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General RM Rings

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Abstract

The concept of central RM rings is introduced in this paper as a generalization of RM rings. Since every RM ring is central RM we study the sufficient condition for a central RM ring to be a RM one. It is shown that every central reversible and hence every central symmetric is central RM ring, however converse implications are wrong. It is also proven that the polynomial ring $R[x]$ is central RM ring if R is central RM and quasi-Armendariz. Also α -RM ring has been studied with it is central. ©2019 All rights reserved.

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1. Introduction

Throughout this paper all rings are associative with unity unless otherwise stated. For a ring R , $M_n(R)$, $T_n(R)$, $Z(R)$, $R[x]$, α and id_R denote the $n \times n$ -full matrix ring over R , $n \times n$ -upper triangular matrix ring over R , the center of R , the polynomial ring over R , an endomorphism and the identity endomorphism of R respectively. Denote $\{(a_{ij}) \in U_n(R) \mid \text{the diagonal entries of } (a_{ij}) \text{ are all equal}\}$ by $D_n(R)$. By $\mathbb{Z}$, we mean the ring of integers unless otherwise mentioned. A ring is called reduced if it has no nonzero nilpotent element, similarly a ring R is called central reduced [16] if every nilpotent element of R is central.

A ring R is said to be symmetric [31] if for any $a, b, c \in R$ with $abc = 0$ implies $acb = 0$ if and only if $abc = 0$ implies $bac = 0$. It can be equivalently defined as whenever a product of any number of elements of R is zero, then any permutation of the factors gives product zero [5]. Kafkas et al. [21] defined that a ring R is called central symmetric if whenever $abc = 0$ for $a, b, c \in R$ then $bac \in Z(R)$. They have shown that the class of symmetric rings is strictly contained in the class of central symmetric rings. A ring R is said to be reversible [12] if $ab = 0$ for $a, b \in R$, then $ba = 0$, similarly a ring R is called central reversible [21] if for any $a, b \in R$, $ab = 0$ implies $ba \in Z(R)$. It is shown that every symmetric is reversible and every central symmetric ring is central reversible, however converses in both occasions are not true. A ring R is called semicommutative if whenever $ab = 0$ for $a, b \in R$, then $aRb = 0$, similarly a ring R is defined to be central semicommutative [34] if $ab = 0$ for $a, b \in R$, $aRb = (0)$ implies $bRa = (0)$. Both the classes of semicommutative rings and reflexive rings contain the class of reversible rings and the inclusions

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are strict, in general. Moreover, semicommutative rings and reflexive rings do not imply each other [26, Example 2.3 and Theorem 2.6(2)]. Motivated by these, we introduce the concept of central RM rings as a generalization of RM rings and central reversible rings. The relation among all the central versions of rings mentioned above can be given as follows:

On the other hand, an endomorphism α of a ring R is called rigid [3] if $\alpha\alpha(a) = 0$ implies $a = 0$ for $a \in R$ and R is called an α -rigid ring [16] if there exists such a rigid endomorphism α of R . Baser et al. defined α -reversible [6] and α -semicommutative rings [7] as a generalization of α -rigid rings. Zhao and Zhu [37] defined a ring to be α -reflexive if there exists an endomorphism $\alpha : R \rightarrow R$ such that for $a, b \in R$ if $aRb = (0)$, then $bR\alpha(a) = 0$.

A ring R is called *semi prime* if $aRa = 0$ for every $a \in R$ implies $a = 0$, every semiprime rings is reduced but clearly the converse is not true.

Generalized reduced rings were extended by ring endomorphisms. According to Krempa [16], an endomorphism α of a ring R is called *rigid* if $\alpha\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring R is called α -rigid [16] if there exists a rigid endomorphism α of R . Note that any rigid endomorphism of a ring is a monomorphism and α -rigid rings are reduced rings by [16, Proposition 5]. By Hashemi and Moussavi [14], a ring R is α -compatible if for each $a, b \in R$, $\alpha\alpha(b) = 0$ if and only if $ab = 0$, every α -rigid ring is α -compatible but the converse is not true.

In [6, Definition 2.1], an endomorphism α of a ring R is called *right skew reversible* if whenever $ab = 0$ for $a, b \in R$, $b\alpha(a) = 0$, and the ring R is called *right α -skew reversible* if there exists a right skew reversible endomorphism α of R . Similarly, left α -skew reversible rings are defined. A ring R is called α -skew reversible if it is both left and right α -skew reversible. Note that R is an α -rigid ring if and only if R is semiprime and right α -skew reversible for a monomorphism α of R by [6, Proposition 2.5(iii)]. (We change over from “an α -reversible ring” in [6] to “an α -skew reversible ring” to cohere with other related definitions.)

The reflexive property was studied by Mason [22]. Mason [22] called a right ideal I of a ring R (possibly without identity) *reflexive* if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$, and a ring R is usually called *reflexive* if 0 is a reflexive ideal (i.e., $aRb = 0$ implies $bRa = 0$ for $a, b \in R$).

In [28, Definition 2.1], an endomorphism α of a ring R is called *right (resp., left) skew reflexive* if whenever $aRb = 0$ for $a, b \in R$, $bR\alpha(a) = 0$, and the ring R is called *right (resp., left) α -skew reflexive* if there exists a right (resp., left) skew reflexive endomorphism α of R . A ring R is α -skew reflexive if it is both right and left α -skew reflexive. A ring R is reflexive if R is 1_R -reflexive ring where 1_R denotes the identity endomorphism of R . Any domain R is obviously α -skew reflexive for any endomorphism α of R , but the converse need not hold by the [28, Example 2.2]. An endomorphism α of a ring R is called *right (resp., left) skew RIP* if whenever $eRf = 0$ for $e, f \in \text{Id}(R)$, $fR\alpha(e) = 0$, and the ring R is called *right (resp., left) α -skew RIP* if there exists a right (resp., left) skew RIP endomorphism α of R . A ring R is α -skew RIP if it is both right and left α -skew RIP.

A ring R is said to have *the reflexivity with maximal ideal* [1] if R satisfies the property that

$$aMb = 0 \text{ whenever } bMa = 0 \text{ for a maximal ideal } M \text{ of } R \text{ and } a, b \in R. \quad (*)$$

A ring is called *RM* if it satisfies the condition $(*)$.

We will freely use these facts without reference.

Motivated by the above facts, the concepts of Central RM and Central α -RM rings have been introduced, as a generalization of α -rigid rings. First basic examples and properties has been found. Also Central RM property of some kind of polynomials have been discussed.

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Central RM property of some kind of polynomials have been discussed.

