On strong CNZ rings and their extensions

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Abstract

T.K. Kwak and Y. Lee called a ring R satisfy the *commutativity of nilpotent elements at zero*[1] if $ab = 0$ for $a, b \in N(R)$ implies $ba = 0$. For simplicity, a ring R is called CNZ if it satisfies the commutativity of nilpotent elements at zero. In this paper we study an extension of a CNZ ring with its endomorphism. An endomorphism α of a ring R is called *strong right (resp., left) CNZ* if whenever $\alpha\alpha(b) = 0$ (resp., $\alpha(a)b = 0$) for $a, b \in N(R)$ $ba = 0$. A ring R is called *strong right (resp., left) α -CNZ* if there exists a strong right (resp., left) CNZ endomorphism α of R , and the ring R is called *strong α -CNZ* if R is both strong left and right α -CNZ. Characterization of strong α -CNZ rings and their related properties including extensions are investigated. In particular, it's shown that a ring R is reduced if and only if $U_2(R)$ is a CNZ ring. Furthermore extensions of strong α -CNZ rings are studied.

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1. Introduction

Throughout this paper R denotes an associative ring with identity. Cohn [7] called a ring *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. A reversible ring is a generalization of a reduced ring (i.e., it has no nonzero nilpotent elements). T.K. Kwak and Y. Lee called a ring R satisfy the *commutativity of nilpotent elements at zero*[1] if $ab = 0$ for $a, b \in N(R)$ implies $ba = 0$. For simplicity, a ring R is called CNZ if it satisfies the commutativity of nilpotent elements at zero. Recently, the concept of commutativity of nilpotent elements at zero is extended to one of both an endomorphism and an element in a ring.

From [2, Definition 2.1], an endomorphism α of a ring R is called a *right (resp., left) skew CNZ* if whenever $ab = 0$ for $a, b \in N(R)$, $b\alpha(a) = 0$ (resp., $\alpha(b)a = 0$), and the ring R is called *right (resp., left) α -skew CNZ* if there exists a right (resp., left) skew CNZ endomorphism α of R . A ring R is called *α -skew CNZ* if it is both left and right α -skew CNZ.

From [5, Definition 2.1], an endomorphism α of a ring R is called *strong right (resp., left) reversible* if whenever $\alpha\alpha(b) = 0$ for $a, b \in N(R)$ (resp., $\alpha(a)b = 0$), we get $ba = 0$ and the ring R is called a *strong right (resp., left) α -CNZ* if there exists a strong right (resp., left) reversible endomorphism α of R . A ring R is called *strong α -reversible* if it is both strong left and right α -reversible. Note that R is an α -rigid ring if and only if R is semiprime and strong right α -reversible for a monomorphism α of R by [5, Proposition 1.2(2)].

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Motivated by above, we extend in this paper the concept of the CNZ ring property to the strong CNZ ring property by ring endomorphisms, introducing the notation of a *strong α -CNZ* ring for an endomorphism α of a ring as a generalization of strong α -reversible rings as well as an extension of CNZ rings, and then study the structure of strong α -CNZ rings and their related properties. Consequently, several known results are obtained as corollaries of our results.

Throughout this paper, α denotes a nonzero endomorphism of a given ring, unless specified otherwise.

2. Basic characterization and properties on strong strong α -CNZ rings

Definition 2.1. An endomorphism α of a ring R is called strong *right* (resp., *left*) CNZ if whenever $\alpha\alpha(b) = 0$ for $a, b \in N(R)$ (resp., $\alpha(a)b = 0$), we get $ba = 0$ and the ring R is called a strong *right* (resp., *left*) α -CNZ if there exists a strong right (resp., left) CNZ endomorphism α of R . A ring R is called strong α -CNZ if it is both strong left and right α -CNZ.

The following results are a direct consequence of routine computations

Remark 2.2. Let α be an endomorphism of a ring R (i) Every subring S with $\alpha(S) \subseteq S$ of a strong right (resp., left) α -CNZ ring is also a strong right (resp., left) α -CNZ ring. (ii) Every α -rigid ring is strong α -CNZ and every strong right (resp., left) α -CNZ ring is an α -skew CNZ. (iii) Any strong α -CNZ ring R is left-right symmetric whenever either $\alpha^2 = \text{id}_R$ or R is a reversible ring, where id_R denotes the identity endomorphism of R .

For a nonempty subset S of a ring R , $r_R(S) = \{c \in R \mid Sc = 0\}$ is called the *right annihilator* of S in R . The left annihilator is defined similarly and written by $\ell_R(S)$. If $S = \{a\}$, then we write $r_R(a)$ (resp., $\ell_R(a)$) instead of $r_R(\{a\})$ (resp., $\ell_R(\{a\})$).

Proposition 2.3. For a ring R with an endomorphism α of R , the following statements are equivalent:

- (1) R is strong right (resp., left) α -CNZ.
- (2) $\ell_{N(R)}(\alpha(S)) = r_{N(R)}(S)$ (resp., $r_{N(R)}(\alpha(S)) = \ell_{N(R)}(S)$) for any nonempty subset S of $N(R)$.
- (3) For each $a \in N(R)$, $\ell_{N(R)}(\alpha(a)) = r_{N(R)}(a)$ (resp., $r_{N(R)}(\alpha(a)) = \ell_{N(R)}(a)$).
- (4) $A\alpha(B) = 0$ (resp., $\alpha(A)B = 0$) if and only if $BA = 0$ for any nonempty subsets A, B of $N(R)$.

