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Elements of Cartesian Analysis in Saidou Spaces

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Abstract

The objective of this research paper is to follow up on the work already started in order to install a new mathematical analysis, the one we called Cartesian analysis see our previous works[1],[2]. During these last two papers, we started with the definition of cartesian geometry and we defined and introduced to a new Cartesian topology which gave birth to new spaces called Saidou spaces. Thus, and to follow up on this way, we propose to study the analytical and functional part of this analysis. We will define the notion of a Cartesian function and then its epigraph before to characterize the analytical properties of these functions, for example the continuity and the differentiability. In an other hand, we will see the relationship between cartesian functions and the convex functions. According to the latest papers and this work we do the asset of the first foundations of the cartesian analysis.

Keywords: Cartesian subsets, Saidou Spaces, Cartesian functions., Epigraphs
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1. Introduction

Among the various aspects of mathematics that have contributed to functional analysis, we must underline the convex analysis, a branch that has had fertile tracks in analytical and even geometric forms. It will be shown that convex analysis and Cartesian analysis coincide in particular cases, but generally they are different. We have to denote that this geometry will be a generalization of the polyhedron and polytopes in affine spaces, see[5],[6]. The regular polytopes and polyhedron are convex sets but cartesian sets are the union not necessary convex of many polytopes see Figure4 below. After defining and characterizing this new notion, you will realize how this track would open interesting doors in the field of the applications of mathematics for the benefit of applied sciences, like medicine, physics, engineering, economics and even sociology.

We will give an original definition of a Cartesian set and then to define a locally cartesian topology which called Saidou Space. At the last part of this paper we start the main study of this paper which is to define a new kind of functions and characterize the relationship between the cartesian topology and the cartesian functions by doing a functional analysis. This is to prepare the elements for the optimization of cartesian problems for future works.

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2. Definitions and Preliminaries

Let E be a Banach space, E' its dual and $\langle \cdot, \cdot \rangle$ the dual product.

1) H is called an hyperplan in E if and only if, there exists a linear form x' belonging to the dual of E such that: $H = \{y \in E, \langle x', y \rangle = \alpha, \alpha \in \mathbb{R}\}$.

2) $D = \{y \in E, \langle x', y \rangle \leq \alpha\}$. is called the lower half space defined by $x' = \alpha$.

3) $D' = \{y \in E, \langle x', y \rangle \geq \alpha\}$. is called the upper half space defined by $x' = \alpha$.

4) A subset C in E is said to be convex if and only if, for each x, y in C , the segment $[x, y]$ is included in C .

Remark 2.1. It is easy to prove that H, D and D' are closed convex sets in E .

Definition 2.2. C is said to be a regular Cartesian set of E if and only if, C is the intersection of a finite family of D_i , with D_i are half-spaces of E . See Figure 1 below.

$$C = \cap (D_i)$$

Figure 1: Regular Cartesian Sets

Example 2.3. . A square is a regular Cartesian set defined by the intersection of four half spaces.

$[0, 1] \times [0, 1]$ is the intersection of the following four half-spaces:

The half space below the line $y = 1$, the half space above $y = 0$, the half space to the left of the vertical axis $x = 1$ and the half space to the right of the vertical axis $x = 0$.

A triangle is a regular Cartesian set defined by the intersection of three half-spaces.

Now, we have all the necessary assets to pronounce the first definition of a Cartesian set. In fact it would be a concatenation of regular Cartesian sets.

Definition 2.4. Let C be a set of E . We say that C is Cartesian if and only if C is a finite union of regular Cartesian sets connected to each other.

$$C = \cup \cap (D_{i,j})$$

See Figure 2.5.

Example 2.5. .

As illustrated in the Figure3 above, we note that set C is an union of regular Cartesian sets as it has been illustrated and defined above.

We could proceed geometrically by inverse process. That is to say, consider the Cartesian set C and subdivide it to find the regular Cartesian subsets. See Figure4 below.