2. Central RM rings

In this section properties of central RM rings and related examples are found .

Definition 2.1. A ring R is said to be central RM for any $a, b \in R$ and for a maximal ideal M of R , whenever $aMb = (0)$, then $bMa \subset Z(R)$.

All RM rings are clearly central RM. But the converse is not true by the following example.

Let K be a field and $A = K\langle x, y, z \rangle$ be the free algebra with non-commuting three indeterminates x, y, z over K . Let I be the ideal of A generated by x^2, y^2, z^2 , all quadratic products of x, y, z except xy, yx, zy, yz , and all quartic products of x, y, z . Set $R = A/I = \{a_0 + a_1x + a_2y + a_3z : a_0, a_1, a_2, a_3 \in K\}$. Identify the elements in A with their images in R for simplicity. The ring R is central RM by straightforward computation. But R is not RM by the following computation. Let M be the ideal of R generated by x, y and z . Then M is the unique maximal ideal of R since $R/M \cong K$. We have $yMx = 0$, but $0 \neq xzy \in xMy$. Thus R is not RM.

Now we find some condition under which a central RM ring is RM. For a nonempty subset S of a ring R , $r_R(S) = \{c \in R \mid Sc = 0\}$ is called the *right annihilator* of S in R . The left annihilator is defined similarly and written by $\ell_R(S)$. If $S = \{a\}$, then we write $r_R(a)$ (resp., $\ell_R(a)$) instead of $r_R(\{a\})$ (resp., $\ell_R(\{a\})$). Recall that a ring R is called right(left) principally quasi-Baer [9] if the right (left) annihilator of a principal right (left) ideal of R generated by an idempotent. A ring R is called right (left) principally projective [10] if the right (left) annihilator of an element of R is generated by an idempotent.

Proposition 2.2. Let R be a central RM ring. Then R is RM if any of the following conditions holds:

- (i) R is a right(left) principally projective ring.
- (ii) R is a right(left) principally quasi-Baer ring.

Proof. Suppose that $aMb = (0)$ for $a, b \in R$ and for a maximal ideal M . Then $bma \in Z(R)$ for any $m \in M$, since R is central RM.

(i) Let R be a right principally projective ring. Then there is an idempotent $e \in R$ such that $r_R(a) = eM$. Thus $ae = 0$. Since $bm \in r_R(a) = eM$, we have $bm = ebm$. It follows that $bma = ebma = bmae = 0$, since $bma \in Z(R)$ and $ae = 0$. Therefore R is RM. A similar proof may be given for left principally projective rings.

(ii) Let R be a right(left) principally quasi-Baer ring. Then there is an idempotent $e \in R$ such that $bm \in r_R(a) = eM$. Thus $ae = 0$. On the other hand, $axbma = abma = 0$ for all $x \in R$, since $aMb = (0)$ and $bma \in Z(R)$. This implies that $bma \in r_R(aM) = (eM)$, and so $bma = ebma = bmae = 0$, by the same method as the proof of (i). Thus R is RM. Similarly it can be shown that every left principally quasi-Baer central RM ring is RM. □

Proposition 2.3. Let R be a ring. Then the following statements are equivalent:

- (1) R is a central RM ring.
- (2) For a maximal ideal M of R , $AMB = 0$ implies $BMA \subset Z(R)$ where A and B are any nonempty subsets of R .
- (3) For a maximal ideal M of R , $IMJ = 0$ implies $JMI \subset Z(R)$ where I and J are any right (or, left) ideals of R .
- (4) For a maximal ideal M of R , $IMJ = 0$ implies $JMI \subset Z(R)$ where I and J are any ideal of R .

Proof. (1) $\Rightarrow$ (2) Assume that the condition (i) holds. Let A and B be two nonempty subsets of R with $AMB = 0$ for a maximal ideal M of R . Then $aMb = 0$ for any $a \in A$ and $b \in B$, and hence $bMa \subset Z(R)$ by assumption. Thus, $BMA = \sum_{a \in A, b \in B} bMa \subset Z(R)$.

(2) $\Rightarrow$ (3) Let I, J be two right ideals of R such that $IMJ = (0)$, for a maximal ideal M of R . By (2) we have $JMI \subset Z(R)$. A similar proof may be given for left ideals of R .

(3) $\Rightarrow$ (4) is straightforward.

(4) $\Rightarrow$ (1) Assume that the condition (4) holds. Let $aMb = 0$ for a maximal ideal M of R and $a, b \in R$. Then $aRMb \subseteq aMb = 0$ and so $RaRMbR = 0$. By assumption, we have $RbRMaR \subset Z(R)$ and hence $bMa \subset Z(R)$. Thus R is a central RM ring. $\square$

We know that every reduced ring is RM. The same follows for their central version also.

Proposition 2.4. *If R is reduced ring, then it is central RM.*

Proof. Let R be reduced ring and let $a, b \in R$ and for a maximal ideal M of R such that $aMb = (0)$. Then $amb = bam = 0$ for $m \in M$ by assumption. Then $bma \subset Z(R)$. Hence $bMa \subset Z(R)$. Thus R is central RM. $\square$

The converse of the above result is however not true as shown by the following example.

The ring $M_2(Z)$ is Reflexive [26, Example 1.6] and hence central RM. But $M_2(Z)$ is not reduced for $0 \neq A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(Z)$ but $A^2 = 0$.

Subring of central RM ring may not be central RM, which can be shown by the following example.

Consider the central RM ring as in Example 2. But $T_n(Z)$ is subring of $M_n(Z)$, which is not central RM as

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0,$$

for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M(T_n)(Z)$, but

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix} \notin Z(T_n(Z)).$$

The ring $M_2(Z)$ is RM [26, Example 1.6] and hence central RM. But $M_2(Z)$ is not reduced for $0 \neq A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(Z)$ but $A^2 = 0$.

One may conjecture that $M_n(Z)$ is central RM if R is central RM. But the following example erases that possibility.