Proof. (1) $\Rightarrow$ (2) For $a \in N(R)$ and $N(S) \subseteq N(R)$, $a \in \ell_{N(R)}(\alpha(S))$ if and only if $\alpha\alpha(N(S)) = 0$ if and only if $N(S)a = 0$ by (1), if and only if $a \in r_{N(R)}(S)$. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are straightforward. (3) $\Rightarrow$ (4) Let A and B be nonempty subsets of $N(R)$. Then $A\alpha(B) = 0$ if and only if $\alpha\alpha(b) = 0$ for all $a \in A$ and $b \in B$ if and only if $a \in \ell_{N(R)}(\alpha(b)) = r_{N(R)}(b)$ if and only if $ba = 0$ by (3), if and only if $BA = \{\sum_{a \in A, b \in B} ba\} = 0$. The case of a strong left α -CNZ ring can be proved similarly. $\square$

A ring R is CNZ if R is strong one-sided id_R -CNZ. Any domain is strong α -CNZ for a monomorphism α of R , but the converse does not hold by [6, Example 1.4], while there is a commutative reduced ring which is not a strong right α -CNZ ring by the next example.

Consider the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, say A . Then A is a commutative reduced ring. Let $\alpha : A \rightarrow A$ be the endomorphism defined by $\alpha((a, b)) = (b, a)$. For

$$x = \begin{pmatrix} (0,0) & (1,0) \\ (0,1) & (0,0) \end{pmatrix}, y = \begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (0,0) \end{pmatrix} \in N(\text{Mat}_2(A)),$$

we have $x\alpha(y) = 0$ but

$$yx = \begin{pmatrix} (0,0) & (0,1) \\ (1,0) & (0,0) \end{pmatrix} x = \begin{pmatrix} (0,0) & (1,0) \\ (0,1) & (0,0) \end{pmatrix} = \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (1,0) \end{pmatrix} \neq 0.$$

Thus $\text{Mat}_2(A)$ is not a strong right $\bar{\alpha}$ -CNZ.

Proposition 2.4. For a reversible ring R with an endomorphism α of R , the following statement are equivalent:

- (1) R is strong α -CNZ.
- (2) R is strong right α -CNZ.
- (3) If either $\alpha\alpha^n(b) = 0$ or $\alpha^n(a)b = 0$ for a positive n and $a, b \in N(R)$, then $ab = 0$. Conversely, $ab = 0$ for $a, b \in N(R)$ implies $\alpha\alpha^m(b) = 0$ and $\alpha^m(a)b = 0$ for any positive integer m .
- (4) For each $a, b \in N(R)$ $ab = 0$ implies $\alpha\alpha(b) = 0$.

Proof. (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are obvious. (2) $\Rightarrow$ (3) is straightforward. suppose that $\alpha\alpha^n(b) = 0$ for a positive n and $a, b \in N(R)$. Since R is reversible and strong right α -CNZ, $\alpha\alpha^n(b) = 0$ implies $0 = \alpha^{n-1}(b)a = \alpha\alpha^{n-1}(b) = \dots = ba = ab$. Similarly, $\alpha^n(a)b = 0$ yields $ab = 0$. The remainder is clear by hypothesis. (4) $\Rightarrow$ (1) follows from definition. □

The following example illuminates that there exists right α -skew CNZ ring which is not strong right α -CNZ.

Consider a ring $R = U_2(\mathbb{Z}_4)$. Then

$$N(R) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, c \in \{0, 2\} \text{ and } b \in \mathbb{Z}_4 \right\}.$$

(1) Let $\alpha : R \rightarrow R$ be an endomorphism defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Assume that $AB = 0$ for

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in N(R).$$

Then $aa' = 0$ and it implies that $B\alpha(A) = 0$ since $\mathbb{Z}_4$ is commutative. Hence R is a right α -skew CNZ, but it is not right strong α -CNZ. Indeed, for

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R),$$

we have $A\alpha(B) = 0$ but

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$$

According to Krempa [17], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ for $a \in R$ implies $a = 0$, and the ring R is called α -*rigid* [11] if there exists a rigid endomorphism α of R . Note that any rigid endomorphism of a ring is a monomorphism and α -rigid rings are reduced rings by [11, Proposition 5].

Following [10], a ring R is said to be α -*compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow \alpha\alpha(b) = 0$. If R is an α -compatible ring, then the endomorphism α is clearly a monomorphism. Note that R is α -compatible if and only if the left version holds (i.e., for each $a, b \in R$, $ab = 0 \Leftrightarrow \alpha(a)b = 0$). Note also that the left version holds then clearly α is a monomorphism. Indeed, if R is α -compatible then $ab = 0 \Leftrightarrow \alpha(ab) = \alpha(a)\alpha(b) = 0 \Leftrightarrow \alpha(a)b = 0$; if the left version holds then $ab = 0 \Leftrightarrow \alpha(ab) = \alpha(a)\alpha(b) = 0 \Leftrightarrow \alpha\alpha(b) = 0$. We use this fact freely in the procedure of this note.

Proposition 2.5. (1) For a ring R with a monomorphism α , R is strong right α -CNZ above if and only if R is right α -skew CNZ.

(2) For a α -compatible ring R , R is strong right α -CNZ if and only if R is right α -skew CNZ.

Proof. (1) Suppose that R is strong right α -CNZ. Let $ab = 0$ for $a, b \in N(R)$. Then $\alpha(a)\alpha(b) = \alpha(ab) = 0$, and so $b\alpha(a) = 0$ by assumption. Thus R is right α -skew CNZ.

Conversely, assume that R is right α -skew CNZ. If $a\alpha(b) = 0$ for $a, b \in N(R)$ then $\alpha(b)\alpha(a) = 0$ since $\alpha(b) \in N(R)$. Hence $ba = 0$ because α is a monomorphism, concluding that R is strong right α -CNZ.

(2) is an immediate consequence of (1) because α is a monomorphism when R is α -compatible. $\square$

The following example shows that the condition “ α -compatibility” is not superfluous.