Figure 2: Cartesian sets as a finite union of regular cartesian sets

Figure 3: A cartesian set in the left subdivided in two regular cartesian sets in the right

We can see this notion of regular Cartesian sets on $\mathbb{R}^n$ which are represented in the form of polyhedra see [5], [6], [7] sets that have already been treated before in the context of convex analysis. The cartesian subsets can be illustrated as the finite union of polyhedra. See Figure 4 below.

Figure 4: Example of Cartesian sets in $\mathbb{R}^n$

Lemma 2.6. . *A regular cartesian set is always cartesian*

Proof. The proof is trivial according to the definitions. □

Lemma 2.7. . *Let E be a Banach space. The finite intersection and the finite union of Cartesian sets are Cartesians.*

Proof. Let C_k a finite family of cartesian sets in E , then for each k , $C_k = \cup(\cap D_{i,j,k})$ then

$$\cup(C_k) = \cup(\cup(\cap D_{i,j,k})) = \cup(\cap D_{i,j,k}).$$

So, $\cup(C_k)$ is also cartesian.

Samely $\cap(C_k) = \cap(\cup(\cap D_{i,j,k})) = \cup(\cap D_{i,j,k})$ which is cartesian. □

Now, we will begin to give characterizations linking with the other notions of convex analysis and the functional analysis, knowing that there would be a multitude of future results and digging in this track for Cartesian sets. The usefulness of the results below is to demonstrate that the "cartesianity" and the convexity are two completely different notions, which gives the utility of this field (Cartesian analysis).

Proposition 2.8. *A regular cartesian set is always closed convex but the reverse is not always true.*

Proof. Let C a regular cartesian subset in E , it is clear that C is closed because C is an intersection of a finite closed sets D_i . Now we have to demonstrate that C is convex.

Let $x, y \in C$ it suffice to prove that $[x, y]$ is included in C .

Let $\alpha \in [0, 1]$, we will prove that,

$$\alpha.x + (1 - \alpha).y \in C.$$

$x \in C$ then $x \in D_i$ for each i . Then for each i , $\langle x'_i, x \rangle \leq \alpha_i$ with $x'_i \in E'$ the dual of E and $\alpha_i \in \mathbb{R}$.

Samely $y \in C$ then for each i , $\langle x'_i, y \rangle \leq \alpha_i$. We can deduce that,

$$\langle x', \alpha.x + (1 - \alpha).y \rangle \leq \alpha_i.$$

Hence C is convex.

The reverse is not true, we consider the circle in $\mathbb{R}^2$. The $C((0, 0), 1)$ is convex. Suppose that $C = \cap (D_i)$. In $\mathbb{R}^2$, each half space can have an equation like $ax + by \leq \alpha$.

We will verify that $(0, 0) \in D_i$ for each i .

We have $(0, 1)$ and $(-1, 0)$ in $C((0, 0), 1)$ then $(0, 1)$ and $(-1, 0)$ in D_i for each i , which implies that $a \leq \alpha_i$ and $-a \leq \alpha_i$, therefore, $\alpha_i \geq 0$. Hence

$$a.0 + b.0 \leq \alpha_i$$

then $(0, 0)$ is in D_i for each i , which means that $(0, 0) \in C$ absurd. □

Proposition 2.9. *Let C is a cartesian set, then $C + a$ is also cartesian with a in E .*

According to the definition, a cartesian subset $C = \bigcup \cap (D_{i,j})$, with $D_{i,j} = \{y \in E, \langle x'_{i,j}, y \rangle \leq \alpha_{i,j}\}$. It is easy to say that:

$$D_{i,j} + a = \{y + a \in E, \langle x'_{i,j}, y + a \rangle \leq \alpha_{i,j}\}.$$

Hence, if we put $\alpha_{i,j} - \langle x'_{i,j}, a \rangle = \beta_{i,j}$ we have:

$$D_{i,j} + a = \{y \in E, \langle x'_{i,j}, y \rangle \leq \beta_{i,j}\}.$$

Therefore we put $D_{i,j} + a = D^1_{i,j}$ then, $C + a = \bigcup \cap (D^1_{i,j})$. Then, $C + a$ is also cartesian.