Consider the ring R in Example 2 in the present study. here R is central RM, but $M_n(R)$ is not so, as

$$\begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 + a_1x + a_2y + a_3z & b_0 + b_1x + b_2y + b_3z \\ c_0 + c_1x + c_2y + c_3z & c_0 + c_1x + c_2y + c_3z \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = 0,$$

for all $\begin{pmatrix} a_0 + a_1x + a_2y + a_3z & b_0 + b_1x + b_2y + b_3z \\ c_0 + c_1x + c_2y + c_3z & c_0 + c_1x + c_2y + c_3z \end{pmatrix} \in M(M_n(R))$ but

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z & z \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix} \notin Z(M_n(Z)).$$

Proposition 2.5. (1) *If R is a central RM ring then so is eRe for all $e \in \text{Id}(R)$.*

(2) *If $M_n(R)$ over any ring R is central RM for $n \geq 2$, then so is R .*

(3) *Let R be a ring and $e \in \text{Id}(R)$. If e is central then R is central RM if and only if eR and $(1 - e)R$ are central RM.*

Proof. (1) Let $e \in \text{Id}(R)$. First, we claim that M is a maximal ideal of eRe if and only if $M = eNe$ for some maximal ideal N of R . Let M be a maximal ideal of eRe . Then clearly $M = eNe$ for some ideal N of R . We show that N is a maximal ideal of R . Assume on the contrary that there exists an ideal S of R such that $N \subsetneq S \subsetneq R$. Take $r_0 \in R$ but $r_0 \notin S$. Then $M = eNe \subsetneq eSe$ and eSe is an ideal of eRe , and so $eSe = eRe$ since M is a maximal ideal of eRe . But it is a contradiction to $er_0e \notin eSe$ for $r_0 \in R$. Thus $M = eNe$ and N is a maximal ideal of R .

One can see the proof of the converse of claim in the literature. Now, let R be a central RM ring and $e \in \text{Id}(R)$. Suppose $aMb = 0$ for $a, b \in eRe$ and a maximal ideal M of eRe . Note that $ae = a = ea$ and $eb = b = be$. By claim, there exists a maximal ideal N of R such that $M = eNe$. Hence, $0 = aMb = aeNb = aNb$, and so $bNa \subset Z(R)$ since R is central RM. Thus $Z(R) \supset bNa = beNea = bMa$, showing that eRe is central RM.

(2) It comes from (1), since $R \cong Re_{11} = e_{11}M_n(R)e_{11}$ where $e_{11} \in \text{Id}(M_n(R))$.

(3) Let $e \in \text{Id}(R)$ be central. Suppose that eR and $(1-e)R$ are central RM and let $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R . Recall that eM and $(1-e)M$ are maximal ideals of eR and $(1-e)R$ respectively, by the proof of (1). Then $a(eM)b = eaMb = 0$ and $a((1-e)M)b = (1-e)aMb = 0$. By hypothesis, we have $b(eM)a = ebMa \subset Z(R)$ and $b((1-e)M)a = (1-e)bMa \subset Z(R)$. Thus $bMa = ebMa + (1-e)bMa \subset Z(R)$, showing that R is central RM.

The converse follows from (1) directly. $\square$

The converse of Proposition 2.5(1) does not hold in general. The non-RM ring $R = T_2(\mathbb{Z})$. For $e = e_{11} \in \text{Id}(R)$, $eRe \cong \mathbb{Z}$ is RM clearly.

Recall that for a ring R and an (R, R) -bimodule M , the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note that $T(R, R) = D_2(R)$.

Recall that for a ring R and an (R, R) -bimodule M , the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note that $T(R, R) = D_2(R)$.

Recall that $T(R, R)$ of a semiprime ring R is ideal-symmetric by help of [11, Corollary 2.9] and thus RM. Since every RM ring is central RM, therefore we can say that if R is semiprime, then $T(R, R)$ is central RM. Moreover we obtain the following.

Theorem 2.6. (1) If R is a simple ring, then $T(R, R)$ is central RM.

(2) If $T(R, R)$ of any ring R is central RM, then so is R .

(3) If $M_n(R)$ over any ring R is central RM for $n \geq 2$, then so is R .

Proof. (1) Let R be simple. Then

$$M = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$$

is the only maximal ideal of $T(R, R)$. Suppose that $AMB = 0$ for

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R, R).$$

Then we have $aRc = 0$. But since R is simple, R is RM and so R is central RM and we get $cRa \subset Z(R)$. This implies that $BMA \subset Z(R)$. Thus $T(R, R)$ is central RM.

(2) Suppose that $T(R, R)$ of any nonzero ring R is central RM and let $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R . Then

$$\begin{pmatrix} M & R \\ 0 & M \end{pmatrix}$$

is a maximal ideal of $T(R, R)$ and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} M & R \\ 0 & M \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & aMb \\ 0 & 0 \end{pmatrix} = 0.$$

Since $T(R, R)$ is RM,

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M & R \\ 0 & M \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & bMa \\ 0 & 0 \end{pmatrix} \subset Z(T(R, R)),$$

and so $bMa \subset Z(R)$. Thus R is central RM.

Theorem 2.7. *If $T(R, R)$ is a central RM ring, then R is central reflexive ring.*

Proof. Let $T(R, R)$ be central RM. First, suppose that $aRb = 0$ for $a, b \in R$. Then $aMb = 0$ for a maximal ideal M of R . Since $\mathcal{M} = \begin{pmatrix} M & R \\ 0 & M \end{pmatrix}$ is a maximal ideal of $T(R, R)$ and $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathcal{M} \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = 0$, we have $\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \mathcal{M} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \subset Z(T(R, R))$, by hypothesis. This yields $bRa \subset Z(R)$, and so R is central reflexive.

The following example shows that there exists a ring R which is not central RM such that for any nonzero proper ideal I of R , R/I is central RM and I is central RM as a ring without identity.

Consider $R = D_3(F)$ over a field F . Then R is not RM by [1, Example 1.7(2)]. All the nonzero proper ideals of R are $I_1 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $I_3 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}$, $I_4 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $I_5 =$

$\left\{ \begin{pmatrix} 0 & \alpha b & c \\ 0 & 0 & \alpha d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d, \alpha \in F \text{ and } b \neq 0, d \neq 0 \text{ are fixed} \right\}$. Each I_i for $i = 1, 2, 3, 4, 5$ is nilpotent of

index ≤ 3 , and so it is clearly RM, but they are not reduced obviously. Note that $R/I_1 \cong F$ is RM and $R/I_5 \cong D_2(F)$ is also RM by help of Theorem 3.10(2). Now we show that every R/I_j for $j = 2, 3, 4$ is RM based on

the condition that F is a field. Let $\bar{R}_j = R/I_j$ for $j = 2, 3, 4$. Then $\bar{R}_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} + I_2 \mid a, d \in F \right\}$, $\bar{R}_3 =$

$\left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + I_3 \mid a, b \in F \right\}$ and $\bar{R}_4 = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} + I_4 \mid a, b, d \in F \right\}$. Each $\bar{M}_j$ in the following is

only a maximal ideal of $\bar{R}_j$ for $j = 2, 3, 4$ respectively, since F is a field. $\bar{M}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} + I_2 \mid d \in F \right\}$, $\bar{M}_3 =$

$\left\{ \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + I_3 \mid b \in F \right\}$ and $\bar{M}_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$. Then it can be easily checked that each $\bar{R}_j =$

R/I_j for $j = 2, 3, 4$ is RM.