Theorem 2.6. (1) Let α_γ be an endomorphism of a ring R_γ for each $\gamma \in \Gamma$. Then the following are equivalent:

- (i) R_γ is a strong right α_γ -CNZ ring for each $\gamma \in \Gamma$.
- (ii) The direct sum $\bigoplus_{\gamma \in \Gamma} R_\gamma$ of R_γ is strong right $\bar{\alpha}$ -CNZ for the endomorphism $\bar{\alpha} : \bigoplus_{\gamma \in \Gamma} R_\gamma \rightarrow \bigoplus_{\gamma \in \Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma \in \Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma \in \Gamma}$.
- (iii) The direct product $\prod_{\gamma \in \Gamma} R_\gamma$ of R_γ is strong right $\bar{\alpha}$ -CNZ for the endomorphism $\bar{\alpha} : \prod_{\gamma \in \Gamma} R_\gamma \rightarrow \prod_{\gamma \in \Gamma} R_\gamma$ defined by $\bar{\alpha}((a_\gamma)_{\gamma \in \Gamma}) = (\alpha_\gamma(a_\gamma))_{\gamma \in \Gamma}$.

(2) Let R be a ring with an endomorphism α . (i) If e is a central idempotent of a ring R with $\alpha(e) = e$ and $\alpha(1-e) = 1-e$, then eR and $(1-e)R$ are strong right α -CNZ if and only if R is strong right α -CNZ.

(ii) Let S be a ring and $\sigma : R \rightarrow S$ be a ring isomorphism. Then R is strong right α -CNZ if and only if S is a strong right $\sigma\alpha\sigma^{-1}$ -CNZ.

Proof. (1) It is enough to show that (i) $\Rightarrow$ (iii) because the class of strong α -CNZ rings is closed under subrings. Note that $N(\prod_{\gamma \in \Gamma} R_\gamma) \subseteq \prod_{\gamma \in \Gamma} N(R_\gamma)$ and $\alpha_\gamma(R_\gamma) \subseteq R_\gamma$ for each $\gamma \in \Gamma$. Suppose that R_γ is strong right α_γ -CNZ for each $\gamma \in \Gamma$ and let $A\bar{\alpha}(B) = 0$ where $A = (a_\gamma)_{\gamma \in \Gamma}, B = (b_\gamma)_{\gamma \in \Gamma} \in N(\prod_{\gamma \in \Gamma} R_\gamma)$. Then $a_\gamma\alpha(b_\gamma) = 0$ for each $\gamma \in \Gamma$ and $b_\gamma a_\gamma = 0$ by hypothesis, since $a_\gamma, b_\gamma \in N(R_\gamma)$ for each $\gamma \in \Gamma$. This implies $BA = 0$, entailing that the direct product $\prod_{\gamma \in \Gamma} R_\gamma$ of R_γ is strong right $\bar{\alpha}$ -CNZ.

(2) (i) It comes from (1) with $R \cong eR \oplus (1-e)R$ and $\alpha(eR) \subseteq eR, \alpha((1-e)R) \subseteq (1-e)R$ since the class of strong right α -CNZ rings is closed under subrings S with $\alpha(S) \subseteq S$.

(ii) First we have $N(S) = \sigma(N(R))$ clearly. Let R be strong right α -CNZ and $a'\sigma\alpha\sigma^{-1}(b') = 0$ for $a', b' \in N(S)$. Since σ is an isomorphism, $a' = \sigma(a), b' = \sigma(b)$ for some $a, b \in N(R)$. Then $0 = \sigma(a)\sigma\alpha\sigma^{-1}(\sigma(b)) = \sigma(a\alpha(b))$, and so $a\alpha(b) = 0$ and $ba = 0$ since R is strong right α -CNZ ring and σ is an isomorphism. Hence, $0 = \sigma(ba) = b'a'$ and therefore S is strong right $\sigma\alpha\sigma^{-1}$ -CNZ.

Conversely, let $a\alpha(b) = 0$ for $a, b \in N(R)$. Then $0 = \sigma(a\alpha(b)) = \sigma(a)\sigma\alpha\sigma^{-1}(\sigma(b))$, and thus $\sigma(b)\sigma(a) = 0$ since S is strong right $\sigma\alpha\sigma^{-1}$ -CNZ. Hence $ba = 0$, entailing R is strong right α -CNZ. The proof is complete. $\square$

Recall that for a ring R with an endomorphism α and an ideal I of R , if I is an α -ideal (i.e., $\alpha(I) \subseteq I$) of R , then $\bar{\alpha} : R/I \rightarrow R/I$ defined by $\bar{\alpha}(a+I) = \alpha(a)+I$ for $a \in R$ is an endomorphism of a factor ring R/I .

Theorem 2.7. Let R be a ring with an endomorphism α and I an α -ideal of R . If R/I is a strong right (resp., left) $\bar{\alpha}$ -CNZ and I is α -rigid as a ring without identity, then R is strong right (resp., left) α -CNZ ring.

Proof. Suppose that R/I is a strong right $\bar{\alpha}$ -CNZ and I is α -rigid as a ring without identity. Let $a\alpha(b) = 0$ for $a, b \in N(R)$. Then $(a+I)(\alpha(b)+I) = I$ and $a+I, b+I \in N(R/I)$. Since R/I is strong right $\bar{\alpha}$ -CNZ, $ba \in I$. Hence $ba\alpha(ba) = ba\alpha(b)\alpha(a) = 0$ and so $ba = 0$ since I is an α -rigid ring. Thus R is strong right α -CNZ. $\square$

The condition “ I is α -rigid as a ring without identity” in Theorem 2.7 is not superfluous by Example 2(2): In fact, $(x+I)\bar{\alpha}(x+I) = I$, where

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \notin I.$$

(1) Let K be a field and $R = K\langle a, b \rangle$ be the free algebra with noncommuting indeterminates a, b over K . Then R is a domain. Define an automorphism α of R by

$$a \mapsto b \text{ and } b \mapsto a.$$

Then R is obviously an strong α -CNZ.

