 1$. From Fig. 2B it follows that $n_{\max} \leq 9$. The time point t_{stop} after which no transitions occur is determined by the following condition:

$$U'(c_{\max}(t_{\text{stop}})) = e^{\rho(t_{\text{stop}}-t_0)} \alpha \Delta_{m^*(t_{\text{stop}})},$$

$$n_{\max} = m^*(t_{\text{stop}}).$$

Evolution of the firms' fractions at different capital-labor ratio levels in this example is shown in Fig. 3⁹. It illustrates two previously described facts. First, the transitions are initially made from the lowest level. Second, there exists the maximal level (level 8) participating in the process of transitions.

It should be noted that in the case under consideration the uniqueness of the solution guarantees its optimality.

Figure 3

Time evolution of the firms' fractions in the Example 1 with decreasing returns to scale

2.5. Solution in case of non-decreasing increments of production function

The case of an arbitrary structure of g_n (e.g. that with non-monotonic increments) is more complicated. Let us analyse the evolution of the system at time t under the condition

⁹ To find the numerical solution the "shooting" method was used. To solve the boundary value problem one should choose such initial values of the costate variables $\lambda_n(0)$, $n \in \mathbb{N}$ as in the solution of the initial value problem with these values $\lambda_n(T) = 0$. Considering (18) one should find only the correct initial value $\alpha(0)$, such that $\alpha(T) = 0$. To solve the initial value problem the fourth-order Runge-Kutta method was used.

$$\Delta_{m^*(t)} < \Delta_{m^*(t)+1}, \quad (25)$$

so that it is preferable to perform transitions from the level $m^*(t)+1$, but there are no firms at it. Let us also assume that $\Delta_{m^*(t)+2} < \Delta_{m^*(t)+1}$. Then at time t a transition will occur from the level $m^*(t)$. However, at time $t+dt$ (with infinitesimal dt) we have $m^*(t+dt) = m^*(t)+1$ and, therefore, $\Delta_{m^*(t+dt)} = \Delta_{m^*(t)+1}$. Thus, the right-hand side of the following differential equation on alpha (from (20)) $\dot{\alpha}(t) = \alpha(t)\Delta_{m^*(t)}$ has an infinite number of simple discontinuities so that its time derivative $\dot{\alpha}$ does not exist.

Qualitatively such dynamics corresponds to transitions bypassing the adjacent level which are forbidden due to the chosen form of the difference-differential analogue of the Burgers' type equation. However, as shown in Appendix A.3, if the value of u_n is not bounded from above then, in the limit of $u_n \rightarrow \infty$, transitions of the type n to the level $n+2$ are possible.

Thus the formally justifiable solution can be found only if one of the following modifications of the initially posed problem are introduced:

- one may look for an suboptimal solution, the ε -approximation, defined as the solution of the initial problem with the additional condition $u_n < \varepsilon$, $n \in \mathbb{N}$, and analyse the limit of $\varepsilon \rightarrow \infty$;
- if in the initial problem $\mathcal{P}$ the domain of the $\phi(\cdot)$ function is bounded from above by u the additional condition on the admissible set of the control variables $u_n < u$, $n \in \mathbb{N}$ is introduced;
- the right-hand side of the equation describing the time-evolution of the firms fractions (5) contains some cumulative operator.

It should be noted that if one considers the analogue of the problem $\mathcal{P}'$ in discrete time the above-described problem won't appear. This allows to analyse the properties of the approximate numerical solution of $\mathcal{P}$ in discrete time.

2.6. Example 2. Numerical solution in the case of production function with non-monotonic increments

Let us consider an example in which:

- the utility function is logarithmic $U(c) = \ln c$;
- initially all firms are concentrated at the first level, i.e.

$$f_1(0) = 1, \quad f_n(0) = 0, \quad n > 1; \quad (26)$$

- the production function g_n , $n \in \mathbb{N}$ (and its increment Δ_n , $n \in \mathbb{N}$) is assumed to have the form shown in Fig. 4. As seen in Fig. 4A the key feature of the production function g_n from is that the main incentive for the firms to transit from the first level to the second is not in the significant growth of the output but in the ability to move to the third level from the second;
- two different modifications of g_n will be analysed:
 - the one with $\Delta_1 = 2$;
 - the one with $\Delta_1 = 1$ (this case is of special interest as the transition from the first level to the second one is "locally" ineffective, but nevertheless the firm should do it to be able to reach the third level).

Let us note that in the case of $\Delta_1 = 1$ the solution of the boundary value problem (11')–(14') is not unique. This solution can be described as follows: one should find the set of initial values $\lambda_n(0)$, $n \in \mathbb{N}$ guaranteeing that $\lambda_n(T) = 0$, $n \in \mathbb{N}$ in the solution of the initial value problem (11')–(13'). From (18) it follows that one should look only for such $\alpha(0)$ for which $\alpha(T) = 0$ as well. The uniqueness of the solution is equivalent to the uniqueness of such $\alpha(0)$. In the case of $\Delta_1 = 1$ there are two different $\alpha(0)$ which lead to $\alpha(T) = 0$ (Fig. 5B). In contrast, in the case when $\Delta_1 = 2$; the solution is unique (Fig. 5A).

Figure 4

Relationship between the capital-labor ratio level n , $n \in \mathbb{N}$, and A: output (g_n) or B: its increment ($\Delta_n = g_{n+1} - g_n$) in the Example 2. The dashed horizontal line shows the identity level

Figure 5

Dependence between $\alpha(0)$ and $\alpha(T)$ in the initial value problem (11')–(13') (where $\alpha(t)$ is defined in (18); case A: $\Delta_1 = 2$; case B: $\Delta_1 = 1$)

Evolution of the firms fractions at different levels is shown in Fig. 6. The fact that there is a small fraction of firms at the first level that does not move to the third one in the case when $\Delta_1 = 1$ is due to finite accuracy of the numerical solution. Qualitatively the evolution in both cases ($\Delta_1 = 1$ and $\Delta_1 = 2$) is the same. As it was already noted, the evolution looks like a transition from the first level to the third.