Therefore every R/I_i (for $i = 1, 2, 3, 4, 5$) is RM.

Proposition 2.8. *Let R/I be a central RM ring for any ideal I of a ring R . If I is a reduced as a ring without identity, then R is central RM.*

Proof. Let $aMb = 0$ with $a, b \in R$ and for a maximal ideal M of R . Since R/I is central RM and M/I is a maximal ideal of R/I . Hence $aMb + I = I \rightarrow bMa + I \subset Z(R/I)$. That is $bMaR - RbMa \subseteq I$. Now let $bmar - rbma \in I$ for all $m \in M$ and $r \in R$. So $(bmar - rbma)^2 \in (bmar - rbma)I = 0$. Since I is reduced we get $bmar - rbma = 0$. Therefore $bmar = rbma$ for all $m \in M$ and $r \in R$ and $bMaR = RbMa$ implies $bMa \subset Z(R)$. Hence R is central RM ring. $\square$

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

A multiplicatively closed subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is shown by [33, Theorem 2.1.12] that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring $Q(R)$ of R with respect to S exists.

Theorem 2.9. *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is central RM if and only if $\Delta^{-1}R$ is central RM.*

Proof. Let R be a central RM ring and let $au^{-1}, bv^{-1} \in \Delta^{-1}R$ such that $au^{-1}\mathcal{M}(\Delta^{-1}R)bv^{-1} = (0)$. Thus for any $ms^{-1} \in \mathcal{M}(\Delta^{-1}R) = \Delta^{-1}(\mathcal{M}(R))$, we have $au^{-1}ms^{-1}bv^{-1} = 0$. This implies $amb = 0$ and so $aMb = 0$ for a maximal ideal M of R , and so $aMb = (0)$. Since R is central RM, therefore $bMa \subset Z(R)$, i.e. $bma \in Z(R)$. Since u^{-1}, v^{-1} and s^{-1} are central elements, therefore $bv^{-1}ms^{-1}au^{-1} \in Z(\Delta^{-1}R)$. Thus $bv^{-1}\Delta^{-1}(\mathcal{M}R)au^{-1} \subset Z(\Delta^{-1}R)$, and hence $\Delta^{-1}R$ is central RM.

Conversely, assume that $\Delta^{-1}R$ is central RM. For $a, b \in R$ and for maximal ideal M of R let $aMb = 0$. Thus $a\Delta^{-1}(\mathcal{M}R)b = 0$. Since $\Delta^{-1}R$ is central RM, therefore $b\Delta^{-1}(\mathcal{M}R)a \subset Z(R)$. Therefore $bMa \subset Z(R)$. Hence R is central RM. $\square$

Proposition 2.10. *Let R be a ring. Then $R[x]$ is central RM if and only if $R[x, x^{-1}]$ is central RM.*

Proof. Let $\Delta = 1, x, x^2, \dots$. Then clearly Δ is a multiplicative closed subset of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$. The rest follows from Theorem 2.15 $\square$

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. It is well-known that a regular ring R is reversible if and only if it is IFP if and only if it is Abelian.

Let R be an algebra over a commutative ring S . Due to Dorroh [13], the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$. We use D to denote the Dorroh extension of R by S .

For an algebra R over a commutative ring S , the Dorroh extension D of R by S is the Abelian group $D = R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Theorem 2.11. *Let R be an algebra over a commutative S . Then R is central RM if and only if the Dorroh extension of R by S is so.*

Proof. First, note that $\mathcal{M} = R \oplus \{0\}$ is the unique maximal ideal of D , since every $(r, s) \in D$ is a unit when $s \neq 0$ by the proof of [19, Proposition 1.5] and that $s \in S$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$.

Suppose that R be central RM and let $(r_1, s_1)\mathcal{M}(r_2, s_2) = 0$ for $(r_1, s_1), (r_2, s_2) \in D$. Since

$$(r_1, s_1)(r, 0)(r_2, s_2) = (r_1rr_2 + s_1rr_2 + s_2r_1r + s_1s_2r, 0) = (0, 0),$$

for $r_1, r, r_2 \in R, s_1, s_2 \in S$ and

$$(r_1 + s_1 1)r(r_2 + s_2 1) = r_1 r r_2 + s_1 r r_2 + s_2 r_1 r + s_1 s_2 r,$$

for $(r_1 + s_1 1), (r_2 + s_2 1) \in R$ and $r \in R$.

For any $(r, 0) \in \mathcal{M}$, we have $(r_1, s_1)R(r_2, s_2) = 0$. Since R is a right α -skew reflexive ring, we get

$$0 = (r_2 + s_2 1)R\alpha(r_1 + s_1 1) = (r_2 + s_2 1)R(\alpha(r_1) + s_1).$$

Hence

$$r_2 r \alpha(r_1) + s_2 r \alpha(r_1) + s_1 r_2 r + s_1 s_2 r = 0,$$

for all $r \in R$. Let $(r, 0) \in \mathcal{M}$. Then

$$(r_2, s_2)(r, 0)(\alpha(r_1), s_1) = ((r_2 r + s_2 r)\alpha(r_1) + s_1(r_2 r + s_2 r), 0) = (r_2 r \alpha(r_1) + s_2 r \alpha(r_1) + s_1 r_2 r + s_1 s_2 r, 0) = (0, 0),$$

showing that $(r_2, s_2)\mathcal{M}\tilde{\alpha}(r_1, s_1) = 0$. Therefore D is a right $\tilde{\alpha}$ -skew RM ring.

Conversely, suppose that D is central RM and let $aMb = 0$ for $a, b \in R$ and for a maximal ideal M of R . Then

$$a(r + s)b = 0$$

for any $(r, s) \in \mathcal{M}(D)$. This yields

$$(a, 0)(r, s)(b, 0) = 0$$

for $(r, s) \in \mathcal{M}(D)$. Since D is central RM, therefore

$$(b, 0)(r, s)(a, 0) \in Z(D)$$

and so, $b(r + s)a \in Z(D)$, proving that R is central RM. □

Rege and Chhawchharia called a ring R *Armendariz* [35, Definition 1.1] if whenever the product of any two polynomials in $R[x]$ over R is zero, then so is the product of any pair of coefficients from two polynomials. It is well-known that any reduced ring is Armendariz, but not conversely. Quasi-Armendariz rings [15] are the generalization of Armendariz rings defined by Hireno. A ring R is called *quasi-Armendariz* [15] if whenever $f(x) = \sum_{i=0}^n a_i x^i, g(x) = \sum_{j=0}^m b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, then $a_i R b_j = 0$ for all i, j .