Now, let I be the ideal of R generated by

$$ab, a^2 \text{ and } b^3.$$

For $a + I, b + I \in N(R/I)$, we get $(a + I)\bar{\alpha}((b + I)) = (a + I)(\alpha(b) + I) = a^2 + I = I$, but $(b + I)(a + I) = ba + I \neq I$ by the construction of I . Thus R/I is not a strong right $\bar{\alpha}$ -CNZ. This concludes that the class of strong right α -CNZ rings is not closed under homomorphic images.

(2) We refer to [2, Example 2.8]. Let A be a reduced ring and consider a ring $R = U_3(A)$ with an endomorphism α defined by

$$\alpha \left(\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

Then

$$N(R) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d \in A \right\}.$$

For

$$x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in N(R),$$

we obtain $x\alpha(y) = 0$, but

$$yx = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Thus R is not a strong right α -CNZ.

Now, for a nonzero proper ideal

$$I = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ 0 & 0 & A \end{pmatrix}$$

of R , $R/I \cong U_2(A)$ is a strong right $\bar{\alpha}$ -CNZ by Proposition 2.4(1) and $\alpha(I) \subseteq I$ obviously.

For a ring R with an endomorphism α and $n \geq 2$, the corresponding $(a_{ij}) \rightarrow (\alpha(a_{ij}))$ induces an endomorphism of $\text{Mat}_n(R)$, $U_n(R)$ and $D_n(R)$, respectively. We denote them by $\bar{\alpha}$.

Notice that for a reduced ring R , both $U_2(R)$ and $D_2(R)$ are strong $\bar{\alpha}$ -CNZ for any endomorphism α of R by Proposition 2.4(1). We will freely use these facts without reference.

However there exists a reduced ring A with an endomorphism α such that $\text{Mat}_2(A)$ is not strong right $\bar{\alpha}$ -CNZ as follows.

Define an automorphism α of $\mathbb{Z}_2$ by

$$0 \mapsto 1 \text{ and } 1 \mapsto 0.$$

Consider $R = \text{Mat}_2(\mathbb{Z}_2)$. For $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in N(R)$, we have

$$a\bar{\alpha}(b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bar{\alpha} \left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha(1) & \alpha(1) \\ \alpha(1) & \alpha(1) \end{pmatrix} = 0,$$

but

$$ba = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0.$$

Thus $\text{Mat}_2(A)$ is not strong right $\bar{\alpha}$ -CNZ.

Remark 2.8. Note that $\text{Mat}_n(R)$, $D_n(R)$ and $U_n(R)$, for $n \geq 3$ are not strong right $\bar{\alpha}$ -CNZ for any ring R with an endomorphism α such that $\alpha(1) \neq 0$ (e.g., α is a monomorphism).

Let R be a ring with an endomorphism α such that $\alpha(1) \neq 0$. For the ring $D_3(R)$, consider $e_{12}, e_{23} \in N(D_3(R))$. Then $e_{23}\bar{\alpha}(e_{12}) = 0$, but

$$e_{12}e_{23} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

showing that $D_3(R)$ is not a strong right $\bar{\alpha}$ -CNZ.

Similarly, we can show that $D_m(R)$ for $m \geq 4$ is not strong right $\bar{\alpha}$ -CNZ.

Consequently, it can be obtained that $\text{Mat}_n(R)$ and $U_n(R)$ for $n \geq 3$ are not strong right $\bar{\alpha}$ -CNZ, since the class of strong α -CNZ rings is closed under subrings S with $\alpha(S) \subseteq S$.

One may ask whether both $D_2(R)$ and $U_2(R)$ are strong right $\bar{\alpha}$ -CNZ when either R is a reversible ring or R is a strong right α -CNZ ring with an endomorphism α . However the answer is negative by the following example.

(1) We apply the ring construction and argument in [16, Example 2.1]

Consider the free algebra $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c \rangle$ with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2$ over $\mathbb{Z}_2$. Define an automorphism δ of A by

$$a_0, a_1, a_2, b_0, b_1, b_2, c \mapsto b_0, b_1, b_2, a_0, a_1, a_2, c,$$

respectively. Let B be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$\begin{aligned} & a_0a_0, a_0a_1 + a_1a_0, a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2, \\ & b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2, \\ & (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2) \text{ and } r_1r_2r_3r_4 \end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \subseteq I$. Set $R = A/I$.

Since $\delta(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. We identify every element of A with its image in R for simplicity. Then R is a reversible ring by the argument in [16, Example 2.1]. Note that R is not strong right α -CNZ, since $a_0\alpha(b_0) = a_0\delta(b_0) = a_0a_0 = 0$ for $a_0, b_0 \in N(R)$, but $b_0a_0 \neq 0$.

Now we show that $D_2(R)$ is not strong right $\bar{\alpha}$ -CNZ. For

$$x = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}, y = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} \in N(D_2(R)),$$

we get $x\bar{\alpha}(y) = 0$ by the construction of I . But

$$yx = \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} = \begin{pmatrix} b_0a_0 & b_0a_1 + b_1a_0 \\ 0 & b_0a_0 \end{pmatrix} \neq 0,$$

entailing that $D_2(R)$ is not a strong right $\bar{\alpha}$ -CNZ.

Therefore we conclude that both $D_n(R)$ and $U_n(R)$ for $n \geq 2$ need not be strong right $\bar{\alpha}$ -CNZ.

(2) We use [2, Example 3.5]. Consider a ring $R = U_2(A)$ over a reduced ring A and an endomorphism α of R is defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Then R is strong right α -CNZ by Proposition 2.4(1). Clearly R is not reversible. For

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(D_2(R))$$

with $A^3 = 0$ and $B^2 = 0$, we have $A\bar{\alpha}(B) = 0$ but

$$BA = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Thus $D_2(R)$ is not strong right $\bar{\alpha}$ -CNZ, and it implies that $D_n(R)$ and $U_n(R)$ for $n \geq 2$ need not be strong right $\bar{\alpha}$ -CNZ, when R is strong right α -CNZ with an endomorphism α .