Figure 6

Evolution of the firms fractions on the different levels in the Example 2 (case A: $\Delta_1 = 2$; case B: $\Delta_1 = 1$)

3. The model with TFP growth

As was mentioned in Introduction, the model presented in the previous section realises a preliminary step in developing an endogenous two-factor growth model with the TFP and capital-labor ratio factors. Development of such a two-factor model is out of the scope of the current study. However, in this section we present some intuitive arguments related to this topic. To illustrate some effects resulting from coexistence of two different growth factors, we present the model in which the first factor (TFP) is exogenous and the second one (capital-labor ratio) is endogenous. In this case expressions for the production function and its increment take the following form:

$$\tilde{g}_n(t) = e^{(t-t_0)} g_n, \quad (27)$$

$$\tilde{\Delta}_n(t) = e^{\gamma(t-t_0)} \Delta_n \quad (28)$$

where γ is an exogenously defined TFP growth rate. Therefore, the equation (9') now reads

$$c(t) = \sum_{n=1}^{\infty} \left(e^{\gamma(t-t_0)} g_n - u_n(t) \right) f_n(t), \quad (9a)$$

while other equations of the $\mathcal{P}'$ problem remain unchanged. Let us denote a two-factor problem by $\mathcal{P}^a$. The details of solution of $\mathcal{P}^a$ are given in Appendix 8. Here we confine ourselves to a qualitative analysis of the numerical solution of a particular modification of the Example 1 (the case of decreasing returns to scale) where g_n , $n \in \mathbb{N}$ is the same as in Example 1 (see Fig. 2). To illustrate the effect of incorporation of exogenous TFP growth we analyse numerical solutions of the problem for different values of γ . In particular, the case of Example 1 presented in the Section 2.4, corresponds to the case of $\gamma = 0$. As seen in Fig. 3, the last level to which transitions occur is the ninth, i.e. $f_9(1) = 1$ and $f_{10}(1) = 0$. As $\tilde{\Delta}_n$ increases with time introduction of exogenous growth rate factor γ results in growing incentives for further

Figure 7

Firms' fractions on ninth and tenth levels at $t = 1$ in case of different values of exogenous growth rate γ (in modification of Example 1)

transitions. The Fig. 7 illustrates the fact that with growing γ the fraction of firms at the tenth level ($f_{10}(1)$) at final time point increases.

The results described in this section demonstrate the importance of simultaneous consideration of various growth factors so that changing one efficiency factor may result in the change of another one.

4. The model with heterogeneous investment effectiveness

In the model described in the Section 2 we assumed that the effectiveness of an investment does not depend on its size and the level of capital-labor ratio of firms in which

this investment is made. However, it is natural to assume that the higher the level of the firm's capital-labor ratio, the more effective is this investment. One of justifications of this assumption is in the fact that a debt service for firms at higher levels of capital-labor ratio is cheaper. In addition, following (Blanchard et al., 1989), one may expect a decline of the investment effectiveness with the increase of its size (i.e. some kind of decreasing returns in the process of transformation of investment into capital). In the next two paragraphs we analyse an extended version of the model described in the previous sections that takes into account the two above-described features.

4.1. Heterogeneity in capital-labor ratio level

Let us start with the assumption concerning the relation between the level of the firm's capital-labor ratio and the effectiveness of an investment. For an amount $u_n f_n$ of investment into firms at the level n the overall capital-labor ratio evolution is given by $dk/dt = \sum_{n=1}^{\infty} v_n u_n f_n$, where v_n , $n \in \mathbb{N}$, is the discrete function quantifying the growth of the investment effectiveness, i.e. $v_{n+1} > v_n$. In turn, evolution of firms' fractions on different levels is described by $\dot{f}_n = -v_n u_n f_n + v_{n-1} u_{n-1} f_{n-1}$.

Incorporation of the investment effectiveness heterogeneity into a problem $\mathcal{P}'$ leads to its modification $\mathcal{P}^b$ described by the following equations:

$$W = \int_{t_0}^T e^{-\rho(t-t_0)} U(c(t)) dt \rightarrow \max_{u_n(\cdot), n \in \mathbb{N}}, \quad (2b)$$

$$c(t) = \sum_{n=1}^{\infty} (g_n - u_n(t)) f_n(t), \quad (9b)$$

$$df_n(d)/dt = -v_n u_n f_n + v_{n-1} u_{n-1} f_{n-1}, \quad n \in \mathbb{N}, \quad (5b)$$

$$f_n(0) = \text{fix}, \quad n \in \mathbb{N}, \quad (6b)$$

$$u_n(t) \geq 0. \quad (7b)$$

As shown in details in the Appendix A.4, qualitative properties of the solution of $\mathcal{P}^b$ are the same as of those of $\mathcal{P}'$. However, the parameter determining the level from which transition occurs (level $m^*(t)$), is now the product $v_n \tilde{\Delta}_n$ (instead of $\tilde{\Delta}_n$ in $\mathcal{P}'$).

From this it follows that for heterogeneous investment effectiveness then even in the case of decreasing returns to scale an evolution of the system may look qualitatively like the one in the Example 2. The reason is that the product $v_n \Delta_n$, $n \in \mathbb{N}$ does not necessarily monotonically decrease with growing n .

4.2. Heterogeneity in investment size

In the model described in the Section 2 we assumed that the ability of an investment to transform into capital does not depend on its size. This was reflected by the quasi-linearity of the dependence (8') between dk/dt and u_n . However, following the ideas put forward in (Blanchard et al., 1989), it is natural to assume that investment effectiveness drops with increasing size. This assumption can be included into the model of Section 2 via replacement of equation (5') in $\mathcal{P}'$ by the following expression:

$$\frac{df_n}{dt} = -\chi(u_n)f_n + \chi(u_{n-1})f_{n-1}, \quad n \in \mathbb{N}, \quad (5c)$$

where the function $\chi(\cdot)$ has the following properties:

$$\chi(x) > 0, \text{ if } x > 0; \quad \chi(0) = 0; \quad \chi'(x) > 0; \quad \chi''(x) < 0.$$

The other properties of the problem $\mathcal{P}'$ remain the same.

Let us denote the problem with heterogeneous investment effectiveness with respect to its size by $\mathcal{P}^c$. The Hamiltonian and evolution of the state and costate variables are in this case described by the following equations:

$$\mathcal{H} = e^{-\rho(t-t_0)} U(c(\mathbf{f}, \mathbf{u})) + \sum_{n=1}^{\infty} \lambda_n (-\chi(u_n)f_n + \chi(u_{n-1})f_{n-1}), \quad (10c)$$

$$\dot{f}_n = -\chi(u_n)f_n + \chi(u_{n-1})f_{n-1}, \quad (11c)$$

$$\dot{\lambda}_n = -\left(e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u})) (g_n - u_n) + \chi(u_n)(\lambda_{n+1} - \lambda_n) \right), \quad (12c)$$

$$f_n(0) = \text{fix}, \quad (13c)$$

$$\lambda_n(T) = 0, \quad (14c)$$

$$u_n \geq 0.$$

If a vector $\mathbf{u} = (u_1, u_2, \dots)'$ is the optimal control for the problem $\mathcal{P}^c$ then it maximises the Hamiltonian (10c). The first order conditions of this problem read $\partial \mathcal{H} / \partial u_n = -f_n \Psi(t, \mathbf{f}, \boldsymbol{\lambda}) + f_n \chi'(u_n)(\lambda_{n+1} - \lambda_n) \leq 0$, with equality in the case of $u_n > 0$. Let us again analyse only the case of $f_n > 0$. Rewriting the previous expression, we get

$$\chi'(u_n)(\lambda_{n+1} - \lambda_n) \leq \Psi(t, \mathbf{f}, \boldsymbol{\lambda}), \quad (29)$$

with equality when $u_n > 0$.