Theorem 2.12. *Let R be a quasi-Armendariz central RM ring. Then $R[x]$ is central RM ring.*

Proof. Let $p(x)M[x]q(x) = 0$ for $p(x) = \sum_{i=0}^n a_i x^i, q(x) = \sum_{j=0}^m b_j x^j \in R[x]$ and for a maximal ideal $M[x]$ of $R[x]$. Since R is a quasi-Armendariz then we obtain $a_i M b_j = 0$ for all i, j and for a maximal ideal M of R . Since R is central RM, therefore $g(x)M[x]p(x) \subset Z(R[x])$ and hence $R[x]$ is central RM. □

The quasi-Armendariz property and central RM property do not imply each other as shown in the next example:

(i) Since $\mathbb{Z}$ is a semiprime ring, therefore by [26, Proposition 2.5(2)(ii)]. $T(\mathbb{Z}, \mathbb{Z})$ is RM. Thus by [26, Example 2.16] the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in T(\mathbb{Z}, \mathbb{Z}) \right\}$ is an RM and hence a central RM ring but not a quasi-Armendariz ring.

(ii) The ring $T_2(\mathbb{Z})$ is quasi-Armendariz by [15, Corollary 3.15] but not central RM by Example 2.

3. α -RM Rings

In this section, the notion of the central α -RM ring and its properties have been introduced. We begin with the following definition.

Definition 3.1. An endomorphism α of a ring R is called RM if whenever $aMb = 0$ for $a, b \in R$ and for a maximal ideal M of R , $bM\alpha(a) = 0$ (resp., $\alpha(b)Ma = 0$). A ring R is called α -RM if there exists an RM endomorphism α of R .

Both simple ring and domains are obviously α -RM ring for any endomorphism α of a given ring R , but the converses need not hold. The following example shows that the α -RM property is not left-right symmetric.

Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2 \right\}$$

(1) Let $\alpha : R \rightarrow R$ be an endomorphism defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

For

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R.$$

We discuss two cases:

Case 1. For a maximal ideal $\mathcal{M} = \begin{pmatrix} 0 & R \\ 0 & R \end{pmatrix}$ of R assume that $AMB = 0$. Then we have $BM\alpha(A) = 0$. It is obvious that R is a right α -RM ring.

Case 2. For a maximal ideal $\mathcal{N} = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$ of R by the definition of α assume that $ANB = 0$. Then for any $\begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \in \mathcal{N}$, we have $ara' = 0$ and so $\alpha(ra') = 0$. This entails that $BN\alpha(A) = 0$.

Conversely, R is not left α -RM. To see this, let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

Then $ANB = 0$, for the maximal ideal $\mathcal{N} = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$ of R , but

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in \alpha(B)\mathcal{N}A.$$

(2) Let $\beta : R \rightarrow R$ be an endomorphism defined by

$$\beta \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

By the similar method to the proof (1), we can show that R is left β -RM, but not right β -RM.

Lemma 3.2. If a ring R is an α -rigid ring, then R is α -RM.

Proof. Let R be α -rigid. Assume that $aMb = 0$ for a maximal ideal M of R and $a, b \in R$. Then $amb = 0$ for all $m \in M$ and so $abm = 0$ since R is reduced and so we obtain $bM\alpha(a)\alpha(bM\alpha(a)) = bM\alpha(abm)\alpha^2(a) = 0$. This shows that $bM\alpha(a) = 0$ and so $bM\alpha(a) = 0$ since R is α -rigid. Hence R is right α -RM. Now by the similar argument to above from $aMb = 0$, we also have $mab = 0$ for all $m \in M$ since R is reduced. Thus we get $\alpha(\alpha(b)ma)\alpha(b)ma = \alpha^2(b)\alpha(mab)ma = 0$. This shows that $\alpha(b)ma = 0$ and so $\alpha(b)Ma = 0$, since R is α -rigid. This shows that R is left α -RM. Therefore R is α -RM. $\square$

The converse of Lemma 4.2 does not hold by the following example.

Let $\mathbb{Z}$ be the ring of integers. Consider the ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

Let $\alpha : R \rightarrow R$ be an endomorphism defined by

$$\alpha \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}.$$

It is clear that α is an automorphism and R is not semiprime and hence R is not α -rigid.

Now for a maximal ideal $\mathcal{M} = \begin{pmatrix} M & R \\ 0 & M \end{pmatrix}$ of R and $A, C \in R$, let $ABC = 0$ for all $B \in \mathcal{M}$ with

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, C = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \text{ and } B = \begin{pmatrix} m & n \\ 0 & m \end{pmatrix}.$$

Then we have $amc = 0$ and $amd + anc + bmc = 0$ for every m . This implies that $a = 0$ or $c = 0$. If $a = 0$, then $bmc = 0$ for every $m \in M$ a maximal ideal of $\mathbb{Z}$, hence $b = 0$ or $c = 0$. If $c = 0$, then $amd = 0$ for each $m \in M$ where M is a maximal ideal of R , hence $a = 0$ or $d = 0$. In both cases, we obtain $CB\alpha(A) = 0$. Hence R is $\bar{\alpha}$ -RM. Therefore R is $\bar{\alpha}$ -RM.

We have the basic equivalence for right α -skew RM rings as follows. For a non-empty subset X of a ring R , we write $r_R(X) = \{c \in R \mid Xc = 0\}$ which is called the *right annihilator* of X in R . Similarly, $\ell_R(X)$ denotes the left annihilator of X in R .

Proposition 3.3. *For a ring R with an endomorphism α , the following are equivalent:*

- (1) R is an α -RM ring.
- (2) For a maximal ideal M of R and $a \in R$, $r_R(aM) \subseteq \ell_R(M\alpha(a))$.
- (3) For a maximal ideal M of R , $AMB = 0$ implies $BM\alpha(A) = 0$, where A and B are any non-empty subsets of R .
- (4) For a maximal ideal M of R , $IMJ = 0$ implies $JM\alpha(I) = 0$, where I and J are any right (or left) ideals of R .
- (5) For a maximal ideal M of R , $IMJ = 0$ implies $JM\alpha(I) = 0$, where I and J are any ideals of R .

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are straightforward.

(2) $\Rightarrow$ (3): Assume that the condition (2) holds. Let A and B be two non-empty subsets of R with $AMB = 0$ for a maximal ideal M of R . Then $aMb = 0$, for any $a \in A$ and $b \in B$ and hence $bM\alpha(a) = 0$ by assumption. Thus, $BM\alpha(A) = \sum_{a \in A, b \in B} bM\alpha(a) = 0$.