3. Extensions of strong α -CNZ rings

Given a ring R and an (R, R) -bimodule M , the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:

$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note that $T(R, R) = D_2(R)$ and For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R , $\bar{\alpha} : T(R, R) \rightarrow T(R, R)$ defined by

$$\bar{\alpha} \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix},$$

is an endomorphism of $T(R, R)$. Since $T(R, 0)$ is isomorphic to R , we can identify the restriction of $\bar{\alpha}$ by $T(R, 0)$ to α .

Proposition 3.1. *Let R be a reduced ring with an endomorphism α . Then R is strong α -CNZ if and only if the trivial extension $T(R, R)$ is a strong $\bar{\alpha}$ -CNZ.*

Proof. It is sufficient to show that the trivial extension $T(R, R)$ of a reduced and strong α -CNZ R is strong $\bar{\alpha}$ -CNZ. Assume that $A\alpha(B) = 0$ for

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in N(T(R, R)).$$

Then, $a\alpha(c) = 0$ and $a\alpha(d) + b\alpha(c) = 0$. Since R is strong α -CNZ, we get $ca = 0$ and $0 = c(a\alpha(d) + b\alpha(c)) = cb\alpha(c)$, and so $c^2b = 0$. Since R is reduced, $cb = 0$ and $bc = 0$, and so $b\alpha(c) = 0$ by Proposition 2.5. Then $a\alpha(d) = 0$ and $da = 0$. Consequently, we have $BA = 0$ and therefore $T(R, R)$ is strong $\bar{\alpha}$ -CNZ. $\square$

Corollary 3.2. *If R is a reduced ring with an endomorphism α . Then $T(R, R)$ is CNZ ring.*

The condition “ R is reduced ring” in Proposition 3.1 cannot be dropped by the next example. Consider a ring $R = U_2(A)$ over a reduced ring A and an endomorphism α of R is defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

Then R is strong right α -CNZ by proposition 2.4(1). But R is not reversible (and so not reduced). For

$$A = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } B = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \in N(D_2(R)) = N(T(R, R))$$

with $A^3 = 0$ and $B^2 = 0$, we have $A\bar{\alpha}(B) = 0$ but

$$BA = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Thus $T(R, R)$ is not strong right $\bar{\alpha}$ -CNZ, and it implies that $T(R, R)$ need not be strong right $\bar{\alpha}$ -CNZ, when R is strong right α -CNZ with an endomorphism α .

For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \dots, n-2$ and $t = 2, \dots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$.

Theorem 3.3. Let R be a ring with an endomorphism α . Let R be a reduced ring and n a positive integer, then R is strong right α -CNZ with $\alpha(1) = 1$ if and only if $\frac{R[x]}{\langle x^n \rangle}$ is strong right $\bar{\alpha}$ -CNZ ring, where $\langle x^n \rangle$ is the ideal generated by x^n .

Proof. Since the class of strong right α -CNZ rings is closed under subrings S with $\alpha(S) \subseteq S$, so R is strong right α -CNZ.

Conversely, assume that R is a reduced and strong right α -CNZ. Let $V_n(R) = \frac{R[x]}{\langle x^n \rangle}$. Note that $N(V_n(R)) = (N(R), R, \dots, R)$. If $n = 1$, then $V_1(R) \cong R$. For $n \geq 2$, if $n = 2$, since $V_2(R) \cong T(R, R)$ it is a strong right $\bar{\alpha}$ -CNZ ring by Proposition 3.1. Now we assume $n \geq 3$. Since $V_n(R) \cong \frac{R[x]}{x^n R[x]}$ let $f = \sum_{i=0}^{n-1} a_i \bar{x}^i$, $g = \sum_{j=0}^{n-1} b_j \bar{x}^j \in N(V_n(R))$ with $f\bar{\alpha}(g) = 0$, where $\bar{x} = x + \langle x^n \rangle$. If $i+j \geq n$, then $a_i \alpha(b_j) \bar{x}^{i+j} = 0$. Hence, we set $i+j \leq n-1$. Then $f\bar{\alpha}g = 0$ implies the following system of equations:

$$(1) \quad a_0 \alpha(b_0) = 0 \text{ and } a_0 \alpha(b_k) + a_1 \alpha(b_{k-1}) + a_2 \alpha(b_{k-2}) + \dots + a_{k-1} \alpha(b_1) + a_k \alpha(b_0) = 0, k = 1, 2, \dots, n-1.$$

Multiplying Eq.(1) by b_0 on the left hand-side, we get $b_0 a_k \alpha(b_0) = 0$ by induction hypothesis and assumption $b_0^2 a_k = 0$ and so $b_0 a_k = 0$ and $a_k \alpha(b_0) = 0$ by the assumption. Hence Eq.(1) becomes

$$(2) \quad a_0 \alpha(b_0) = 0 \text{ and } a_0 \alpha(b_k) + a_1 \alpha(b_{k-1}) + \dots + a_{k-1} \alpha(b_1) = 0,$$

Multiplying Eq.(2) by b_1 on the left hand-side, by a similar way we get $b_1 a_{k-1} \alpha(b_1) = 0$ and hence $b_1 a_{k-1} = 0$ and $a_{k-1} \alpha(b_1) = 0$ Eq.(2) becomes

$$(3) \quad a_0 \alpha(b_0) = 0 \text{ and } a_0 \alpha(b_k) + a_1 \alpha(b_{k-1}) + \dots + a_{k-2} \alpha(b_2) = 0$$

Multiplying Eq. (3) by b_2 on the left hand-side, we get $b_2 a_{k-2} \alpha(b_2) = 0$ and so $b_2 a_{k-2} = 0$ by the same argument as above. Continuing this process yields $b_j a_i = 0$ for $i + j = k$, and consequently $b_j a_i = 0$ for all i and j . Thus $gf = 0$, proving that $V_n(R)$ is a strong right $\bar{\alpha}$ -CNZ.