2.1. Saidou Spaces and its Topology

in this section, we define a new topology in a Banach space and then the locally cartesian spaces which called Saidou spaces. As defined in the previous work see [1], a locally cartesian topology is compounded by a fundamental system of neighborhood compact cartesian of 0.

Definition 2.10. *A Banach space E is called Locally cartesian space (Saidou Space) if and only if it is generated by the locally cartesian topology.*

Example 2.11. *The space $\mathbb{R}^n$ with the norm $\|x\| = \sum |x_i|$ is a Saidou space because the unit ball is cartesian. Samely, the space $\mathbb{R}^n$ with the norm $\|x\| = \max |x_i|$ is a Saidou space because the unit ball is also cartesian.*

Now, we will install a new cartesian analysis in the saidou spaces by studying several properties like the studies of cartesian functions and their properties: continuity, its epigraphs, differentiability. Then, we will see the optimization criteria. Thus a comparison between cartesian optimization and convex optimization will be given.

2.2. The cartesian hull of A

For all the next results, we consider E a Saidou space.

Definition 2.12. Let A a subset of E . The cartesian hull of A is the smalest cartesian subset including A , denoted by: $\text{Car}(A)$

Lemma 2.13. . Let E be a saidou space. The cartesian hull of a subset A is the intersection of all cartesian subset containing A $\text{Car}(A) = \bigcap_{F \supset A} F$ with F cartesian.

Proof. The proof is clear because $\bigcap_{F \supset A} F$ is the smalest subset containing A . According to the lemma 2.8 above the $\text{Car}(A)$ is also cartesian. $\square$

Corollary 2.14. Let A a subset of E . If A is cartesian, then : $\text{Car}(A) = A$

The proof is clear from the definition.

We will use this cartesian hull in the future in order te cartesianize a subset in E .

3. Cartesian Functions

Definition 3.1. Let E a Saiou space, A function $f : E \longrightarrow \mathbb{R}$ is called a cartesian function if and only if, for all x in E there exists y in E such that: $\forall \alpha \in [0, 1]$.

$$f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)$$

see Figure below

Remark 3.2. . We have to denote that there is no relation between a cartesian function and a convex function.

Example.

The function $f(x) = x^2$ is a convex function but it is not cartesian.

The function $f(x) = 0$ if $x \in]-\infty, 0[$ and $f(x) = x$ if $x \in [0, 2[$ and $f(x) = 2$ if $x \in [2, +\infty[$, is cartesian but not convex.

3.1. Operation of Cartesian Functions

Theorem 3.3. Let f and g two cartesian functions defined in E with values in $[-\infty, +\infty]$. Then, we have:

- 1) $f + g$ is cartesian.
- 2) For all $\lambda \in \mathbb{R}$, λf is cartesian.
- 3) $\text{Sup}(f, g)$ is cartesian.
- 4) $\text{Min}(f, g)$ is cartesian.

Proof. Suppose that f et g are cartesian, then for all x in Df there exists y_1 and y_2 such that $\forall \alpha \in [0, 1]$.

$$f(\alpha x + (1 - \alpha)y_1) = \alpha f(x) + (1 - \alpha)f(y_1)$$

and,

$$g(\alpha x + (1 - \alpha)y_2) = \alpha g(x) + (1 - \alpha)g(y_2)$$

Let $x \in Df \cap Dg$. So we will treat two cases.

1) If $y_1 \in [x, y_2]$ then, we can say that:

$$\begin{aligned} (f + g)(\alpha x + (1 - \alpha)y_1) &= f(\alpha x + (1 - \alpha)y_1) + g(\alpha x + (1 - \alpha)y_1) \\ &= \alpha f(x) + (1 - \alpha)f(y_1) + \alpha g(x) + (1 - \alpha)g(y_1) \end{aligned}$$

Hence,

$$(f + g)(\alpha x + (1 - \alpha)y_1) = \alpha(f + g)(x) + (1 - \alpha)(f + g)(y_1)$$