(5) $\Rightarrow$ (1): Assume that the condition (5) holds. Let $aMb = 0$, for a maximal ideal M of R and $a, b \in R$. Then $aMRb \subseteq aMb = 0$ and so, $(RaR)M(RbR) = 0$. By assumption, we have $(RbR)M\alpha(RaR) = 0$ and hence $bM\alpha(a) = 0$. Thus, R is right α -skew RM. $\square$ $\square$

Proposition 3.4. *Let α be an endomorphism of a ring R . If R is an α -skew RM ring, then $aMb = 0$ for a maximal ideal M of R and $a, b \in R$ implies $aM\alpha^{2k}(b) = 0$ and $\alpha^{2k-1}(b)Ma = 0$, for a positive integer k .*

Proof. Suppose that R is α -skew RM and $aMb = 0$ for a maximal ideal M of R and $a, b \in R$. $\alpha(b)Ma = 0$, which implies $aM\alpha^2(b) = 0$ and $\alpha^3(b)Ma = 0$ and $aM\alpha^n(b) = 0, \dots$, and so we get $aM\alpha^{2k}(b) = 0$ and $\alpha^{2k-1}(b)Ma = 0$ for $k \geq 1$ inductively. $\square$

Proposition 3.5. *Let α be an endomorphism of a ring R , S a ring and suppose that $\sigma : R \rightarrow S$ is a ring isomorphism. Then R is an α -RM ring if and only if S is an $\sigma\alpha\sigma^{-1}$ -RM ring.*

Proof. Let R be an α -RM and let $a'N'b' = 0$, for a maximal ideal N' of S and $a', b' \in S$, then there exist $a, b \in R$ such that $\sigma(a) = a'$, $\sigma(b) = b'$ and a maximal ideal N of R such that $\sigma(N) = N'$.

So $a'N'b' = 0$ implies $\sigma(a)\sigma(N)\sigma(b) = \sigma(0)$ and $\sigma(aNb) = \sigma(0)$ so $aNb = 0$ since σ is an isomorphism. Then $bN\alpha(a) = 0$ by hypothesis. This entails that

$$0 = \sigma(0) = \sigma(bN\alpha(a)) = \sigma(bN\alpha((\sigma^{-1}\sigma)(a))) = \sigma(b)\sigma(N)\sigma(\alpha(\sigma^{-1}(\sigma(a)))) = b'N'(\sigma\alpha\sigma^{-1})(a') = 0.$$

Thus S is an $\sigma\alpha\sigma^{-1}$ -RM ring.

Conversely, let S be an $\sigma\alpha\sigma^{-1}$ -RM, let $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R . Set $\sigma(a) = a'$, $\sigma(b) = b'$ and $\sigma(M) = M'$. Then $a', b' \in S$ and M' a maximal ideal of S such that $a'M'b' = \sigma(aMb) = 0$. Since S is an $\sigma\alpha\sigma^{-1}$ -RM we have $0 = b'M'(\sigma\alpha\sigma^{-1})(a') = 0$ and so $\sigma(bM\alpha(a)) = 0$. Since σ is an isomorphism $bM\alpha(a) = 0$. Shows that R is an α -RM ring. $\square$

For an endomorphism α and an idempotent e of a ring R let $\alpha(e) = e$. Then we have an endomorphism $\bar{\alpha} : eRe \rightarrow eRe$ defined by $\bar{\alpha}(ere) = e\alpha(r)e$.

Proposition 3.6. *Let R be a ring with an endomorphism α such that $\alpha(e) = e$ for $e^2 = e \in R$.*

(1) *If R is an α -RM ring then eRe is an $\bar{\alpha}$ -RM ring for all $e \in \text{Id}(R)$.*

(2) *If e is central for $e \in \text{Id}(R)$, then R is an right α -RM ring if and only if eR and $(1-e)R$ are $\bar{\alpha}$ -RM rings.*

Proof. (1) Let R be an α -RM ring and $e \in \text{Id}(R)$. Suppose $aMb = 0$ for $a, b \in eRe$ and a maximal ideal M of eRe . Note that $eae = a$ and $ebe = b$. By claim, there exists a maximal ideal N of R such that $M = eNe$. Hence, $0 = aMb = aeNeb = eae(eNe)ebe = eaeNebe = aNb = bN\alpha(a)$, since R is an α -RM ring. Thus $0 = bN\alpha(a) = ebe(eNe)e\alpha(a)e = ebeM\bar{\alpha}(eae) = bM\bar{\alpha}(a)$. Showing that eRe is an $\bar{\alpha}$ -RM ring.

(2) Let $e \in \text{Id}(R)$ be central. Suppose that eR and $(1-e)R$ are $\bar{\alpha}$ -RM rings, let $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R . Recall that eM and $(1-e)M$ are maximal ideals of eR and $(1-e)R$, respectively. By the proof of (1). Then $ea(eM)eb = 0$ and $(1-e)a((1-e)M)(1-e)b = 0$. By hypothesis, we have $0 = eb(eM)\bar{\alpha}(ea) = eb(eM)e\alpha(a) = ebM\alpha(a)$ and $0 = (1-e)b((1-e)M)\bar{\alpha}((1-e)a) = (1-e)b((1-e)M)((1-e)\alpha(a)) = (1-e)bM\alpha(a)$.

Thus $bM\alpha(a) = ebM\alpha(a) + (1-e)bM\alpha(a) = 0$, showing that R is an α -RM ring.

The converse part follows from (1) directly. $\square$

The next Theorem gives the relationship between RM and α -RM rings under the condition that R is α -compatible.

Theorem 3.7. *Let R be an α -compatible ring.*

(1) *R is reflexive ring if and only if R is α -skew reflexive ring.*

(2) *R is RM ring if and only if R is α -RM ring.*

Proof. (1) Let R be a reflexive ring and $aRb = 0$ for $a, b \in R$ then $bRa = 0$. So $arb = 0$ for all $r \in R$ and $bra = 0$ by hypothesis and $bc\alpha(a) = 0$ for all $c \in R$ (by α -compatibility), hence $bR\alpha(a) = 0$, and so R is right α -skew reflexive ring. By the same method in above we get R is left α -skew reflexive ring. Therefore R is α -skew reflexive ring.

Conversely, let R be an α -skew reflexive ring and $aRb = 0$ for $a, b \in R$, so $acb = 0$ for all $c \in R$. Then $bc\alpha(a) = 0$ by hypothesis and $bca = 0$ for all $c \in R$ (by α -compatibility), so $bRa = 0$. Therefore R is reflexive ring.