To simplify the further analysis let us focus only on the case of functions $U(x)$ and $\chi(x)$ having the form $U(x) = \ln(x)$, $\chi(x) = \ln(1+x)$.

It is easy to show that the Hamiltonian (10c) is concave with respect to $\mathbf{u} = (u_1, u_2, \dots)'$ (for details see Appendix A.5). Therefore fulfilment of first order conditions is sufficient to guarantee maximisation of Hamiltonian with respect to $\mathbf{u}$.

Let us define two sets of capital-labor ratio levels $\mathcal{C}^0$ and $\mathcal{C}^+$ characterised by the following expressions: $\mathcal{C}^0 = \{n \in \mathbb{N} : u_n = 0\}$, $\mathcal{C}^+ = \{n \in \mathbb{N} : u_n > 0\} = \mathbb{N} \setminus \mathcal{C}^0$.

In the Appendix A.6 it is shown that

- there exists an algorithm of determination of the sets $\mathcal{C}^0$ and $\mathcal{C}^+$;
- the vector of optimal controls $\mathbf{u} = (u_1, u_2, \dots)'$ for the problem $\mathcal{P}^c$ is given by the following expression

$$u_n = \begin{cases} 0, & \text{if } n \in \mathcal{C}^0, \\ (\lambda_{n+1} - \lambda_n) \frac{c_{\max} + \sum_{k \in \mathcal{C}^+} f_k}{e^{-\rho(t-t_0)} + \sum_{k \in \mathcal{C}^+} f_k (\lambda_{k+1} - \lambda_k)} - 1, & \text{otherwise.} \end{cases} \quad (30)$$

In the next paragraph we will analyse the properties of the numerical solution of the boundary value problem (11c)–(14c), where the optimal control is defined by (30).

4.3. Example 3. Numerical solution in the case of investment effectiveness heterogeneity with respect to size

Example 3a. Decreasing returns to scale. In this example we analyse an evolution of a system with the production function g_n and its increment Δ_n of the form shown in Fig. 8.

Figure 8

Relationship between the capital-labor ratio level n , $n \in \mathbb{N}$, and A: output (g_n) or B: increment ($\Delta_n = g_{n+1} - g_n$) in the Example 3a

Evolution of the system in the case of two different initial distributions of firms over the capital-labor ratio levels is shown in Fig. 9. The first row in this figure (Fig. 9 A1 and B1)) corresponds to the case in which at the

time $t = 0$ all firms are at the first capital-labor ratio level. The second one (Fig. 9 (A2 and B2)) illustrates the evolution of the distribution in the case when the initial distribution is Poissonian with the parameter equal to 2. In both cases it is obvious that distributions widen in time. From Fig. 9 (B1 and B2) it is clear that transitions appear from every non-empty level. The reason for such a dynamics is obvious. As the effectiveness of an investment drops with increasing size (see the properties of the function $\chi(\cdot)$) it is optimal to distribute total investment over different levels. Therefore, investment effectiveness decreasing with respect to size can be one of possible explanations for the observable capital-labor diversity growth.

Figure 9

Evolution of the distribution of the firms by the capital-labor ratio levels in the Example 3a. A1, A2: at different times; B1, B2: Temporal evolution of the firms' fractions $f_n(t)$, $n \in \mathbb{N}$. The two lines correspond to the cases with different initial conditions

Example 3b. Production function with non-monotonic increments. In the Example 3b we analyse an evolution of a system in which increments of the production function are non-monotonic. As described in the Paragraph 2.5, in the case of a linear dependence of capital increase on investment size (problem $\mathcal{P}'$) a solution of the problem in continuous time does not exist. In the corresponding discrete numerical approximation (Example 2) one observes an unnatural evolution.

One of the possible modifications of the formulation of the initial problem proposed in the Paragraph 2.5 is in introduction of some positive

exogenous constant u and of an additional constraint on u_n , $n \in \mathbb{N}$ $u_n < u$. However, such a modification does not seem reasonable. The case of heterogeneous investment effectiveness described in the previous paragraph provides a more natural realisation of the similar idea. That is, in the case of decreasing effectiveness of investment with respect to its size an investment of enormous size can't be optimal. Therefore, qualitatively, in this case the investment size on the optimal trajectory is bounded automatically.

Let us analyse the numerical solution of the problem $\mathcal{P}^c$ in which the production function g_n (and its increments Δ_n) have the form shown in Fig. 10. The Fig. 11 displays the evolution of the distribution $\{f_n(t), n \in \mathbb{N}\}$. We see that the dynamics is characterised only by sequential transitions so that, e.g., there are no transitions from the first level to the third one. The optimal solution is described by some non-trivial distribution of the firms over different levels.

Figure 10

Relationship between the capital-labor ratio level n , $n \in \mathbb{N}$, and A: output (g_n) or B: its increment ($\Delta_n = g_{n+1} - g_n$) in the Example 3b

Figure 11

Evolution of the distribution of the firms by the capital-labor ratio levels in the Example 3b. A: in different time-points; B: Time-evolution of the firms' fractions $f_n(t)$, $n \in \mathbb{N}$

Example 3c. Increasing returns to scale. As mentioned in Introduction, the arguments put forward by K. Arrow in his seminal paper (Arrow, 1962) motivate an interest to study the case in which g_n , $n \in \mathbb{N}$ is characterised by increasing returns to scale. It should be noted that, as it follows from the results of Section 2.5, the basic model $\mathcal{P}'$ is in this case not applicable. However, introduction of decreasing investment effectiveness with respect to its size makes such an analysis possible.

In this section the results of numerical solution of the problem $\mathcal{P}^c$ with the production function g_n , $n \in \mathbb{N}$ shown in Fig. 12 are presented. Initial distribution of firms with respect to the capital-labor ratio levels is Poissonian with the parameter 2. Fig. 13 illustrates time evolution of the firms distribution.