2) If $y_2 \in [x, y_1]$ then, we do the same replacing y_1 by y_2

Finally, we conclude that for all x in $Df \cap Dg$ there exists y such that $\forall \alpha \in [0, 1]$.

$$(f + g)(\alpha x + (1 - \alpha)y) = \alpha(f + g)(x) + (1 - \alpha)(f + g)(y)$$

Which means that $f + g$ is cartesian. For the second statement, it is clear that if f is cartesian for all x in Df there exists y_1 such that $\forall \alpha \in [0, 1]$.

$$f(\alpha x + (1 - \alpha)y_1) = \alpha f(x) + (1 - \alpha)f(y_1)$$

which is true for λf . Just multiply by λ .

For the Statement 3. We have to prove that $\text{sup}(f, g)$ is cartesian knowing that f and g are both cartesian. We know that for all x in Df there exists y_1 such that $\forall \alpha \in [0, 1]$.

$$f(\alpha x + (1 - \alpha)y_1) = \alpha f(x) + (1 - \alpha)f(y_1)$$

On $[x, y_1]$ if $f = \text{sup}(f, g)$ then $\text{sup}(f, g)$ is cartesian. otherwise, there exist a in $[x, y_1]$ such that $f = \text{sup}(f, g)$ on $[x, a]$ and $g = \text{sup}(f, g)$ on $[a, y_1]$. Then for x there exist a in Df such that,

$$f(\alpha x + (1 - \alpha)a) = \alpha f(x) + (1 - \alpha)f(a).$$

which mean that:

$$\text{sup}(f, g)(\alpha x + (1 - \alpha)a) = \alpha \text{sup}(f, g)(x) + (1 - \alpha) \text{sup}(f, g)(a).$$

□

We can generalize these statements to a finite subset of functions.

Theorem 3.4. Let $(f_i)_{i \in I}$, $I \subset \mathbb{N}$ a finite family of cartesian functions defined in E with values in $] -\infty, +\infty]$. Then we have:

- 1) $\sum_{i \in I} f_i$ is cartesian .
- 2) $\forall \lambda_{i \in I}, \sum_{i \in I} \lambda_i f_i$ is cartesian .
- 3) $\sup_{i \in I} (f_i)$ is cartesian.
- 4) $\min_{i \in I} (f_i)$ is cartesian.

The proof is direct consequence of the theorem 3.3 above. The demonstration can be given using the recurrence.

Following these results and if we denote $\text{Car}(E, K)$ the set of cartesian functions defined on E and its values in K , then we can confirm the theorem below.

Theorem 3.5. $(\text{Car}(E, K), +, \cdot)$ is a vector space on $K \in \mathbb{R} \text{ or } \mathbb{C}$.

The proof is clear because according to the theorem 3.3 above $(\text{Car}(E, K), +, \cdot)$ is a vector subspace of the vector space of functions.

Remark 3.6. In the convex analysis field, the function $\delta_C = 0$ on C and $\delta_C = +\infty$ otherwise is called the support function on C . It is an important function. So this function is a lower semi continuous convex function. Specially, we can say that it is a lower semi continuous cartesian function. We have to signal that the function is the fundamental link between the convex subsets and the convex functions.

Theorem 3.7. The composition gof of two cartesian functions is also cartesian.

Proof. . Let x in $Df \subset E$, then there exists y in Df such that, $\forall \alpha \in [0, 1]$.

$$f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)$$

g is also cartesian, for each $z = f(x)$ there exist $t = f(y)$ such that, $\forall \alpha \in [0, 1]$.

$$g(\alpha z + (1 - \alpha)t) = \alpha g(z) + (1 - \alpha)g(t)$$

Then, we can conclude that for each x there exist t such that, $\forall \alpha \in [0, 1]$.

$$\text{gof}(\alpha x + (1 - \alpha)y) = \alpha \text{gof}(x) + (1 - \alpha)\text{gof}(y)$$

then, gof is cartesian. □

3.2. Epigraph of Cartesian Functions

In this section we characterize the epigraph of cartesian. As you know, the convexity of convex function is equivalent to the convexity of its epigraphs. In this section we will show the same equivalence for cartesian functions. Before, we will recall the definition of epigraph of f and give the characterization of the cartesian functions with its epigraphs.

$$\text{epif} = \{(x, r) \in E \times \mathbb{R} / f(x) \leq r\}.$$

Theorem 3.8. A function f is cartesian if and only if its epigraph is cartesian in $E \times \mathbb{R}$.