(2) Let R be an RM ring, $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R , then $amb = 0$ for all $m \in M$. Then $bma = 0$ for all $m \in M$ by hypothesis and $bm\alpha(a) = 0$ for all $m \in M$ (by α -compatibility), hence $bM\alpha(a) = 0$, and so R is an α -RM ring. and hence R is α -RM ring.

Conversely, let R be an α -RM ring, $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R , so $amb = 0$ for all $m \in M$. Then $bm\alpha(a) = 0$ by hypothesis and $bma = 0$ for all $m \in M$ (by α -compatibility), so $bMa = 0$. Therefore R is an RM ring. □

According to the the following Example, we see that the condition “ α -compatibility” is not superfluous.

(1) Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and it is maximal ideal $\mathcal{M} = \mathbb{Z}_2 \oplus \{0\}$ with endomorphism $\alpha : R \rightarrow R$ defined by $\alpha((a, b)) = (b, a)$ with the usual addition and multiplication α is not compatible. The ring R is a commutative semiprime ring and hence it is an RM ring. However, R is not an α -RM ring. Indeed, for $a = (0, 1), b = (1, 0) \in R$ and $(1, 0) \in \mathcal{M}$, we have $(0, 1)(1, 0)^2 = 0$, but $(1, 0)^3 = (1, 0) \neq 0 \in bM\alpha(a)$.

(2) Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with endomorphism $\alpha : R \rightarrow R$ defined by $\alpha((a, b)) = (b, a)$ with the usual addition and multiplication α is not compatible. The ring R is a commutative semiprime ring and hence it is reflexive. However, R is not an α -skew reflexive. Indeed, for $a = (0, 1), b = (1, 0) \in R$ and $(1, 1) \in R$, we have $(0, 1)(1, 1)(1, 0) = 0$, but $(1, 0)(1, 1)(1, 0) = (1, 0) \neq 0 \in bR\alpha(a)$.

Proposition 3.8. *Let R be a ring with endomorphism α . Then R is an α -RM ring if and only if $\text{Mat}_n(R)$ is an $\bar{\alpha}$ -RM ring for all $n \geq 2$.*

Proof. Suppose that R is an α -RM ring. Let $S = M_n(R)$ and $ANB = 0$ for ideals A, B of S and for a maximal idea N of S . Using an elementary ring theoretic argument, there exists ideals I and J and a maximal ideal M of R such that $A = M_n(I), B = M_n(J)$ and $N = M_n(M)$. Then

$$M_n(IMJ) = M_n(I)M_n(M)M_n(J) = ANB = 0,$$

implies $IMJ = 0$. Since R is an α -RM ring, $JM\alpha(I) = 0$ by Proposition 3.3(5). This yields $BN\bar{\alpha}(A) = 0$, and so S is an $\bar{\alpha}$ -RM ring for all $n \geq 2$ by Proposition 3.3(5).

Conversely, Since $R \cong Re_{11} = e_{11}M_n(R)e_{11}$, where $e_{11} \in \text{Id}(M_n(R))$ so R is an α -RM ring by Proposition 3.6(1). □

Given a ring R and an (R, R) -bimodule M , the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

For an endomorphism α of a ring R and the trivial extension $T(R, R)$ of R , $\bar{\alpha} : T(R, R) \rightarrow T(R, R)$ defined by

$$\bar{\alpha} \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right) = \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix},$$

is an endomorphism of $T(R, R)$. Since $T(R, 0)$ is isomorphic to R , we can identify the restriction of $\bar{\alpha}$ by $T(R, 0)$ to α .

The converse of Proposition 3.6 does not hold in general.

Corollary 3.9. Let R be a reduced ring with an endomorphism α . Then the following are equivalent.

- (1) R is an α -RM ring.
- (2) The trivial extension $T(R, R)$ of R is an $\bar{\alpha}$ -RM ring.
- (3) $\frac{R[x]}{x^n R[x]}$ is an $\bar{\alpha}$ -RM ring, for $n \geq 2$.

Proof. (1) $\Rightarrow$ (3). It follows by the [2, Theorem 3.1]. (3) $\Rightarrow$ (2). The proof is trivial. (2) $\Rightarrow$ (1). Suppose that $T(R, R)$ of any non-zero ring R is an $\bar{\alpha}$ -RM ring and let $aMb = 0$ for $a, b \in R$ and a maximal ideal M of R . Then,

$$\begin{pmatrix} M & R \\ 0 & M \end{pmatrix}$$

is a maximal ideal of $T(R, R)$ and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} M & R \\ 0 & M \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & aMb \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $T(R, R)$ is an $\bar{\alpha}$ -RM then,

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M & R \\ 0 & M \end{pmatrix} \begin{pmatrix} \alpha(a) & 0 \\ 0 & \alpha(a) \end{pmatrix} = \begin{pmatrix} 0 & bM\alpha(a) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and so $bM\alpha(a) = 0$. Thus, R is an α -RM ring. □

Proposition 3.10. If R is a division ring, then $T(R, R)$ is an $\bar{\alpha}$ -RM.

Proof. Let R be a division ring. Then

$$M = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$$

is the only maximal ideal of $T(R, R)$. Suppose that $AMB = 0$, for

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R, R).$$

Then, we have

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

implies $arc = 0$. Since R is division ring so R is domain and $arc = 0$ implies $a = 0$ or $rc = 0$ so we have $a = 0$ or $r = 0$ or $c = 0$ and so from

$$\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha(a) & \alpha(b) \\ 0 & \alpha(a) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we get $c\alpha(a) = 0$ and this implies that $BM\bar{\alpha}(A) = 0$. Thus, $T(R, R)$ is an $\bar{\alpha}$ -RM ring. □

Let R be an algebra over a commutative ring S . Due to Dorroh [13], the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$. We use D to denote the Dorroh extension of R by S .

For an endomorphism α of R and the Dorroh extension D of R by S , $\bar{\alpha} : D \rightarrow D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an S -algebra homomorphism. Recall that an algebra R in which every element of the algebra is nilpotent a *nil algebra*.

The next result shows one way to build new α -RM rings from old once.

Theorem 3.11. Let R be a nil algebra over a field K and D the Dorroh extension of R by K . Let α be an endomorphism of R with $\alpha(1) = 1$. Then R is a right α -skew reflexive ring if and only if D is an $\bar{\alpha}$ -RM ring.