Interestingly, comparison of the results of the current example and of the Example 3a (see Fig. 9 (A2, B2)) shows that increasing returns to scale may cause stronger distribution widening than in the case of decreasing returns. This is clearly seen in Fig. 14, where time evolution of the mean capital-labor ratio level ($k(t) = \sum_{n=1}^{\infty} n f_n(t)$) and the variance ($V_k(t) = \sum_{n=1}^{\infty} (n - k(t))^2 f_n(t)$) for the two cases are presented. As it shown in Fig. 14A, the mean capital-labor ratio is greater for the case with decreasing returns, whereas the variance is greater for the case with increasing returns (Fig. 14B). The reason for this is that increasing returns lead to a growth of incentives for transitions from the levels in the right tail of the distribution resulting in a relative growth of the variance.

Figure 12

Relationship between the capital-labor ratio level n , $n \in \mathbb{N}$, and A: output (g_n) or B: its increment ($\Delta_n = g_{n+1} - g_n$) in the Example 3c

Figure 13

Evolution of the distribution of the firms by the capital-labor ratio levels in the Example 3c. A: in different time-points; B: Time-evolution of the firms' fractions $f_n(t)$, $n \in \mathbb{N}$

Figure 14

Evolution of A mean capital-labor ratio level $k(t)$ and B variance of capital-labor ratio levels $V_k(t)$ in two cases: the case of decreasing returns (Example 3a) and the case of increasing ones (Example 3c)

5. Conclusion

The model describing evolution of the distribution of firms on an efficient path with respect to capital-labor ratio is presented. The most interesting feature of the model is in providing a possible explanation for the observable growth of the firms variability with respect to capital-labor ratio levels.

The present paper covers only a part of an analysis that should be done. In particular, we focused only on a simplified version of the central planner's problem proposed in (Polterovich, 2017). We do not formulate and analyze a decentralized analogue of the model. However, follow-

ing (Lucas, Moll, 2014), we consider the two versions as distinct challenging problems. Formulation and analysis of a decentralised version and a comparison of the two approaches represent perspective directions for the future research.

As a concluding remark, we would like to stress that the role of the capital-labor ratio is of special interest in two contexts representing possible directions for future model extension. The first one, described in the recent papers (Furman et al., 2016; Aghion et al., 2017; Korinek, Stiglitz, 2017), is related to studying an impact of the progress in artificial intelligence with the focus on the problem of substitution of human labor with the machine one. The side effect of such a phenomenon is a growth of unemployment and inequality.

Another possible direction is related to one of the models of economic growth described in (Sargent, Rodriguez, 2000) where the capital-labor ratio k is interpreted as a ratio of some measure of effective capital to labor $k = E\bar{K} / L$, where $\bar{K}$ is a physical amount of capital units and E stands for capital effectiveness. With capital effectiveness increasing with time, the ratio k does grow. Therefore, a growth of the capital-labor ratio may be viewed as a consequence of that of the effectiveness of each unit of capital.

Additionally, it should be noted that the setting used in the present study is stylised and does not allow to address some important issues, in particular:

- an influence of the network of firms communications structure on the process of knowledge diffusion;
- an account of depreciation in the process of physical capital accumulation.

These topics also represent perspective directions for future research.

APPENDIX

A. Proofs

A.1. Change of variable in the problem $\mathcal{P}$

1. Let us assume that $\mathbf{u} = \tilde{\mathbf{u}}$ maximises the Hamiltonian $\mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{u})$ of $\mathcal{P}'$ with respect to $\mathbf{u}$. Turning to the problem in which u_n is replaced by $\phi(F_n, s_n)$, let us denote the Hamiltonian of this problem by $\mathcal{H}_\phi$ and assume that $\mathbf{s} = \hat{\mathbf{s}}$ maximises it.

Let us assume that $\phi(F_n, \hat{s}_n) \neq \tilde{u}_n$. Then

$$\mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \phi(F_n, \hat{s}_n)) = \mathcal{H}_\phi(\mathbf{f}, \boldsymbol{\lambda}, \hat{\mathbf{s}}) > \mathcal{H}_\phi(\mathbf{f}, \boldsymbol{\lambda}, \phi^{-1}(\tilde{u}_n | F_n)) = \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \tilde{\mathbf{u}}),$$

which contradicts to the fact that $\tilde{\mathbf{u}}$ maximises the Hamiltonian problem $\mathcal{P}'$.

2. Let us show that such a replacement does not change the differential equations on state and costate variables. For the state variables $(f_n, n \in \mathbb{N})$ this follows from the following expression:

$$\dot{f}_n = \frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \tilde{\mathbf{u}})}{\partial \lambda_n} = \frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \phi(F_n, \hat{s}_n))}{\partial \lambda_n} = \frac{\partial \mathcal{H}_\phi(\mathbf{f}, \boldsymbol{\lambda}, \hat{\mathbf{s}})}{\partial \lambda_n}.$$

The differential equations for the costate variables $(\lambda_n, n \in \mathbb{N})$ in the problem $\mathcal{P}'$ have the following form:

$$\dot{\lambda}_n = -\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \bar{\mathbf{u}}) / \partial f_n. \quad (31)$$

If u_n is replaced by $\phi(F_n, s_n)$ the following expression describes the evolution of the costate variables:

$$\begin{aligned} \dot{\lambda}_n = & -\frac{\partial \mathcal{H}_\phi(\mathbf{f}, \boldsymbol{\lambda}, \hat{\mathbf{s}})}{\partial f_n} - \sum_{n=1}^{\infty} \frac{\partial \mathcal{H}_\phi(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{s})}{\partial s_n} \bigg|_{s_n} \frac{\partial \hat{s}_n}{\partial f_n} = -\frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{u})}{\partial f_n} \bigg|_{\mathbf{u}=\bar{\mathbf{u}}} - \\ & - \sum_{n=1}^{\infty} \frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{u})}{\partial u_n} \bigg|_{\mathbf{u}=\bar{\mathbf{u}}} \frac{\partial \phi(F_n, s_n)}{\partial f_n} \bigg|_{s=\hat{\mathbf{s}}} - \sum_{n=1}^{\infty} \frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{u})}{\partial u_n} \bigg|_{\mathbf{u}=\bar{\mathbf{u}}} \frac{\partial \phi(F_n, s_n)}{\partial s_n} \bigg|_{s=\hat{\mathbf{s}}} \frac{\partial \hat{s}_n}{\partial f_n}. \end{aligned} \quad (32)$$

Let us note that if the solution is interior then

$$\frac{\partial \mathcal{H}(\mathbf{f}, \boldsymbol{\lambda}, \mathbf{u})}{\partial u_n} \bigg|_{\mathbf{u}=\bar{\mathbf{u}}} = 0, \quad n \in \mathbb{N},$$

i.e. (30) is equivalent to (31). If the solution is on the boundary $\phi(F_n, s_n) = 0$ (or $u_n = 0$) then

$$\frac{d}{df_n} \phi(F_n, s_n) = \frac{\partial \phi}{\partial f_n} + \frac{\partial \phi}{\partial s_n} \frac{\partial s_n}{\partial f_n} = 0,$$

i.e. expressions (30) and (31) are also equivalent.