□

For a ring R and $n \geq 2$, let

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ 0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \mid a_1, a_2, \dots, a_n \in R \right\}.$$

The next corollary is follows directly from Theorem 3.3.

Corollary 3.4. Assume that R is a reduced ring with an endomorphism α . Then R is strong right α -CNZ ring. if and only if $V_n(R)$ is a strong right $\bar{\alpha}$ -CNZ ring. for any $n \geq 2$.

Proof. Note that $V_n(R) \cong R[x]/\langle x^n \rangle$ by [24].

□

Corollary 3.5. If R is a reduced ring. Then $V_n(R) \cong R[x]/\langle x^n \rangle$ is a CNZ ring for any positive integer n .

Rege and Chhawchharia called a ring R Armendariz [23, Definition 1.1] if whenever the product of any two polynomials in $R[x]$ over R is zero, then so is the product of any pair of coefficients from two polynomials. It is well-known that any reduced ring is Armendariz, but not conversely. The concept of an Armendariz ring is extended to power series rings over general noncommutative rings. A ring R is called power-serieswise Armendariz [15] if $a_i b_j = 0$ for all i, j whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i, g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$ satisfy $f(x)g(x) = 0$. Power-serieswise Armendariz rings are Armendariz by definition, but not conversely by [15, Example 2.4].

Following [4], a ring R is called skew power-serieswise α -Armendariz if $a_i b_j = 0$ for all i and j whenever $p(x)q(x) = 0$ for $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \alpha]]$. It is shown that R is a α -rigid ring if and only if R is reduced and skew power-serieswise α -Armendariz in [4, Theorem 3.3(1)]. It is obvious that skew power-serieswise α -Armendariz property of a ring is inherited to its subrings, and α is clearly a monomorphism by help of [4, Theorem 3.3(3)]. (We also change over from “a skew power series Armendariz ring with the endomorphism α ” in [4] to “a skew power-serieswise α -Armendariz ring”.) Due to [22, Definition 2.1], a ring R with an endomorphism α is called skew powerserieswise Armendariz if for every skew power series $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \alpha]]$, $p(x)q(x) = 0$ if and only if $a_i b_j = 0$ for all i, j . It is clear that every skew powerserieswise Armendariz ring is skew power-serieswise α -Armendariz. Note that the exists a ring R with an endomorphism α such that $ab = 0$ but $\alpha x b \neq 0$, in general. For example, consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the endomorphism α of R defined by $\alpha((a, b)) = (b, a)$ as in Example 2. Then $(0, 1)(1, 0) = 0$ but $(0, 1)x(1, 0) = (0, 1)x \neq 0$. However, the two concept of skew power-serieswise α -Armendariz ring property of [4] and skew powerserieswise Armendariz ring property of [22] are actually coincided, since every skew power-serieswise α -Armendariz ring is α -compatible by help of [14, Proposition 3.14].

Lemma 3.6. Let R be a skew power-serieswise α -Armendariz ring and α an endomorphism of R . If we let S is one of symbols $[x; \alpha], [x, x^{-1}; \alpha], [[x; \alpha]]$ or $[[x, x^{-1}; \alpha]]$, then $N(RS) = N(R)S$, in case α is an automorphism.

Proof. It directly follows from [22, Theorem 2.13].

□

Theorem 3.7. *Let R be a skew power-serieswise α -Armendariz ring and α an automorphism of R . Then the following are equivalent:*

- (1) R is strong right α -CNZ ring.
- (2) $R[x; \alpha]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (3) $R[x, x^{-1}; \alpha]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (4) $R[[x; \alpha]]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (5) $R[[x, x^{-1}; \alpha]]$ is a strong right $\bar{\alpha}$ -CNZ ring.

Proof. It suffices to show that (1) $\Rightarrow$ (5): Assume that (1) holds R is a strong right α -CNZ ring. Let $p(x)\bar{\alpha}(q(x)) = 0$ for $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in N(R[[x, x^{-1}; \alpha]])$. Then $a_i, b_j \in N(R)$ by Lemma 3.6 and so $a_i \alpha(b_j) = 0$ for all i, j . Thus $b_j a_i = 0$ by (1), since R is α -compatible as noted above. This yields $q(x)p(x) = 0$, and thus $R[[x, x^{-1}; \alpha]]$ is strong right $\bar{\alpha}$ -CNZ ring. $\square$

Corollary 3.8. *Let R be a power-serieswise Armendariz ring. The following are equivalent:*

- (1) R is strong right α -CNZ ring.
- (2) $R[x]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (3) $R[x, x^{-1}]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (4) $R[[x]]$ is a strong right $\bar{\alpha}$ -CNZ ring.
- (5) $R[[x, x^{-1}]]$ is a strong right $\bar{\alpha}$ -CNZ ring.

For a ring R with endomorphism α , the corresponding $\sum a_i x^i \rightarrow \sum \alpha(a_i) x^i$ induces an endomorphism of $R[x; \alpha], R[x, x^{-1}; \alpha], R[[x; \alpha]]$ and $R[[x, x^{-1}; \alpha]]$, respectively. We denote them by $\bar{\alpha}$.

The concept of a strong right α -CNZ ring. ring does not go up to skew polynomial rings (skew power series rings) by next example.

We adapt the ring in [14, Example 2.8], based on [16, Example 2.1]. We take the same A and the automorphism δ of A as in Example 2.