The proof of the theorem requires some preliminary results.

Proposition 3.9. . Let f a lower semi continuous function, then the interior of its epigraph:

$$\text{int}(\text{epif}) = \{(x, r) \in E \times \mathbb{R} / f(x) < r\}.$$

Proof. Let f a lower semi continuous function, we have to recall that the norm in $E \times \mathbb{R}$ is $\|(x, \lambda)\| = \sup(\|x\|, |\lambda|)$ with x in E and λ in $\mathbb{R}$. In the first we prove that $\text{int}(\text{epif}) \supset \{(x, r) \in E \times \mathbb{R} / f(x) < r\}$. Let (a, λ) such that $f(a) < \lambda$. We put $\varepsilon = \frac{(\lambda - f(a))}{2} > 0$. Using the lower semi continuity of f at a we can say that for the ε considered above there exists δ such that,

$$\|y - a\| \leq \delta \Rightarrow |f(y) - f(a)| \leq \varepsilon.$$

Now, we consider the ball $B((a, \lambda), r)$ with $r = \inf(\varepsilon, \delta)$. We can confirm that

$$B((a, \lambda), r) \subset \text{epif}.$$

For this, let (y, β) in $B((a, \lambda), r)$, this means that

$$\|y - a\| \leq r \leq \varepsilon$$

and

$$|\beta - \lambda| \leq r \leq \varepsilon$$

Using all the inequation above and replacing ε , we have the two inequalities:

$$\frac{(\lambda + f(a))}{2} \leq \beta.$$

and

$$f(y) \leq \frac{(\lambda + f(a))}{2}$$

So, if we combine these two inequalities we conclude that

$$f(y) \leq \beta.$$

Consequently, (y, β) is in epif which means that $B((a, \lambda), r) \subset \text{epif}$. Finally,

$$(a, \lambda) \in \text{int}(\text{epif}).$$

The other inclusion. Let $(x, \lambda) \in \text{int}(\text{epif})$, then there exist $\varepsilon > 0$ such that the ball $B((x, \lambda), \varepsilon) \subset \text{epif}$. It is clear that

$$(x, \lambda - \frac{\varepsilon}{2}) \in B((x, \lambda), \varepsilon)$$

Hence, $(x, \lambda - \frac{\varepsilon}{2}) \in \text{epif}$, which means that :

$$f(x) \leq \lambda - \frac{\varepsilon}{2} < \lambda.$$

Therefore, $f(x) < \lambda$. Finally,

$$\text{int}(\text{epif}) \subset \{(x, r) \in E \times \mathbb{R} / f(x) < r\}.$$

□

Proposition 3.10. *The boundary of an epigraph of a lower or upper semi continuous function is its graph. which means. $\text{Fr}(\text{epif}) = \text{Gr}(f)$ Fr : designates the boundary.*

Proof. We recall that the boundary of a subset C is $\text{Fr}(C) = \text{cl}(C) \setminus \text{int}(C)$ where $\text{cl}(C)$ respectively $\text{int}(C)$ are the closure respectively the interior of C .

So, let $(x, f(x))$ an element of $\text{Gr}(f)$. Show that $(x, f(x))$ is not in $\text{int}(\text{epif})$. This because epif is colsed (f is lower semi continuous).

Suppose that $(x, f(x))$ is in $\text{int}(\text{epif})$ then $f(x) > f(x)$ because $(x, f(x)) \in \text{epif}$. Absurd. For the reverse inclusion, suppose that $(x, r) \in \text{Fr}(\text{epif})$ then using the definition above and the proposition 3.10, we can say that $(x, r) \in (\text{epif})$ and (x, r) is not in $\{(x, a) / f(x) < a\}$ which means that $f(x) \geq r$. Hence $f(x) = r$.