A.2. Existence of the function $\alpha(t)$

From (17) it follows that $\dot{\lambda}_n / g_n = \dot{\lambda}_k / g_k \quad \forall n, k \in \mathbb{N}$, i.e. $\lambda_n / g_n - \lambda_k / g_k = \text{const}$. Moreover, from the transversality condition (14') it follows that

$$\lambda_n(T) / g_n = \lambda_k(T) / g_k = 0,$$

therefore

$$\lambda_n(\tau) / g_n = \lambda_k(\tau) / g_k \quad \forall n, k \in \mathbb{N}, \quad \tau \in [t, T].$$

Put differently, there exists the function $\alpha(\tau)$, $\tau \in [t, T]$, such as $\lambda_n(\tau) = g_n \alpha(\tau) \quad \forall n \in \mathbb{N}, \quad \forall \tau \in [t, T]$.

A.3. Transitions over the level

Let us fix the value of $f_{n+1}(t)$ at time t : $f_{n+1}(t) = \varepsilon$. The evolution of the firms fractions at the levels $n+1$ and $n+2$ is described by the following equations: $\dot{f}_{n+1} = -u_{n+1}f_{n+1} + u_n f_n$, $\dot{f}_{n+2} = -u_{n+2}f_{n+2} + u_{n+1}f_{n+1}$.

Let us also introduce the following condition $\dot{f}_{n+1} = 0$, which holds if $u_{n+1}(\varepsilon) = u_n f_n / \varepsilon$, then $\dot{f}_n = -u_n f_n + u_{n-1} f_{n-1}$, $\dot{f}_{n+1} = 0$, $\dot{f}_{n+2} = -u_{n+2} f_{n+2} + u_n f_n$. For all ε one can find the corresponding $u_{n+1}(\varepsilon)$. Therefore, in a limiting case $\varepsilon \rightarrow 0$ a transition looks like a transition from the level n to the level $n+2$. However, in such a case $u_{n+1} \rightarrow \infty$, in other words, the exact solution is unattainable.

A.4. Model with investment heterogeneity with respect to capital-labor ratio levels

The Hamiltonian and the differential equations for the state and costate variables for the problem $\mathcal{P}^b$ are given by the following expressions:

$$\mathcal{H} = e^{-\rho(t-t_0)} U(c(\mathbf{f}, \mathbf{u})) + \sum_{n=1}^{\infty} \lambda_n (-v_n u_n f_n + v_{n-1} u_{n-1} f_{n-1}), \quad (10b)$$

$$\dot{f}_n = -u_n v_n f_n + u_{n-1} v_{n-1} f_{n-1}, \quad (11b)$$

$$\dot{\lambda}_n = -\left(e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u})) (g_n - u_n) + u_n v_n (\lambda_{n+1} - \lambda_n) \right), \quad (12b)$$

$$f_n(0) = \text{fix}, \quad (13b)$$

$$\lambda_n(T) = 0, \quad (14b)$$

$$u_n \geq 0, \quad n \in \mathbb{N}.$$

The first order conditions on the optimal control maximizing the Hamiltonian take the following form:

$$\frac{\partial \mathcal{H}}{\partial u_n} = e^{-\rho(t-t_0)} U'(c(\mathbf{f}, \mathbf{u})) (-f_n) + (\lambda_{n+1} - \lambda_n) v_n f_n \leq 0. \quad (15b)$$

The analysis below is similar to the one for the problem $\mathcal{P}'$ (see Paragraph 2.2) and we use the same notation as in Paragraph 2.2.

If $f_n = 0$ the condition (15b) holds for arbitrary values of u_n . Without loss of generality we assume that in this case $u_n = 0$.

Let us analyse the case of $f_n > 0$. The first order conditions (15b) can be rewritten in the following form $v_n (\lambda_{n+1} - \lambda_n) \leq \Psi(\cdot)$, with strict equality for $u_n > 0$. The last expression leads to the following conclusions:

- if $u_n > 0$ then $n \in \text{Argmax}_{m \in \mathbb{N}: f_m > 0} v_m (\lambda_{m+1} - \lambda_m)$;
- the differential equation for λ_n , $n \in \mathbb{N}$ takes the following form $\dot{\lambda}_n = -g_n \Psi(t, \mathbf{f}, \boldsymbol{\lambda})$.

Therefore, there exists $\alpha(t)$ (see Appendix 6.2) such that $\lambda_n(t) / g_n = \alpha(t)$, $n \in \mathbb{N}$ and $\alpha(T) = 0$.

Analogously to Paragraph 2.2, the solution is characterised by the following properties:

- if $n \in \text{Argmax}_{m \in \mathbb{N}: f_m > 0} v_m \Delta_m$ then $u_n \geq 0$, else $u_n = 0$;
- the optimal value of the consumption c^* can be obtained from the following expression

$$c^* = \begin{cases} e^{-\rho(t-t_0)} U'(c^*) = \alpha \max_{m \in \mathbb{N}: f_m > 0} v_m \Delta_m, & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}: f_m > 0} v_m \Delta_m; \\ c_{\max}, & \text{otherwise;} \end{cases}$$

- the vector of optimal control $\mathbf{u}$, as previously, fulfils the following condition $\mathbf{u}: \sum_{n=1}^{\infty} (g_n - u_n) f_n = c^*$.

The differential equation for $\alpha(t)$ has the following form:

$$\dot{\alpha} = -\Psi(t, \mathbf{f}, \boldsymbol{\lambda}) \equiv \begin{cases} -e^{-\rho(t-t_0)} U'(c_{\max}), & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}: f_m > 0} v_m \Delta_m; \\ -\alpha \max_{m \in \mathbb{N}: f_m > 0} v_m \Delta_m, & \text{otherwise.} \end{cases}$$

Summarising, one observes that the solution in the case of heterogeneous investment effectiveness with respect to capital-labor ratio level is qualitatively equivalent to the solution of $\mathcal{P}'$, but the parameter Δ_n is replaced with the product $v_n \Delta_n$.