Let C be the set of all polynomials with zero constant terms in A and consider the ideal I of A generated by

$$\begin{aligned} & a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, a_1 b_2 + a_2 b_1, a_2 b_2, a_0 r b_0, a_2 r b_2, \\ & b_0 a_0, b_0 a_1 + b_1 a_0, b_0 a_2 + b_1 a_1 + b_2 a_0, b_1 a_2 + b_2 a_1, b_2 a_2, b_0 r a_0, b_2 r a_2, \\ & (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \\ & a_0 a_0, a_2 a_2, a_0 r a_0, a_2 r a_2, b_0 b_0, b_2 b_2, b_0 r b_0, b_2 r b_2, r_1 r_2 r_3 r_4, \\ & a_0 a_1 + a_1 a_0, a_0 a_2 + a_1 a_1 + a_2 a_0, a_1 a_2 + a_2 a_1, b_0 b_1 + b_1 b_0, b_0 b_2 + b_1 b_1 + b_2 b_0, b_1 b_2 + b_2 b_1, \\ & (a_0 + a_1 + a_2)r(a_0 + a_1 + a_2), (b_0 + b_1 + b_2)r(b_0 + b_1 + b_2), \end{aligned}$$

where $r, r_1, r_2, r_3, r_4 \in C$. Then clearly $C^4 \subseteq I$. Set $R = A/I$. Since $\delta(I) \subseteq I$, we can obtain an automorphism α of R by defining $\alpha(s + I) = \delta(s) + I$ for $s \in A$. We identify every element of A with its image in R for simplicity.

For $p(x) = a_0 + a_1 x^2 + a_2 x^4, q(x) = b_0 c + b_1 c x^2 + b_2 c x^4 \in N(R[x; \alpha])$ since $C^4 \subseteq I$, we have

$$p(x)\bar{\alpha}(q(x)) = (a_0 + a_1 x^2 + a_2 x^4)(a_0 c + a_1 c x^2 + a_2 c x^4) = 0$$

but

$$q(x)p(x) = (b_0 c + b_1 c x^2 + b_2 c x^4)(a_0 + a_1 x^2 + a_2 x^4) \neq 0.$$

Thus $R[x; \alpha]$ is not strong right $\bar{\alpha}$ -CNZ ring.

Notice that R is reversible and right α -skew CNZ by [16, Example 2.1] and [2, Example 3.6], respectively. Thus R is a strong right α -CNZ ring by Proposition 2.6(2-ii), since α is an automorphism of R .

Let R be a ring and α a monomorphism of R . Now we consider the Jordan's construction of an over-ring of R by α (see [13] for more details). Let $A(R, \alpha)$ be the subset $\{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. Note that for $j \geq 0$, $x^j r = \alpha^j(r)x^j$ implies $rx^{-j} = x^{-j}\alpha^j(r)$ for $r \in R$. This yields that for each $j \geq 0$ we have $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$. It follows that $A(R, \alpha)$ forms a subring of $R[x, x^{-1}; \alpha]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $r, s \in R$ and $i, j \geq 0$. Note that $A(R, \alpha)$ is an over-ring of R , and the map $\bar{\alpha} : A(R, \alpha) \rightarrow A(R, \alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$ is an automorphism of $A(R, \alpha)$. Jordan showed, with the use of left localization of the skew polynomial $R[x; \alpha]$ with respect to the set of powers of x , that for any pair (R, α) , such an extension $A(R, \alpha)$ always exists in [13]. This ring $A(R, \alpha)$ is usually said to be the *Jordan extension* of R by α .

Proposition 3.9. *For a ring R with a monomorphism α , R is a strong right α -CNZ if and only if the Jordan extension $A = A(R, \alpha)$ of R by α is a strong right $\bar{\alpha}$ -CNZ.*

Proof. It is enough to show the necessity. Suppose that R is a strong right α -CNZ and $c\bar{\alpha}(d) = 0$ for $c = x^{-i}rx^i, d = x^{-j}sx^j \in N(A)$ for $i, j \geq 0$. Then $r, s \in N(R)$ obviously. From $c\bar{\alpha}(d) = 0$, we get $\alpha^j(r)\alpha^i(s) = 0$ and so $\alpha^i(s)\alpha^j(r) = 0$ by assumption. Hence

$$\begin{aligned} dc &= (x^{-j}sx^j)(x^{-i}rx^i) \\ &= x^{-(j+i)}\alpha^i(s)\alpha^j(r)x^{i+j} = x^{-(j+i)}\alpha^i(s)\alpha^j(r)x^{i+j} = 0. \end{aligned}$$

Thus the Jordan extension A of R by α is a strong right $\bar{\alpha}$ -CNZ. □

Let R be an algebra over a commutative ring S . Due to Dorroh [8], the *Dorroh extension* of R by S is the Abelian group $R \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$. We use D to denote the Dorroh extension of R by S . For an S -endomorphism α of R and the Dorroh extension D of R by S , $\bar{\alpha} : D \rightarrow D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S -algebra homomorphism.

Theorem 3.10. *Let R be an algebra over a commutative reduced ring S with an S -endomorphism α . Then R is a strong right α -CNZ ring if and only if the Dorroh extension D of R by S is a strong right $\bar{\alpha}$ -CNZ.*

Proof. It can be easily checked that $N(D) = (N(R), 0)$ since S is a commutative reduced ring. Then every nilpotent element D is of the form $(r, 0)$ for some nilpotent element r of R . Suppose that R is a strong right α -CNZ ring. Let $(r_1, 0), (r_2, 0) \in N(D)$ with $(r_1, 0)\bar{\alpha}((r_2, 0)) = 0$. Then $r_1\alpha(r_2) = 0$. Then $r_2r_1 = 0$ so $(r_2, 0)(r_1, 0) = 0$.