Proof. First, note that $\mathcal{M} = R \oplus \{0\}$ is the unique maximal ideal of D , since every $(r, s) \in D$ is a unit when $s \neq 0$ by the proof of [19, Proposition 1.5] and that $s \in K$ is identified with $s1 \in R$ and so $R = \{r + s \mid (r, s) \in D\}$.

Suppose that R is a right α -skew reflexive ring with $\alpha(1) = 1$ and let $(r_1, s_1)\mathcal{M}(r_2, s_2) = 0$ for $(r_1, s_1), (r_2, s_2) \in D$. Since

$$(r_1, s_1)(r, 0)(r_2, s_2) = (r_1rr_2 + s_1rr_2 + s_2r_1r + s_1s_2r, 0) = (0, 0),$$

for $r_1, r, r_2 \in R, s_1, s_2 \in K$ and

$$(r_1 + s_11)r(r_2 + s_21) = r_1rr_2 + s_1rr_2 + s_2r_1r + s_1s_2r,$$

for $(r_1 + s_11), (r_2 + s_21) \in R$ and $r \in R$.

For any $(r, 0) \in \mathcal{M}$, we have $(r_1, s_1)R(r_2, s_2) = 0$. Since R is a right α -skew reflexive ring, we get

$$0 = (r_2 + s_21)R\alpha(r_1 + s_11) = (r_2 + s_21)R(\alpha(r_1) + s_1).$$

Hence

$$r_2r\alpha(r_1) + s_2r\alpha(r_1) + s_1r_2r + s_1s_2r = 0,$$

for all $r \in R$. Let $(r, 0) \in \mathcal{M}$. Then

$$(r_2, s_2)(r, 0)(\alpha(r_1), s_1) = ((r_2r + s_2r)\alpha(r_1) + s_1(r_2r + s_2r), 0) = (r_2r\alpha(r_1) + s_2r\alpha(r_1) + s_1r_2r + s_1s_2r, 0) = (0, 0),$$

showing that $(r_2, s_2)\mathcal{M}\bar{\alpha}(r_1, s_1) = 0$. Therefore D is a right $\bar{\alpha}$ -RM ring.

Conversely, suppose that D is a right $\bar{\alpha}$ -RM and let $aRb = 0$ for $a, b \in R$. Then

$$(a, 0)\mathcal{M}(b, 0) = (aRb, 0) = (0, 0).$$

Since D is a right $\bar{\alpha}$ -RM ring,

$$(b, 0)\mathcal{M}\bar{\alpha}(a, 0) = (b, 0)\mathcal{M}(\odot(-), \iota) = \iota,$$

and it implies that $bR\alpha(a) = 0$. Thus R is a right α -skew reflexive. □

4. Central α -RM Rings

In this section, the notion of the central α -RM ring and its properties have been introduced. We begin with the following definition

Definition 4.1. An endomorphism α of a ring R is called central RM if whenever $aMb = 0$ for $a, b \in R$ and for a maximal ideal M of R , $bM\alpha(a) \subset Z(R)$. A ring R is called *central α -RM* if there exists a central RM endomorphism α of R .

From definition it is clear that every central RM ring R is central id_R -RM ring. Every α -RM ring is central α -RM ring. But the converse is not true by the following example.

Consider the ring R as in Example 1. Clearly R is central id_R -RM ring but not id_R -RM ring.

Recall that if α is an endomorphism of a ring R , then the map $\bar{\alpha} : R[x] \Rightarrow R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^m a_i x^i) = \sum_{i=0}^m \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extend α . Let R be a ring and Δ be a multiplicative monoid in R consisting of central regular elements, and let $\Delta^{-1}R = u^{-1}a \mid u \in \Delta, a \in R$, then $\Delta^{-1}R$ is a ring. We have

Proposition 4.2. *Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements and $F(x)H(x)G(x) = 0$. Then $R[x]$ is central α -RM if and only if $\Delta^{-1}R[x]$ is central $\bar{\alpha}$ -RM.*

Proof. It sufficed to show that $\Delta^{-1}R[x]$ is central $\bar{\alpha}$ -RM if $R[x]$ is central α -RM. Let $f(x) = \sum_{i=0}^m u_i^{-1} a_i x^i$, $g(x) = \sum_{j=0}^n v_j^{-1} b_j x^j \in \Delta^{-1}R[x]$ and for $h(x) \in \Delta^{-1}(\mathcal{M}R)[x]$ where $\Delta^{-1}(\mathcal{M}R)[x]$ is a maximal ideal of $\Delta^{-1}R[x]$. Then $F(x) = (u_m, u_{m-1}, \dots, u_0)f(x)$, $H(x) = (\lambda_t, \lambda_{t-1}, \dots, \lambda_0)h(x)$ and $G(x) = (v_n, v_{n-1}, \dots, v_0)g(x) \in R[x]$. Since $R[x]$ is central α -RM and $F(x)H(x)G(x) = 0$, this implies that $G(x)H(x)\alpha(F(x)) \subset Z(F(x)H(x)G(x))$, and so $g(x)h(x)\alpha(f(x)) \subset Z(R[x])$ since Δ is a multiplicative monoid in R consisting of central regular elements and $u_i, v_j \in \Delta$ for all i, j . This shows that $\Delta^{-1}R[x]$ is central $\bar{\alpha}$ -RM. $\square$

Proposition 4.3. *For a ring R . Then $R[x]$ is central α -RM if and only if $R[x, x^{-1}]$ is central α -RM.*

Proof. It suffices to prove the necessity only, Suppose $R[x]$ is central α -RM. Let $\Delta = 1, x, x^2, \dots$. Then clearly Δ is a multiplicative closed subset of $R[x]$ and $R[x, x^{-1}] = \Delta^{-1}R[x]$. It follows that $R[x, x^{-1}]$ is central α -RM, by Proposition 4.2. $\square$

Theorem 4.4. *Let R be a quasi-Armendariz ring. Then the following statements are equivalent:*

- (1) R is central α -RM.
- (2) $R[x]$ is a central $\bar{\alpha}$ -RM.
- (3) $R[x, x^{-1}]$ is a central $\bar{\alpha}$ -RM.

Proof. It is enough to show that (i) $\Rightarrow$ (ii): Let $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^n b_j x^j \in R[[x, x^{-1}]]$, and $h(x) \in (\mathcal{M}R)[x]$ such that Let $f(x)h(x)g(x) = 0$. Since R is quasi-Armendariz, then $a_i R b_j = 0$ for all i, j and so $b_j R \alpha(a_i) \in Z(R)$. Therefore $g(x)(\mathcal{M}R)[x]\alpha(f(x)) \in Z(R[x])$. Hence, $R[x]$ is central α -skew RM. $\square$

If you'd like to thank anyone, place your comments here.

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