A.5. Concavity of the Hamiltonian in the problem $\mathcal{P}^c$

The Hamiltonian of a problem $\mathcal{P}^c$ is given by the expression (10c) which can be rewritten in the following form:

$$\mathcal{H} = e^{-\rho(t-t_0)} U \left(\sum_{n=1}^{\infty} (g_n - u_n) f_n \right) + \sum_{n=1}^{\infty} (\lambda_{n+1} - \lambda_n) \chi(u_n) f_n.$$

We confine our analysis to particular expressions for $U(\cdot)$ and $\chi(\cdot)$: $U(x) = \ln(x)$, $\chi(x) = \ln(1+x)$.

In this case $U \left(\sum_{n=1}^{\infty} (g_n - u_n) f_n \right) \equiv \ln \left(\sum_{n=1}^{\infty} (g_n - u_n) f_n \right)$ has the following Hessian matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{pmatrix} -f_1^2 / S & -f_1 f_2 / S & \dots & -f_1 f_n / S \\ -f_1 f_2 / S & -f_2^2 / S & \dots & -f_2 f_n / S \\ \dots & \dots & \dots & \dots \\ -f_1 f_n / S & -f_2 f_n / S & \dots & -f_n^2 / S \end{pmatrix}, \quad (33)$$

where $S = \left(\sum_{n=1}^{\infty} (g_n - u_n) f_n \right)^2$. Therefore for any vector $\mathbf{x} = (x_1, \dots, x_n)^T$: $\mathbf{x}^T \mathbf{H} \mathbf{x} = -(f_1 x_1 + \dots + f_n x_n)^2 / S \leq 0$, i.e. the Hessian matrix $\mathbf{H}$ is negative semi-definite, which means the concavity of the function $\ln \left(\sum_{n=1}^{\infty} (g_n - u_n) f_n \right)$ with respect to $\mathbf{u} = (u_1, u_2, \dots)$.

Therefore, if $\forall n \in \mathbb{N}: u_n > 0 \rightarrow (\lambda_{n+1} - \lambda_n) f_n \geq 0$, then the Hamiltonian (10c) is concave, as it is the sum of several concave functions.

In the case of $f_n = 0$ the above property does hold. Let us analyse the case of $f_n > 0$. The first order conditions in the problem of maximisation of the Hamiltonian with respect to u_n , $n \in \mathbb{N}$ have the following form:

$$\frac{\partial \mathcal{H}}{\partial u_n} = -e^{-\rho(t-t_0)} U' \left(c(\mathbf{f}, \mathbf{u}) \right) f_n + (\lambda_{n+1} - \lambda_n) f_n \chi'(u_n) \leq 0,$$

with equality in the case when $u_n > 0$. Let us analyse the case when $u_n > 0$, as $\chi(0) = 0$ then the term $(\lambda_{n+1} - \lambda_n) f_n \chi(u_n)$ does not appear in the Hamiltonian when $u_n = 0$. From the first order conditions it follows that $\chi'(u_n)(\lambda_{n+1} - \lambda_n) = \Psi(t, \mathbf{f}, \boldsymbol{\lambda})$. Analysing the case $\chi(x) = \ln(x+1)$ one obtains $\lambda_{n+1} - \lambda_n = \Psi(t, \mathbf{f}, \boldsymbol{\lambda})(u_n + 1) > 0$. Therefore, the Hamiltonian is concave with respect to $\mathbf{u} = (u_1, u_2, \dots)$.

A.6. Optimal control in the model with investment effectiveness heterogeneity with respect to size

In Appendix 6.5 it was shown that in the case when:

- $U(x) = \ln(x)$;
- $\chi(x) = \ln(x+1)$;

- $f_n > 0$;
- $u_n > 0$;

the following simplification of the first order conditions can be obtained:

$$\lambda_{n+1} - \lambda_n = \Psi(t, \mathbf{f}, \boldsymbol{\lambda})(u_n + 1). \quad (34)$$

Therefore, if $u_n, u_k > 0$ than

$$\frac{u_k + 1}{u_n + 1} = \frac{\lambda_{k+1} - \lambda_k}{\lambda_{n+1} - \lambda_n}.$$

Hence, the consumption $c(\mathbf{f}, \mathbf{u})$ is determined with the following expression:

$$\begin{aligned} c &\equiv c_{\max} - \sum_{n=1}^{\infty} u_n f_n = c_{\max} - \sum_{n: u_n > 0} \left((u_{n_0} + 1) \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n_0+1} - \lambda_{n_0}} - 1 \right) f_n = \\ &= c_{\max} + \sum_{n: u_n > 0} f_n - \frac{u_{n_0} + 1}{\lambda_{n_0+1} - \lambda_{n_0}} - \sum_{n: u_n > 0} (\lambda_{n+1} - \lambda_n) f_n, \end{aligned}$$

where n_0 is some level for which $u_{n_0} > 0$.

Substituting the last expression into the first order conditions (34) one obtains the expression for any $u_n > 0$:

$$u_n = (\lambda_{n+1} - \lambda_n) \frac{c_{\max} + \sum_{n: u_n > 0} f_n}{e^{-\rho(t-t_0)} + \sum_{n: u_n > 0} (\lambda_{n+1} - \lambda_n) f_n} - 1. \quad (35)$$

Therefore, to obtain the values of u_n one should first determine the set of levels with positive controls $\mathcal{C}^+ = \{n \in \mathbb{N} : u_n > 0\}$. The values of these controls are determined by (35). For convenience, let us also denote by $\mathcal{C}^0$ the set of levels with zero controls, i.e. $\mathcal{C}^0 = \{n \in \mathbb{N} : u_n = 0\} = \mathbb{N} \setminus \mathcal{C}^+$. Obviously, the level n corresponding to negative value $\lambda_{n+1} - \lambda_n$ lies in $\mathcal{C}^0$: $\{n \in \mathbb{N} : \lambda_{n+1} - \lambda_n < 0\} \subseteq \mathcal{C}^0$. Therefore, one should determine which levels $n \in \mathbb{N}$ with non-negative values of $(\lambda_{n+1} - \lambda_n)$ lie in $\mathcal{C}^0$ and which in $\mathcal{C}^+$.