Thus, we have $\text{Fr}(\text{epif}) = \text{Gr}(f)$.

□

In the next proposition below we try to demonstrate that a function f is cartesian if and only if its graph is an union of the segments see figure 5.

We denote by $[(x, f(x)), (y, f(y))]$ a segment in $E \times \mathbb{R}$

Proposition 3.11. . Let f a lower semi continuous function in a Saidou space E . The following statements are equivalent.

1) f is cartesian.

2) $\text{Gr}(f) = \cup_{x \neq y \in Df} [(x, f(x)), (y, f(y))]$

Proof. Suppose that f is cartesian, we consider an any point $(x, f(x)) \in \text{Gr}(f)$ and we choose $y_x \neq x$. So, it easy to say that $(x, f(x)) \in [(x, f(x)), (y_x, f(y_x))]$ because the segment is closed. Then $(x, f(x)) \in \cup_{x \neq y_x \in Df} [(x, f(x)), (y_x, f(y_x))]$ which means that:

$$\text{Gr}(f) \subset \cup_{x \neq y_x \in Df} [(x, f(x)), (y_x, f(y_x))]$$

For the other inclusion, let $(a, b) \in \cup_{x \neq y_x \in Df} [(x, f(x)), (y, f(y))]$, then there exists x and $y_x \neq x$ such that $(a, b) \in [(x, f(x)), (y_x, f(y_x))]$. Then there exist $\alpha \in [0, 1]$ and,

$$a = \alpha x + (1 - \alpha)y_x.$$

and,

$$b = \alpha f(x) + (1 - \alpha)f(y_x)$$

Or f is cartesian , then $f(a) = b$. Hence $(a, b) \in \text{Gr}(f)$. Finally

$$\text{Gr}(f) = \cup_{x \neq y \in Df} [(x, f(x)), (y, f(y))]$$

Now, suppose that $\text{Gr}(f) = \cup_{x \neq y \in Df} [(x, f(x)), (y, f(y))]$. Let $a \in Df$ then $(a, f(a)) \in \text{Gr}(f)$. According to 2) we can say that :

$$(a, f(a)) \in \cup_{x \neq y_x \in Df} [(x, f(x)), (y_x, f(y_x))]$$

We can say that there exists $x \in Df$ and $y_x \neq x$ such that:

$$a = \alpha x + (1 - \alpha)y_x$$

and,

$$f(a) = \alpha f(x) + (1 - \alpha)f(y_x)$$

The last inequality is

$$f(\alpha x + (1 - \alpha)y_x) = \alpha f(x) + (1 - \alpha)f(y_x)$$

which means that f is cartesian. □

According to all this results above, the demonstration of the theorem 3.9 above will be a consequence using the main theorem in our previous work See [1].

Proof. Let f a lower semi continuous cartesian function in a saidou space. We have to prove that its epigraph is a cartesian subset in $E \times \mathbb{R}$. According to the proposition .. above the graph $\text{Gr}(f)$ of f is an union of segments which are linear subsets in $E \times \mathbb{R}$. Using the proposition 3.13 above we can say that the boundary $\text{Fr}(\text{epif})$ of epigraph of f is an union of linear subsets. Therefore, according to the main theorem... see [1], we conclude that epigraph of f is cartesian subset in $E \times \mathbb{R}$. If epif is cartesian, we can process by equivalence which let us to confirm that f is cartesian. □

4. Conclusion

This work is a continuity of the previous works see [1], [2]. In order to install the first fundements of a New analysis, new topology and new spaces. In this paper we gave some elements about functional analysis. We defined the cartesian function and its epigraphs. Then we proved some results which characterize the cartesianity of functions in term of cartesian subsets in $E \times R$. You remark that in this paper we did not treat the continuity and differentiability of cartesian functions and the characterization of the last notion. So, the differentiability will be the subject of our futur studie and our futur work. Finally, we have to denote that there is a power similarity between cartesianity and convexity. This one, will be used to open the way for the optimization in case of cartesian functions.

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