This implies that R is a strong right α -CNZ if and only if the Dorroh extension D is a strong right $\bar{\alpha}$ -CNZ. □

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* is defined, and *regular* means if it is both left and right regular (and hence not a zero divisor). Assume that M is a multiplicatively closed subset of R consisting of central regular elements. Let α be an automorphism of R and assume $\alpha(m) = m$ for every $m \in M$. Then $\alpha(m^{-1}) = m^{-1}$ in $M^{-1}R$ and the induced map $\bar{\alpha} : M^{-1}R \rightarrow M^{-1}R$ defined by $\bar{\alpha}(u^{-1}a) = u^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.11. *Let R be a ring with an automorphism α and assume that there exists a multiplicatively closed subset M of R consisting of central regular elements and $\alpha(m) = m$ for every $m \in M$. Then R is a strong right α -CNZ ring if and only if $M^{-1}R$ is a strong right $\bar{\alpha}$ -CNZ.*

Proof. It suffices to prove the necessary condition. First, note that $N(M^{-1}R) = M^{-1}N(R)$. Suppose that R is strong right α -CNZ. Let $A\bar{\alpha}(B) = 0$ with $A = u^{-1}a, B = v^{-1}b \in N(M^{-1}R)$ where $u, v \in M$ and $a, b \in N(R)$. Then $\alpha\alpha(b) = 0$ and so $ba = 0$ by assumption. Thus

$$BA = (v^{-1})b(u^{-1}a) = v^{-1}u^{-1}ba = 0,$$

showing that $M^{-1}R$ is a strong right $\bar{\alpha}$ -CNZ. □

Let R be a ring with an endomorphism α . Recall that the map $R[x] \rightarrow R[x]$ (resp., $R[x, x^{-1}] \rightarrow R[x, x^{-1}]$) defined by $\sum_{i=0}^m a_i x^i \mapsto \sum_{i=0}^m \alpha(a_i) x^i$ (resp., $\sum_{i=0}^{\infty} a_i x^i \mapsto \sum_{i=0}^{\infty} \alpha(a_i) x^i$) is an endomorphism of $R[x]$ (resp., $R[x, x^{-1}]$), and clearly the map extends α . We still denote the extended maps $R[x] \rightarrow R[x]$ and $R[x, x^{-1}] \rightarrow R[x, x^{-1}]$ by $\bar{\alpha}$.

Corollary 3.12. *Let R be a ring with an endomorphism α such that $\alpha(1) = 1$. Then $R[x]$ is a strong right $\bar{\alpha}$ -CNZ. if and only if $R[x; x^{-1}]$ is a strong right $\bar{\alpha}$ -CNZ.*

Proof. It directly follows from Proposition 3.11. For, letting $M = \{1, x, x^2, \dots\}$, M is a multiplicatively closed subset of $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$ and $\bar{\alpha}(x) = x$ since $\alpha(1) = 1$. □

A ring R is called right Ore if for given $a, b \in R$ with b is regular, there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known fact that R is a right Ore ring if and only if the classical right quotient ring $Q(R)$ of R exists.

Let R be a ring with the classical right quotient ring $Q(R)$. Then each automorphism α of R extends to $Q(R)$ by setting $\bar{\alpha}(ab^{-1}) = \alpha(a)(\alpha(b))^{-1}$ for $a, b \in R$, assuming that $\alpha(b)$ is regular for each regular element $b \in R$.

Theorem 3.13. *Let R be a right Ore ring with the classical right quotient ring $Q(R)$ of R and α an automorphism of R . If $Q(R)$ is an NI ring, then R is a strong right α -CNZ ring if and only if $Q(R)$ is a strong right $\bar{\alpha}$ -CNZ.*

Proof. It suffices to establish the necessity. Let $Q(R)$ be an NI ring and R be a strong right α -CNZ. Then R is NI by [12, Lemma 2.1]. We freely use these assumption without reference in the following procedure. Let $A\bar{\alpha}(B) = 0$ for $A = ab^{-1}$, $B = cd^{-1} \in Q(R)$, where $a, b, c, d \in R$ with b, d regular. Set I and J be the ideals of $Q(R)$ generated by A and B in $N(Q(R))$, respectively. Then both I and J are nil with $a = Ab \in I$ and $c = Bd \in J$, and so $a, c \in N(R)$. Since R is right Ore, there exist $c_1, b_1 \in R$ with b_1 regular such that $bc_1 = \alpha(c)b_1$ and $c_1b_1^{-1} = b^{-1}\alpha(c)$. Here note that $c_1 \in N(R)$. Indeed, $bc_1 = \alpha(c)b_1 \in J$ and so $c_1 = b^{-1}(bc_1) \in J$. From $0 = A\bar{\alpha}(B) = ab^{-1}\bar{\alpha}(cd^{-1}) = ab^{-1}\alpha(c)\alpha(d)^{-1} = ac_1b_1^{-1}\alpha(d^{-1})$, we have $0 = ac_1 = \alpha\alpha(c')$ putting $c_1 = \alpha(c')$ for some $c' \in N(R)$ and so $c'a = 0$ implies $\alpha(c')\alpha(ab) = 0$ then $abc_1 = 0$ so $\alpha\alpha(c)b_1 = 0$ implies $\alpha\alpha(c) = 0$ and $ca = 0$. Now for $a \in N(R)$, $d \in R$ with d regular, there exist $a_1 \in N(R)$, $d_1 \in R$ with d_1 regular such that $da_1 = ad_1$ and $a_1d_1^{-1} = d^{-1}a$ by the same computation as above. Then $a_1 = d^{-1}ad_1 \in N(R)$ and

$$0 = ca = cad_1 = cda_1 = \alpha(cd)\alpha(a_1) = a_1\alpha(cd) = a_1\alpha(c)\alpha(d) = a_1\alpha(c) = ca_1.$$

Thus

$$BA = cd^{-1}ab^{-1} = c(d^{-1}ab^{-1} = ca_1d_1^{-1}(b)^{-1} = 0,$$

concluding that $Q(R)$ is a strong right $\bar{\alpha}$ -CNZ. □

Corollary 3.14. *Let R be a right Ore ring with the classical right quotient ring $Q(R)$ of R . Then R is CNZ if and only if $Q(R)$ is CNZ.*

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