Let us assume that the set $\mathcal{C}^+$ is known and also that $k_0 \in \mathbb{N}$ is such that $u_{k_0} = 0$, $f_{k_0} > 0$, $(\lambda_{k_0+1} - \lambda_{k_0}) \geq 0$ and $k_+ \in \mathbb{N}$ is such that $u_{k_+} > 0$. One can notice, that from the first order conditions it follows that $\lambda_{k_0+1} - \lambda_{k_0} < \Psi(t, \mathbf{f}, \boldsymbol{\lambda}) = (\lambda_{k_++1} - \lambda_{k_+}) / (u_{k_+} + 1) < \lambda_{k_++1} - \lambda_{k_+}$. Therefore, if

$$u_{k_+} = (\lambda_{k_++1} - \lambda_{k_+}) \frac{c_{\max} + \sum_{n \in \mathcal{C}^+} f_n}{e^{-\rho(t-t_0)} + \sum_{n \in \mathcal{C}^+} (\lambda_{n+1} - \lambda_n) f_n} - 1 > 0,$$

then¹⁰

$$(\lambda_{k_++1} - \lambda_{k_+}) \frac{c_{\max} + \sum_{n \in \mathcal{C}^+} f_n + f_{k_0}}{e^{-\rho(t-t_0)} + \sum_{n \in \mathcal{C}^+} (\lambda_{n+1} - \lambda_n) f_n + (\lambda_{k_0+1} - \lambda_{k_0}) f_{k_0}} - 1 > u_{k_+} > 0,$$

which means that if one “occasionally” adds the level k_0 (corresponding to zero control) to the set $\mathcal{C}^+$, the values of true positive controls will remain positive. This remark leads to the following algorithm of determination of $\mathcal{C}^0$ and $\mathcal{C}^+$.

10 This follows immediately from inequalities specified below. Let $a > b > 0$, $x > y > 0$. Then $(a+x)/(b+y) > (a+x)/(b+x) > a/b$. In the main text the following mapping is used:

$a = (\lambda_{k_++1} - \lambda_{k_+}) (c_{\max} + \sum_{n \in \mathcal{C}^+} f_n)$, $b = e^{-\rho(t-t_0)} + \sum_{n \in \mathcal{C}^+} (\lambda_{n+1} - \lambda_n) f_n$, $x = (\lambda_{k_++1} - \lambda_{k_+}) f_{k_0}$, $y = (\lambda_{k_0+1} - \lambda_{k_0}) f_{k_0}$.


```

 $i \leftarrow 0;$ 
 $\mathcal{C}^{0(0)} \leftarrow \{n \in \mathbb{N} : \lambda_{n+1} - \lambda_n < 0\};$ 
 $\mathcal{C}^{+(0)} \leftarrow \mathbb{N} \setminus \mathcal{C}^{0(0)};$ 

$$u_n^{(0)} = \begin{cases} 0, & \text{if } n \in \mathcal{C}^{0(0)}, \\ (\lambda_{n+1} - \lambda_n) \frac{c_{max} + \sum_{n \in \mathcal{C}^{+(0)}} f_n}{e^{-\rho(t-t_0)} + \sum_{n \in \mathcal{C}^{+(0)}} (\lambda_{n+1} - \lambda_n) f_n} - 1, & \text{otherwise;} \end{cases}$$

while  $\exists n \in \mathcal{C}^{+(i)} : u_n^{(i)} < 0$  do
     $\mathcal{C}^{0(i+1)} \leftarrow \{n : u_n^{(i)} < 0\} \cup \mathcal{C}^{0(i)};$ 
     $\mathcal{C}^{+(i+1)} \leftarrow \mathcal{C}^{+(i)} \setminus \mathcal{C}^{0(i+1)};$ 
    
$$u_n^{(i+1)} = \begin{cases} 0, & \text{if } n \in \mathcal{C}^{0(i+1)}, \\ (\lambda_{n+1} - \lambda_n) \frac{c_{max} + \sum_{n \in \mathcal{C}^{+(i+1)}} f_n}{e^{-\rho(t-t_0)} + \sum_{n \in \mathcal{C}^{+(i+1)}} (\lambda_{n+1} - \lambda_n) f_n} - 1, & \text{otherwise;} \end{cases}$$

     $i \leftarrow i + 1.$ 
end
 $\mathcal{C}^0 \leftarrow \mathcal{C}^{0(i)};$ 
 $\mathcal{C}^+ \leftarrow \mathcal{C}^{+(i)};$ 
 $u_n \leftarrow u_n^{(i)}.$ 

```

The algorithm output determines the sets $\mathcal{C}^0$ and $\mathcal{C}^+$ which makes it possible to determine the value of u_n for any $n \in \mathbb{N}$ from the formulae (29).

B. Solution in the case of decreasing returns to scale

Let us show how the boundary value problem (11')–(14') can be rewritten in the form of the system of non-linear equations.

To simplify the analysis we will analyse the logarithmic utility $U(c) = \ln(c)$. Let:

- the initial distribution have the form $f_n(0) = f_n^0$, $n \in \mathbb{N}$;
- $f_1^0 > 0$ (in other case we can renumber the levels);
- $t_0 = 0$.

The boundary value problem for $\mathcal{P}'$ is defined by (21)–(24).

Let us divide the time interval $[0, T]$ into a set of disjoint intervals: $[0, t_1)$, $[t_1, t_2)$, ..., $[t_s, T]$, where t_i is the moment when all firms from the levels lower than $(i+1)$ are moved to the level $(i+1)$. When $t \in [t_{i-1}, t_i)$ $\Delta_{m^*}(t)$ is equal to Δ_i .

We will refer to t_s as to the “stop time”, i.e. $t > t_s : u_n(t) = 0 \quad \forall n \in \mathbb{N}$. The moments $t_1, \dots, t_{s-1}$ we will call “transition time points”.

From (20) it follows that $\alpha(\tau) = C_i e^{-\Delta_i \tau}$, $\tau \in [t_{i-1}, t_i]$, $i \leq s$, where C_i is some constant. To ensure the continuity of $\alpha(t)$ the following

condition on C_i , $i \leq s$ should hold $C_i e^{-\Delta_i t_{i-1}} = C_{i-1} e^{-\Delta_{i-1} t_{i-1}}$, or, equivalently, $C_i = C_1 e^{(\Delta_2 - \Delta_1)t_1} e^{(\Delta_3 - \Delta_2)t_2} \dots e^{(\Delta_i - \Delta_{i-1})t_{i-1}}$. In addition let us note that $f_{i+1}(t) + f_i(t) = \sum_{k=1}^{i+1} f_k^0$, $t \in [t_{i-1}, t_i]$.

The evolution of the fraction of the firms on the level i takes the following form: $\dot{f}_i = -f_i \Delta_i + F_i + (C_i \Delta_i)^{-1} e^{(\Delta_i - \rho)\tau}$, $\tau \in [t_{i-1}, t_i]$, where $F_i = \sum_{k=i+2}^{\infty} f_k^0 g_k + g_{i+1} \sum_{k=1}^{i+1} f_k^0$. The solution of the inhomogeneous linear equation (B) is given by the following expression:

$$f_i = \frac{F_i}{\Delta_i} + G_i e^{-\Delta_i \tau} + \frac{(C_i \Delta_i)^{-1}}{2\Delta_i - \rho} e^{(\Delta_i - \rho)\tau},$$

where G_i is a constant that is determined from the following condition: $f_i(t_{i-1}) = \sum_{k=1}^i f_k^0$. The transition time point t_i can be found from the following condition t_i : $f_i(t_i) = 0$.

The stop time t_s is determined by the following condition:

$$C_i e^{-\Delta_i t_s} \Delta_i = \frac{e^{-\rho t_s}}{c_{\max}(t_s)} \equiv \frac{e^{-\rho t_s}}{\sum_{k=i+2}^{\infty} g_k f_k^0 + g_i f_i(t_s) + g_{i+1} f_{i+1}(t_s)},$$

or, equivalently,

$$\Delta_i \left\{ C_i \left(\sum_{k=i}^{\infty} g_k f_k^0 + g_i \sum_{k=1}^{i-1} f_k^0 \right) e^{-(2\Delta_i - \rho)t_s - \Delta_i t_{i-1}} + \left(e^{-\Delta_i(t_i - t_{i-1})} - 1 \right) / (2\Delta_i - \rho) \right\} = 1.$$

When $t > t_s$ the differential equation on $\alpha(t)$ has the following form $\dot{\alpha} = -e^{-\rho\tau} / c_{\max}(t_s)$, $\tau \in [t_s, T]$, from which it follows $\alpha(\tau) = e^{-\rho\tau} / (\rho c_{\max}(t_s)) + C^T$, where C^T can be found from the transversality condition $\alpha(T) = 0$: $C^T = e^{-\rho T} / (\rho c_{\max}(t_s)) + C^T$. In addition, to ensure continuity of the function $\alpha(t)$ in $t = t_s$ the following condition should hold $C_i e^{-\Delta_i t_s} = (e^{-\rho t_s} - e^{-\rho T}) / \rho c_{\max}(t_s)$, or, considering (B), $e^{-\rho T} = e^{-\rho t_s} (1 - \rho / \Delta_i)$.

Therefore, the solution of boundary value problem (21)–(24) is the same as one of the following system of non-linear equations:

$$f_1(C_1, t_1) = 0; \quad f_2(C_1, t_1, t_2) = 0; \quad \dots; \quad f_{i-1}(C_1, t_1, \dots, t_{i-1}) = 0;$$

$$C_i e^{-\Delta_i t_s} \Delta_i = e^{-\rho t_s} / c_{\max}(t_s); \quad e^{-\rho T} = e^{-\rho t_s} (1 - \rho / \Delta_i); \quad t_s < t_i.$$

The number of equations in the above system is determined by the parameters of the problem.

C. Solution in the case with exogenous TFP growth

The problem with two growth factors, exogenous TFP and endogenous capital-labor ratio level (problem $\mathcal{P}^a$), can be formalised by the following series of equations:

$$W = \int_{t_0}^T e^{-\rho(t-t_0)} U(c(t)) dt \rightarrow \max_{u_n(\cdot), n \in \mathbb{N}}, \quad (2a)$$

$$c(t) = \sum_{n=1}^{\infty} (e^{\gamma(t-t_0)} g_n - u_n(t)) f_n(t), \quad (3a)$$

$$\frac{df_n(d)}{dt} = -u_n f_n + u_{n-1} f_{n-1}, \quad n \in \mathbb{N}, \quad (5a)$$

$$f_n(0) = \text{fix}, \quad n \in \mathbb{N}, \quad (6a)$$

$$u_n(t) \geq 0. \quad (7a)$$

The analysis of this case is completely analogous to the one presented in Section 2.2. The solution is characterised by the following properties:

- if $n \in \text{Arg max}_{m \in \mathbb{N}; f_m > 0} \Delta_m$ then $u_n \geq 0$, else $u_n = 0$;
- optimal value of consumption c^* can be obtained from the following expression

$$c^* = \begin{cases} U'^{-1} \left(e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}; f_m > 0} \Delta_m \right), & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}; f_m > 0} \Delta_m; \\ c_{\max}, & \text{otherwise;} \end{cases}$$

where $c_{\max} = \sum_{n=1}^{\infty} e^{\gamma(t-t_0)} g_n f_n$;

- the vector of optimal control $\mathbf{u}$ fulfils the following condition
- $$\mathbf{u}: \sum_{n=1}^{\infty} (e^{\gamma(t-t_0)} g_n - u_n) f_n = c^*.$$

The differential equation for $\alpha(t) \equiv \lambda_n(t) / g_n \quad \forall n \in \mathbb{N}$ has the following form:

$$\dot{\alpha} = -e^{\gamma(t-t_0)} \Psi(t, \mathbf{f}, \boldsymbol{\lambda}) \equiv \begin{cases} -e^{(\gamma-\rho)(t-t_0)} U'(c_{\max}), & \text{if } U'(c_{\max}) < e^{\rho(t-t_0)} \alpha \max_{m \in \mathbb{N}; f_m > 0} \Delta_m; \\ -e^{\gamma(t-t_0)} \alpha \max_{m \in \mathbb{N}; f_m > 0} \Delta_m, & \text{otherwise.} \end{cases}$$

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Шумпетерианская эволюция распределения фирм по величине капиталовооруженности

Аннотация. В соответствии с идеей Й. Шумпетера существует два ключевых драйвера экономического роста. В ряде работ эта идея реализована в формализме кинетических уравнений. В этих работах предполагается, что общая факторная производительность (ОФП) – единственный фактор эффективности работы фирм. Более полным является описание процесса экономического развития, включающее разные факторы эффективности, например ОФП и капиталовооруженность. В данной работе сделан предварительный шаг к построению модели с двумя факторами. Рассмотрена модель центрального планирования, эндогенный рост в которой обеспечивается капиталовооруженностью. Представленная модель описывает эволюцию распределения фирм по уровням капиталовооруженности вдоль оптимальной траектории в форме дифференциально-разностного аналога уравнения Бюргерса. Показано, что в случае, когда эффективность инвестиций не зависит от объема, а производство описывается убывающей отдачей от масштаба, фирмы концентрируются на одном уровне капиталовооруженности. Однако, в предположении убывания эффективности инвестиций с ростом объема, ширина распределения фирм по уровням капиталовооруженности увеличивается во времени. Кроме того, показано, что данный результат может сохраняться для случая возрастающей отдачи от масштаба.

Ключевые слова: шумпетерианская динамика, экономический рост, капиталовооруженность, распределение фирм, уравнение Бюргерса.

Классификация JEL: O33, E22, C61, C65.

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